TOWARDS NON–ABELIAN P–ADIC HODGE THEORY IN THE GOOD REDUCTION CASE

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ABSTRACT. We develop a non–abelian version of P–adic Hodge Theory for varieties (possible open with "nice compactification") with good reduction. This theory yields in particular a comparison between smooth p–adic sheaves and F–isocrystals on the level of certain Tannakian categories, p–adic Hodge theory for relative Malcev completions of fundamental groups and their Lie algebras, and gives information about the action of Galois on fundamental groups.

CONTENTS

1. INTRODUCTION

1.1. The aim of this paper is to study p–adic Hodge theory for non–abelian invariants. Let us begin, however, by reviewing some of the abelian theory.

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Let k be a perfect field of characteristic $p > 0$, V the ring of Witt vectors of k, K the field of fractions of V, and $K \hookrightarrow \overline{K}$ an algebraic closure of K. Write G_K for the Galois group $\mathrm{Gal}(\overline{K}/K)$. Let X/V be a smooth proper scheme, and denote by X_K the generic fiber of K. Let $D \subset X$ be a divisor with normal crossings relative to V, and set $X^o := X - D$. Denote by $H^*_{\mathrm{et}}(X^o_{\overline{K}})$ the étale cohomology of $X^o_{\overline{K}}$ with coefficients $\mathbb{Q}_p$ and by $H^*_{\mathrm{dR}}(X^o_K)$ the algebraic de Rham cohomology of X^o_K. The vector space $H^*_{\mathrm{et}}(X^o_{\overline{K}})$ has a natural action $\rho_{X^o_{\overline{K}}}$ of G_K, and the space $H^*_{\mathrm{dR}}(X^o_K)$ is a filtered F–isocrystal. That is, the space $H^*_{\mathrm{dR}}(X^o_K)$ comes equipped with a filtration $\mathrm{Fil}^\cdot_{X^o_K}$ and a semi–linear (with respect to the canonical lift of Frobenius to V) Frobenius automorphism $\varphi_{X^o_K} : H^*_{\mathrm{dR}}(X^o_K) \rightarrow H^*_{\mathrm{dR}}(X^o_K)$. The theory of p–adic Hodge theory implies that the two collections of data $(H^*_{\mathrm{et}}(X^o_{\overline{K}}), \rho_{X^o_{\overline{K}}})$ and $(H^*_{\mathrm{dR}}(X^o_K), \mathrm{Fil}^\cdot_{X^o_K}, \varphi_{X^o_K})$ determine each other [Fa1, Fa2, Fa3, Ts1].

1.2. More precisely, let $B_{\mathrm{cris}}(V)$ denote the ring defined by Fontaine [Fo1, Fo2, Fo3], MF_K the category of K–vector spaces M with a separated and exhaustive filtration $\mathrm{Fil}^\cdot$ and a semi–linear automorphism $\varphi_M : M \rightarrow M$, and let $\mathrm{Rep}^{\mathrm{cts}}_{\mathbb{Q}_p}(G_K)$ be the category of continuous representations of G_K on $\mathbb{Q}_p$–vector spaces. The ring $B_{\mathrm{cris}}(V)$ comes equipped with an action of G_K, a semi–linear Frobenius automorphism, and a filtration. There is a functor

$$\mathbf{D} : \mathrm{Rep}^{\mathrm{cts}}_{\mathbb{Q}_p}(G_K) \longrightarrow MF_K \tag{1.2.1}$$

sending a representation L to $(L \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V))^{G_K}$ with the semi–linear automorphism and filtration induced by that on $B_{\mathrm{cris}}(V)$. For any $L \in \mathrm{Rep}(G_K)$ there is a natural transformation

$$\alpha_L : \mathbf{D}(L) \otimes_K B_{\mathrm{cris}}(V) \longrightarrow L \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \tag{1.2.2}$$

which by [Fo1, 5.1.2 (ii)] is always injective. The representation L is called *crystalline* if the map α_L is an isomorphism, in which case L and $\mathbf{D}(L)$ are said to be *associated*. The precise statement of the comparison between between étale and de Rham cohomology in the above situation is then that $(H^*_{\mathrm{et}}(X^o_{\overline{K}}), \rho_{X^o_{\overline{K}}})$ and $(H^*_{\mathrm{dR}}(X^o_K), \mathrm{Fil}^\cdot_{X^o_K}, \varphi_{X^o_K})$ are associated. In particular there is a natural isomorphism

$$H^*_{\mathrm{dR}}(X^o_K) \otimes_K B_{\mathrm{cris}}(V) \simeq H^*_{\mathrm{et}}(X^o_{\overline{K}}) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \tag{1.2.3}$$

compatible with the actions of G_K, the filtrations, and the Frobenius automorphisms.

There is also a version of the comparison 1.2.3 with coefficients. In [Fa1, Chapter V (f)] Faltings defines a notion of a *crystalline sheaf* on the scheme X^o_K, which is a smooth $\mathbb{Q}_p$–sheaf L on $X^o_{K,\mathrm{et}}$ which is associated in a suitable sense to a filtered log F–isocrystal $(E, \mathrm{Fil}_E, \varphi_E)$ on X_k/K (see 6.13 for a precise definition). For such a sheaf L, the étale cohomology $H^*(X^o_{\overline{K},\mathrm{et}}, L_{\overline{K}})$ is a Galois representation (here $L_{\overline{K}}$ denotes the restriction of L to $X^o_{\overline{K},\mathrm{et}}$) and the log de Rham cohomology $H^*_{\mathrm{dR}}(X_K, E)$ is naturally viewed as an object of MF_K. In [Fa1, 5.6], Faltings shows that the representation $H^*(X^o_{\overline{K},\mathrm{et}}, L_{\overline{K}})$ is crystalline and is associated to $H^*_{\mathrm{dR}}(X_K, E)$.

The main goal of this paper is to generalize to the level of certain homotopy types the above comparisons between cohomology.

1.3. Before explaining the main results of the paper, let us discuss an example which provided motivation for this work and hopefully helps put the more technical results below in context. This example is not discussed in the body of the paper and the reader who so desires can skip to 1.5. Let $g \geq 3$ be an integer and $\mathcal{M}^o_g$ the moduli space of smooth curves of genus g. If we fix a point $p \in \mathcal{M}^o_g(\mathbb{C})$ corresponding to a curve C, the fundamental group $\pi_1(\mathcal{M}^o_g(\mathbb{C}), p)$ (where $\mathcal{M}^o_g(\mathbb{C})$ is viewed as an orbifold) is naturally identified with the so–called *mapping class group* Γ_g which has a long and rich history (see [Ha1, Ha3, H-L, Na] for further discussion). The first homology of the universal curve over $\mathcal{M}^o_{g,\mathbb{C}}$ defines a local system on $\mathcal{M}^o_{g,\mathbb{C}}$ which by our choice of base point defines a representation

$$\Gamma_g \longrightarrow \mathrm{Aut}(H_1(C, \mathbb{Z})). \tag{1.3.1}$$

The kernel is the so–called *Torelli group* denoted T_g. Associated to the representation 1.3.1 is a pro–algebraic group $\mathcal{G}$ called the *relative Malcev completion* and a factorization

$$\Gamma_g \to \mathcal{G} \to \mathrm{Aut}(H_1(C, \mathbb{C})). \tag{1.3.2}$$

The kernel of $\mathcal{G} \to \mathrm{Aut}(H_1(C, \mathbb{C}))$ is a pro–unipotent group which we denote by $\mathcal{U}_g$. If $\mathcal{T}_g$ denotes the Malcev completion of T_g, then there is a natural map $\mathcal{T}_g \to \mathcal{U}_g$ whose kernel is isomorphic to $\mathbb{Q}$ [Ha3] (here the assumption $g \geq 3$ is used). Let $\mathfrak{u}_{\mathfrak{g}}$ denote the Lie algebra of $\mathcal{U}_g$. The interest in the group $\mathfrak{u}_{\mathfrak{g}}$ derives from the fact that it carries a natural mixed Hodge structure which Hain and others have used to obtain information about the group T_g.

The construction of the mixed Hodge structure on $\mathfrak{u}_g$ suggests that the Lie algebra $\mathfrak{u}_g$ should in a suitable sense be a motive over $\mathbb{Z}$. In particular, there should be an étale realization, de Rham realization, and a p–adic Hodge theory relating the two similar to the cohomological theory. A consequence of the work in this paper is that this is indeed the case.

To explain this, choose the curve C to be defined over $\mathbb{Q}_p$ and assume C has good reduction. Let $\pi : \mathcal{C}^o \to \mathcal{M}^o_{g,\mathbb{Q}_p}$ be the universal curve and let $E = R^1\pi_*(\Omega^\bullet_{\mathcal{C}^o/\mathcal{M}^o_{g,\mathbb{Q}_p}})$ be the relative de Rham cohomology of $\mathcal{C}^o$ which is a module with connection on $\mathcal{M}^o_g$, and let $L = R^1\pi^o_*\mathbb{Q}_p$ be the relative p–adic étale cohomology. The module with connection E has a natural structure of a filtered log F–isocrystal $(E, \mathrm{Fil}_E, \varphi_E)$ and is associated to the smooth $\mathbb{Q}_p$–sheaf L [Fa1, 6.3]. Let $\widetilde{\langle E\rangle}_\otimes$ be the smallest Tannakian subcategory of the category of modules with connection on $\mathcal{M}^o_{g,\mathbb{Q}_p}$ which is closed under extensions and contains E, and let $\widetilde{\langle L_{\overline{\mathbb{Q}}_p}\rangle}_\otimes$ denote the smallest Tannakian subcategory of the category of smooth $\mathbb{Q}_p$–sheaves on $\mathcal{M}^o_{g,\overline{\mathbb{Q}}_p}$ which is closed under extensions and contains $L_{\overline{\mathbb{Q}}_p}$ (the restriction of L to $\mathcal{M}^o_{g,\overline{\mathbb{Q}}_p}$). The point $p \in \mathcal{M}^o_{g,\mathbb{Q}_p}(\mathbb{Q}_p)$ defined by C defines fiber functors for these Tannakian categories, and using Tannaka duality we obtain pro–algebraic groups $\pi_1(\widetilde{\langle E\rangle}_\otimes)$ and $\pi_1(\widetilde{\langle L_{\overline{\mathbb{Q}}_p}\rangle}_\otimes)$ over $\mathbb{Q}_p$. The group $\pi_1(\widetilde{\langle E\rangle}_\otimes)$ comes equipped with a Frobenius automorphism φ (really a semi–linear automorphism but we are working over $\mathbb{Q}_p$) and the group $\pi_1(\widetilde{\langle L_{\overline{\mathbb{Q}}_p}\rangle}_\otimes)$ comes equipped with an action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Let $\mathfrak{u}^{\mathrm{dR}}_g$ (resp. $\mathfrak{u}^{\mathrm{et}}_g$) denote the Lie algebra of the pro–unipotent radical of $\pi_1(\widetilde{\langle E\rangle}_\otimes)$ (resp. $\pi_1(\widetilde{\langle L_{\overline{\mathbb{Q}}_p}\rangle}_\otimes)$). The Frobenius automorphism φ induces an automorphism $\varphi_{\mathfrak{u}^{\mathrm{dR}}_g}$ of $\mathfrak{u}^{\mathrm{dR}}_g$ and the Galois action on $\pi_1(\widetilde{\langle L_{\overline{\mathbb{Q}}_p}\rangle}_\otimes)$ induces a Galois action $\rho_{\mathfrak{u}^{\mathrm{et}}_g}$ on $\mathfrak{u}^{\mathrm{et}}_g$. By [Ha2, 3.1] any embedding $\mathbb{Q}_p \hookrightarrow \mathbb{C}$ induces a natural isomorphisms of Lie algebras $\mathfrak{u}_g \simeq \mathfrak{u}^{\mathrm{dR}}_g \otimes_{\mathbb{Q}_p} \mathbb{C}$ and $\mathfrak{u}_g \simeq \mathfrak{u}^{\mathrm{et}}_g \otimes_{\mathbb{Q}_p} \mathbb{C}$. It is therefore natural to call $\mathfrak{u}^{\mathrm{dR}}_g$ (resp. $\mathfrak{u}^{\mathrm{et}}_g$) the de Rham (resp. étale) realization of $\mathfrak{u}_g$. The p–adic Hodge theory studied in this paper (in particular 1.10 below) now yields the following result:

Theorem 1.4. *The* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*–representation* $\mathfrak{u}_g^{\mathrm{et}}$ *is pro–crystalline, and there is a natural isomorphism* $\mathbf{D}(\mathfrak{u}_g^{\mathrm{et}}) \simeq \mathfrak{u}_g^{\mathrm{dR}}$ *compatible with the Frobenius automorphisms.*

Similar results also hold for the moduli spaces $\mathcal{M}_{g,n}^o$ of n–pointed genus g curves. Note that $\mathcal{M}_{0,4}^o \simeq \mathbb{P}^1 - \{0, 1, \infty\}$ is the case studied by Deligne in [De2]. Theorem 1.4 for $\mathbb{P}^1 - \{0, 1, \infty\}$ had previously been obtained by Hain and Matsumoto [H-M, 9.8], as well as Shiho [Sh3] and Tsuji. Pridham has also obtained some of the results of this paper using a different method [Pr].

1.5. As in [Ol1] we work in this paper systematically with simplicial presheaves and stacks [Bl, H-S, Ja, To1]. For any ring R, let $\mathrm{SPr}(R)$ denote the category of simplicial presheaves on the category of affine R–schemes, and let $\mathrm{SPr}_*(R)$ denote the category of pointed objects in $\mathrm{SPr}(R)$. By [To1, 1.1.1], there is a natural model category structure on $\mathrm{SPr}(R)$ and $\mathrm{SPr}_*(R)$, and we write $\mathrm{Ho}(\mathrm{SPr}(R))$ and $\mathrm{Ho}(\mathrm{SPr}_*(R))$ for the resulting homotopy categories.

Let X/V be as in 1.1, and assume given a section $x : \mathrm{Spec}(V) \to X^o$. Let L be a crystalline sheaf on X_K^o associated to some filtered log F–isocrystal $(E, \mathrm{Fil}_E, \varphi_E)$. Let $\mathcal{C}_{\mathrm{et}}$ denote the smallest full Tannakian subcategory of the category of smooth $\mathbb{Q}_p$–sheaves on $X_{\overline{K}}^o$ closed under extensions and containing $L_{\overline{K}}$. Similarly let $\mathcal{C}_{\mathrm{dR}}$ denote the smallest full Tannakian subcategory of the category of modules with integrable connection on X_K^o/K closed under extensions and containing E. Also, let $\langle L_{\overline{K}}\rangle_\otimes$ (resp. $\langle E\rangle_\otimes$) denote the Tannakian subcategory of the category of smooth sheaves on $X_{\overline{K}}^o$ generated by $L_{\overline{K}}$ (resp. the Tannakian subcategory of the category of modules with integrable connection on X_K^o/K generated by E), and let $\pi_1(\langle L_{\overline{K}}\rangle_\otimes, \bar{x})$ (resp. $\pi_1(\langle E\rangle_\otimes, x)$) denote the Tannaka dual of $\langle L_{\overline{K}}\rangle_\otimes$ (resp. $\langle E\rangle_\otimes$) with respect to the fiber functor defined by x.

Assumption 1.6. *Assume that the groups* $\pi_1(\langle L_{\overline{K}}\rangle_\otimes, \bar{x})$ *and* $\pi_1(\langle E\rangle_\otimes, x)$ *are reductive and that* E *has unipotent local monodromy (see* 4.14 *for what this means).*

In section 4, we explain a construction of certain pointed stacks $X_{\mathcal{C}_{\mathrm{dR}}} \in \mathrm{Ho}(\mathrm{SPr}_*(K))$ and $X_{\mathcal{C}_{\mathrm{et}}} \in \mathrm{Ho}(\mathrm{SPr}_*(\mathbb{Q}_p))$. The fundamental group of $X_{\mathcal{C}_{\mathrm{dR}}}$ (resp. $X_{\mathcal{C}_{\mathrm{et}}}$) is the Tannaka dual of $\mathcal{C}_{\mathrm{dR}}$ (resp. $\mathcal{C}_{\mathrm{et}}$) and cohomology of local systems (in the sense of Toen [To1, 1.3]) agrees with de Rham cohomology (resp. étale cohomology). The pointed stack $X_{\mathcal{C}_{\mathrm{dR}}}$ comes equipped with a Frobenius automorphism $\varphi_{X_{\mathcal{C}_{\mathrm{dR}}}} : X_{\mathcal{C}_{\mathrm{dR}}} \otimes_{K,\sigma} K \to X_{\mathcal{C}_{\mathrm{dR}}}$, where $\sigma : K \to K$ denotes the canonical lift of Frobenius, and $X_{\mathcal{C}_{\mathrm{et}}}$ has a natural action of the group G_K.

We will use Faltings' approach to p-adic Hodge theory using "almost mathematics" to compare $X_{\mathcal{C}_{\mathrm{dR}}}$ and $X_{\mathcal{C}_{\mathrm{et}}}$. As the referee points (and as explained for example in [Ol3]) this approach naturally leads one to consider a certain localization $\widetilde{B}_{\mathrm{cris}}(V)$ of $B_{\mathrm{cris}}(V)$ (see 6.8 for a precise definition). The main result can now be stated as follows.

Theorem 1.7. *There is a natural isomorphism in* $\mathrm{Ho}(\mathrm{SPr}_*(\widetilde{B}_{\mathrm{cris}}(V)))$

$$\iota : X_{\mathcal{C}_{\mathrm{dR}}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq X_{\mathcal{C}_{\mathrm{et}}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \tag{1.7.1}$$

compatible with the Frobenius automorphisms and the action of G_K*, where* $X_{\mathcal{C}_{\mathrm{dR}}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V)$ *(resp.* $X_{\mathcal{C}_{\mathrm{et}}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V)$*) denotes the restriction of* $X_{\mathcal{C}_{\mathrm{dR}}}$ *(resp.* $X_{\mathcal{C}_{\mathrm{et}}}$*) to the category of affine schemes over* $\widetilde{B}_{\mathrm{cris}}(V)$*.*

We also prove an unpointed version of 1.7 in section 8.

To illustrate the utility of this theorem, let us mention some consequences for homotopy groups of 1.7 and its proof.

Theorem 1.8. *There is an isomorphism of group schemes over* $B_{\mathrm{cris}}(V)$

$$\pi_1(\mathcal{C}_{\mathrm{dR}}, x) \otimes_K B_{\mathrm{cris}}(V) \simeq \pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x}) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \tag{1.8.1}$$

compatible with the natural actions of Frobenius and G_K*.*

For any $i \geq 1$ one can define homotopy groups $\pi_i(X_{\mathcal{C}_{\mathrm{dR}}})$ (resp. $\pi_i(X_{\mathcal{C}_{\mathrm{et}}})$) which are algebraic groups over K (resp. $\mathbb{Q}_p$). The Frobenius automorphism on $X_{\mathcal{C}_{\mathrm{dR}}}$ (resp. the G_K–action on X_{et}) induces a semi–linear automorphism of $\pi_i(X_{\mathcal{C}_{\mathrm{dR}}})$ (resp. an action of G_K on $\pi_i(X_{\mathcal{C}_{\mathrm{et}}})$). This action induces a Frobenius automorphism (resp. G_K–action) on the Lie algebra $\mathrm{Lie}(\pi_i(X_{\mathcal{C}_{\mathrm{dR}}}))$ (resp. $\mathrm{Lie}(\pi_i(X_{\mathcal{C}_{\mathrm{et}}}))$).

Theorem 1.9. *For every* $i \geq 1$*, the* G_K*–representation* $\mathrm{Lie}(\pi_i(X_{\mathcal{C}_{\mathrm{et}}}))$ *is a pro–object in the category of crystalline representations, and the* K*–space with Frobenius automorphism underlying* $\mathbf{D}(\mathrm{Lie}(\pi_i(X_{\mathcal{C}_{\mathrm{et}}})))$ *is canonically isomorphic to* $\mathrm{Lie}(\pi_i(X_{\mathcal{C}_{\mathrm{dR}}}))$*.*

As mentioned above, in the case $i = 1$ the group $\pi_1(X_{\mathcal{C}_{\mathrm{et}}})$ (resp. $\pi_1(X_{\mathcal{C}_{\mathrm{dR}}})$) is canonically isomorphic to $\pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x})$ (resp. $\pi_1(\mathcal{C}_{\mathrm{dR}}, x)$). Hence in this case, 1.9 gives:

Theorem 1.10. *The* G_K*–representation* $\mathrm{Lie}(\pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x}))$ *is a pro–object in the category of crystalline representations and the* K*–vector space with Frobenius automorphism underlying* $\mathbf{D}(\mathrm{Lie}(\pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x})))$ *is canonically isomorphic to the pro–*F*–isocrystal* $\mathrm{Lie}(\pi_1(\mathcal{C}_{\mathrm{dR}}, x))$*.*

On the other hand, if for some embedding $K \hookrightarrow \mathbb{C}$ the complex manifold $X^o(\mathbb{C})$ is simply connected, then 1.9 yields p-adic Hodge theory for certain motivic realizations of the higher rational homotopy groups $\pi_i(X^o(\mathbb{C})) \otimes \mathbb{Q}$.

In 8.27–8.32 we also prove a generalization of 1.10 for spaces of paths. Suppose $x, y \in X^o(V)$ are two points defining two fiber functors $\omega_x^{\mathrm{dR}}\omega_y^{\mathrm{dR}}$ (resp. ω_x^{et}, ω_y^{et}) for $\mathcal{C}_{\mathrm{dR}}$ (resp. $\mathcal{C}_{\mathrm{et}}$). Define schemes over K and $\mathbb{Q}_p$ respectively

$$P_{x,y}^{\mathrm{dR}} := \underline{\mathrm{Isom}}^{\otimes}(\omega_x^{\mathrm{dR}}, \omega_y^{\mathrm{dR}}), \quad P_{x,y}^{\mathrm{et}} := \underline{\mathrm{Isom}}^{\otimes}(\omega_x^{\mathrm{et}}, \omega_y^{\mathrm{et}}). \tag{1.10.1}$$

The scheme $P_{x,y}^{\mathrm{dR}}$ has a natural semi–linear automorphism and $P_{x,y}^{\mathrm{et}}$ has a natural action of G_K.

Theorem 1.11. *The Galois representation* $\mathcal{O}_{P_{x,y}^{\mathrm{et}}}$ *is ind–crystalline and the vector space with semi–linear automorphism underlying* $\mathbf{D}(\mathcal{O}_{P_{x,y}^{\mathrm{et}}})$ *is canonically isomorphic to* $\mathcal{O}_{P_{x,y}^{\mathrm{dR}}}$*.*

Theorem 1.7 also has implications for cohomology. Recall that $\mathcal{C}_{\mathrm{dR}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V)$ (resp. $\mathcal{C}_{\mathrm{et}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V)$) is the category of pairs (M, α) (resp. (S, β)), where M (resp. S) is an ind–object in $\mathcal{C}_{\mathrm{dR}}$ (resp. $\mathcal{C}_{\mathrm{et}}$) and $\alpha : \widetilde{B}_{\mathrm{cris}}(V) \to \mathrm{End}_K(M)$ (resp. $\beta : \widetilde{B}_{\mathrm{cris}}(V) \to \mathrm{End}_{\mathbb{Q}_p}(S)$) is a K–algebra (resp. $\mathbb{Q}_p$–algebra) homomorphism. In particular, for such an object (M, α) (resp. (S, β)) we can form its de Rham cohomology $H^*_{\mathrm{dR}}(M)$ (resp. étale cohomology $H^*_{\mathrm{et}}(S)$) which is a $\widetilde{B}_{\mathrm{cris}}(V)$–module.

Theorem 1.12. *Let $(S, \beta) \in \mathcal{C}_{\mathrm{et}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V)$ be an object corresponding under the equivalence in* 1.8 *to $(M, \alpha) \in \mathcal{C}_{\mathrm{dR}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V)$. Then there is a natural isomorphism of $\widetilde{B}_{\mathrm{cris}}(V)$–modules*

$$H^*_{\mathrm{dR}}(M) \simeq H^*_{\mathrm{et}}(S). \tag{1.12.1}$$

In particular, we can recover the cohomological p–adic Hodge Theory from 1.7.

In the case when $D = \emptyset$ and k is a finite field, the formality theorem of [Ol1, 4.25] can be applied. Let G_{et} denote the pro–reductive completion of $\pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x})$ and let $\mathcal{O}_{G_{\mathrm{et}}}$ be its coordinate ring. Right translation induces a left action of G_{et} on $\mathcal{O}_{G_{\mathrm{et}}}$ which by Tannaka duality gives rise to an ind–object $\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})$ in the category of smooth sheaves. Left translation induces a right action of G_{et} on $\mathcal{O}_{G_{\mathrm{et}}}$ which commutes with the left action and hence induces a right action of G_{et} on $\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})$.

Theorem 1.13. *Assume $D = \emptyset$ and that (E, φ_E) is ι–pure in the sense of* [Ke]. *Then the Galois representation $\mathrm{Lie}(\pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x}))) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$ is determined by the cohomology ring $H^*(X_{\overline{K}}, \mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}}))$ with its natural actions of G_{et} and G_K. Furthermore, if $\mathbb{L}H_1(X_{\overline{K}}, \mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}}))$ denotes the free pro–Lie algebra on the dual of $H^1(X_{\overline{K}}, \mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}}))$, then there exists a surjection of pro–Lie algebras*

$$\pi : \mathbb{L}H_1(X_{\overline{K}}, \mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \longrightarrow \mathrm{Lie}(\pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x})) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \tag{1.13.1}$$

compatible with the Galois actions whose kernel is generated in degree 2.

For a stronger version of this result see 7.22.

In the case when L and E are the trivial sheaves, the category $\mathcal{C}_{\mathrm{et}}$ (resp. $\mathcal{C}_{\mathrm{dR}}$) is the category of unipotent smooth sheaves (resp. unipotent modules with integrable connection) and various versions of 1.8, 1.9, 1.10, and 1.11 have been obtained by Shiho [Sh3], Tsuji, and Vologodsky [Vo].

1.14. On a technical level, this paper is in many ways a fusing of the ideas of [Ol1] (and in turn those of [KPT]) with Faltings' work in [Fa1] (the necessary aspects of Faltings' work is also discussed in detail in [Ol3]). The main point is that the ideas of [Ol1] imply that to obtain the above theorems it suffices to carry out Faltings construction of the comparison isomorphism between de Rham and étale cohomology on the level of certain equivariant differential graded algebras without passing to cohomology. The main ingredient in carrying out this comparison is systematic use of various standard constructions and result from homotopical algebra, most notably the functor of Thom–Sullivan cochains. We review the necessary homotopical algebra (which can be found in [H-S], see also [B-K]) in section 2. In section 3 we review some the aspects of the convergent topos that we need. In section 4 we review the basic techniques and results from [Ol1], as well as a mild generalization to take into account a boundary. In section 5 we discuss the étale pointed stack associated to a smooth sheaf and some basic properties. In section 6 we work through Falting's construction of the comparison isomorphism, keeping track of various differential graded algebra structures. In section 7 we then put it all together to prove 1.7–1.13. In section 8 we explain how to remove the dependence on the base point and also prove 1.11. In section 9 we explain how to replace the point $x \in X^o(V)$ in the above with a tangential base point. This requires a rather detailed study of p–adic Hodge theory on the log point. We conclude in section 10 by briefly discussing a conjecture of Toen which we feel would be a natural extension of the work discussed in this paper.

The paper also includes four appendices discussing some technical points which arise in the paper. Appendix A discusses a generalization of Kato's "exactification of the diagonal" [Ka, 4.10 (2)]. Appendix B collects some basic observations about localization in proper model categories, and in appendix C we discuss a version of the coherator (see [T-T]) for algebraic stacks. Finally in appendix D we discuss how to pass from a comparison isomorphism over $\widetilde{B}_{\text{cris}}(V)$ to a comparison isomorphism over $B_{\text{cris}}(V)$.

Finally let us remark that throughout this paper we assume X is defined over the ring of Witt vectors of a perfect field of characteristic $p > 0$. In fact, it suffices to have X defined over a possibly ramified extension of such a ring. However, in the interest of improving the exposition we make this simplifying hypothesis.

1.15 (Conventions)**.** We assume familiarity with the basics of logarithmic geometry [Ka].

We also assume some familiarity with model categories for which our reference is [Ho].

For the applications we have in mind, such as 1.3 above, it is important to work with Deligne–Mumford stacks rather than schemes. However, for the sake of exposition we work only with schemes below. The reader who so desires can freely replace "scheme" by "Deligne–Mumford stack" in what follows (in a couple of places our arguments may seem strange to the reader only interested in schemes, but we have taken some care in writing the arguments in such a way that they also apply to Deligne–Mumford stacks; in particular, we work exclusively with the étale topology as opposed to the Zariski topology).

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We recently learned from T. Tsuji that he is developing a generalization of the theory of crystalline sheaves to schemes with hollow log structure (such as the log point), which presumably encompasses also the foundational work we do in section 9. Tsuji's work might also help remove the linear reductivity assumption 1.6, which probably is unnecessary.

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2. Review of some homotopical algebra

We review in this section some well–known constructions and results of homotopical algebra. We learned the main results from [H-S]. A discussion of the functor of Thom-Sullivan cochains can also be found in [B-K].

Let $(\mathcal{T}, \mathcal{O})$ be a ringed topos, and assume $\mathcal{O}$ is a commutative $\mathbb{Q}$–algebra. In what follows, we write $\text{Mod}_{\mathcal{O}}$ for the category of $\mathcal{O}$–modules in $\mathcal{T}$.

Review of normalization.

2.1. Let Δ denote the category of finite totally ordered sets with order preserving maps. We usually identify the category with the full sub–category with objects the sets

$$[i] := \{0, 1, \dots, i\}, \quad i \in \mathbb{N}. \tag{2.1.1}$$

If $\mathcal{C}$ is any category, a *cosimplicial object* in $\mathcal{C}$ is a functor $X : \Delta \to \mathcal{C}$. The cosimplicial objects in $\mathcal{C}$ form a category, denoted $\mathcal{C}^{\Delta}$, with morphisms being morphisms of functors. If $X \in \mathcal{C}^{\Delta}$, we write X^i for $X([i])$. There are natural maps

$$d_i : X^{n-1} \longrightarrow X^n, \quad s_i : X^{n+1} \longrightarrow X^n \quad 0 \leq i \leq n, \tag{2.1.2}$$

where d_i (resp. s_i) is induced by the unique injective (resp. surjective) map $d_i : [n-1] \to [n]$ (resp. $s_i : [n+1] \to [n]$) for which $i \notin d_i([n-1])$ (resp. $s_i(i) = s_i(i+1)$). We shall sometimes think of an object of $\mathcal{C}^{\Delta}$ as a collection of objects $X^i \in \mathcal{C}$ together with maps d_i and s_i satisfying the standard simplicial identities [G-J, I 1.3].

2.2. Let $\mathcal{A}$ be an abelian category, and let $C^{\geq 0}(\mathcal{A})$ denote the category of complexes $M_\bullet$ in $\mathcal{A}$ for which $M_i = 0$ for $i < 0$. The *normalization functor* is the functor

$$N : \mathcal{A}^{\Delta} \longrightarrow C^{\geq 0}(\mathcal{A}) \tag{2.2.1}$$

which sends $A \in \mathcal{A}^{\Delta}$ to the complex whose i–th term is

$$\mathrm{Coker}((d_0, \dots, d_{i-1}) : \oplus_{j=0}^{n-1} A^{i-1} \longrightarrow A^i) \tag{2.2.2}$$

and whose differential is induced by $(-1)^i d_i$.

2.3. Given $A \in \mathcal{A}^{\Delta}$, we can also form the *chain complex* of A, denoted $\widetilde{A}$, which is the object of $C^{\geq 0}(\mathcal{A})$ whose i–th term is A^i and whose differential is given by

$$\partial := \sum_{j=0}^{i} (-1)^i d_j : A^{i-1} \longrightarrow A^i. \tag{2.3.1}$$

There is a natural surjective map of complexes

$$\widetilde{A} \longrightarrow N(A) \tag{2.3.2}$$

which is a quasi–isomorphism [G-J, III.2.4].

In fact, the map 2.3.2 is split. Let $D(A) \subset \widetilde{A}$ be the sub–complex whose i–th term is

$$\bigcap_{j=0}^{i-1} \mathrm{Ker}(s_j : A^i \longrightarrow A^{i-1}). \tag{2.3.3}$$

The complex $D(A)$ is called the *denormalization of* A.

Then the composite

$$D(A) \longrightarrow \widetilde{A} \longrightarrow N(A) \tag{2.3.4}$$

is an isomorphism of complexes [G-J, III.2.1]. In particular, the following holds:

Corollary 2.4. (i) *If $\mathcal{O} \to \mathcal{O}'$ is a morphism of rings, then there is a natural isomorphism $N(A) \otimes_{\mathcal{O}} \mathcal{O}' \simeq N(A \otimes_{\mathcal{O}} \mathcal{O}')$.*

(ii) *If $A \in \mathrm{Mod}_{\mathcal{O}}^{\Delta}$ is an object with each A^i flat over $\mathcal{O}$, then $N(A) \in C^{\geq 0}(\mathcal{O})$ is a complex of flat $\mathcal{O}$–modules.*

2.5. There is a third description of $N(A)$ which is important in the context of the functor of Thom–Sullivan cochains discussed below. Let $\mathrm{Mod}_{\mathcal{O}}^{\Delta^o}$ denote the category of simplicial $\mathcal{O}$–modules. That is, $\mathrm{Mod}_{\mathcal{O}}^{\Delta^o}$ is the category of functors from the opposite category Δ^o of Δ to the category of $\mathcal{O}$–modules. Taking $\mathcal{A}$ in the above discussion to be the opposite category of the category of $\mathcal{O}$–modules, we obtain a normalization functor

$$N^o : \mathrm{Mod}_{\mathcal{O}}^{\Delta^o} \longrightarrow C^{\leq 0}(\mathcal{O}), \tag{2.5.1}$$

from $\mathrm{Mod}_{\mathcal{O}}^{\Delta^o}$ to the category of complexes of $\mathcal{O}$–modules with support in degrees ≤ 0.

Let

$$Y : \Delta \longrightarrow \mathrm{Mod}_{\mathcal{O}}^{\Delta^o} \tag{2.5.2}$$

be the functor which sends $[n]$ to the simplicial $\mathcal{O}$–module which sends $[m] \in \Delta$ to the free $\mathcal{O}$–module on $\mathrm{Hom}_\Delta([m],[n])$, and let

$$Z : \Delta \longrightarrow C^{\leq 0}(\mathcal{O}) \tag{2.5.3}$$

be the composite of Y with the normalization functor N^o. For $[n] \in \Delta$, we write

$$(\cdots \to Z_p^n \to Z_{p+1}^n \to \cdots) \tag{2.5.4}$$

for $Z([n])$. By the dual of the isomorphism 2.3.4, we have

$$Z_{-p}^n = \mathcal{O}^{\mathrm{Hom}_\Delta([p],[n])}/\mathrm{Im}(\oplus s_j : \bigoplus \mathcal{O}^{\mathrm{Hom}_\Delta([p-1],[n])} \longrightarrow \mathcal{O}^{\mathrm{Hom}_\Delta([p],[n])}). \tag{2.5.5}$$

Lemma 2.6. *If $A \in \mathrm{Mod}_{\mathcal{O}}^{\Delta}$, then the complex*

$$\cdots \to \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}}^{\Delta}}(Z_{-p+1}^{\bullet}, A) \to \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}}^{\Delta}}(Z_{-p}^{\bullet}, A) \to \cdots \tag{2.6.1}$$

is isomorphic to the normalization of A.

Proof. Let Set denote the category of sets. By definition of normalization, $Z_{-p}^{\bullet}$ is the cokernel in the category of simplicial $\mathcal{O}$–modules of the map

$$\oplus s_j : \bigoplus_j \mathcal{O}^{\mathrm{Hom}_\Delta([p-1],\cdot)} \longrightarrow \mathcal{O}^{\mathrm{Hom}_\Delta([p],\cdot)}. \tag{2.6.2}$$

By the universal property of the free module on a set and Yoneda's lemma, for every i there are natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Mod}^\Delta}(\mathcal{O}^{\mathrm{Hom}_\Delta([i],\cdot)}, A) \simeq \mathrm{Hom}_{\mathrm{Set}^\Delta}(\mathrm{Hom}_\Delta([i],\cdot), A) \simeq A^i. \tag{2.6.3}$$

It follows that

$$\mathrm{Hom}_{\mathrm{Mod}^\Delta}(Z_{-p}^{\bullet}, A) \simeq \mathrm{Ker}(\oplus s_j : \bigoplus_j A^{p-1} \longrightarrow A^p), \tag{2.6.4}$$

and the result follows. □

Review of total complex as an inverse limit.

2.7. Let $C(\mathcal{O})$ denote the category of all complexes of $\mathcal{O}$–modules, and let $A \in C(\mathcal{O})^\Delta$. Taking the normalization of A, we obtain a double complex of $\mathcal{O}$–modules, and we denote by $\mathrm{Tot}(A)$ the resulting (sum) total complex. We now explain another description of $\mathrm{Tot}(A)$ which will be used below.

2.8. Let $\mathcal{M}_\Delta$ denote the category whose objects are morphisms $f : [i] \to [j]$ (sometimes denoted just f) in Δ, and for which a morphism from f to $g : [i'] \to [j']$ is a commutative diagram

$$\begin{array}{ccc} [i] & \xrightarrow{f} & [j] \\ \uparrow & & \downarrow \\ [i'] & \xrightarrow{g} & [j'] \end{array} \tag{2.8.1}$$

in Δ.

If $A, B \in C(\mathcal{O})^\Delta$, define

$$\mathrm{hom}(A, B) : \mathcal{M}_\Delta \longrightarrow C(\mathcal{O}) \tag{2.8.2}$$

by sending $f : [i] \to [j]$ to the complex

$$\mathrm{Hom}^\bullet_{C(\mathcal{O})}(A([i]), B([j])). \tag{2.8.3}$$

Similarly, if $A \in C(\mathcal{O})^{\Delta^o}$ and $B \in C(\mathcal{O})^\Delta$, define

$$A \otimes B : \mathcal{M}_\Delta \longrightarrow C(\mathcal{O}) \tag{2.8.4}$$

by sending $f : [i] \to [j]$ to the complex $A([i]) \otimes B([j])$. We define $\mathrm{hom}_\leftarrow(A, B)$ and $A \otimes_\leftarrow B$ to be the inverse limits over the category $\mathcal{M}_\Delta$ of the functors $\mathrm{hom}(A, B)$ and $A \otimes B$.

Proposition 2.9. *Let Z be as in* 2.5*. For $A \in C(\mathcal{O})^\Delta$, there are natural isomorphisms of complexes*

$$\mathrm{Tot}(A) \simeq \mathrm{hom}_\leftarrow(Z, A) \simeq Z^* \otimes_\leftarrow A, \tag{2.9.1}$$

where $Z^* : \Delta^o \to C(\mathcal{O})$ denotes the functor which sends $[i]$ to the complex $\mathrm{Hom}^\bullet_{C(\mathcal{O})}(Z([i]), \mathcal{O})$.

Proof. The second isomorphism in 2.9.1 follows from the fact that for every $[i] \in \Delta$, the complex $Z([i])$ is by definition a complex of flat and finitely generated $\mathcal{O}$–modules, and hence for any j, there is a natural isomorphism

$$\mathrm{Hom}_{C(\mathcal{O})}(Z([i]), A([j])) \simeq \mathrm{Hom}_{C(\mathcal{O})}(Z([i]), \mathcal{O}) \otimes A([j]). \tag{2.9.2}$$

To see the first isomorphism, note that the degree k term of $\mathrm{hom}_\leftarrow(Z, A)$ is equal to

$$\lim_{([n]\to[m])\in\mathcal{M}_\Delta} \bigoplus_{p+q=k} \mathrm{Hom}(Z^n_{-p}, A^m_q), \tag{2.9.3}$$

which by the following lemma and 2.6 is equal to

$$\bigoplus_{p+q=k} \mathrm{Hom}(Z^\bullet_{-p}, A^\bullet_q) \simeq \bigoplus_{p+q=k} (N(A^\bullet_q))_p. \tag{2.9.4}$$

□

Lemma 2.10. *Let $X, Y \in \mathrm{Mod}^\Delta_\mathcal{O}$. Then there is a natural isomorphism*

$$\lim_{([n]\to[m])\in\mathcal{M}_\Delta} \mathrm{Hom}(X^n, Y^m) \to \mathrm{Hom}_{\mathrm{Mod}^\Delta_\mathcal{O}}(X, Y). \tag{2.10.1}$$

Proof. Given $\delta \in \lim_{([n]\to[m])\in\mathcal{M}_\Delta} \mathrm{Hom}(X^n, Y^m)$, define $\bar{\delta} : X \to Y$ to be the map of cosimplicial $\mathcal{O}$–modules which in degree n is the map $\delta_n : X^n \to Y^n$ obtained from $(\mathrm{id} : [n] \to [n]) \in \mathcal{M}_\Delta$. We leave it to the reader to verify that this is well–defined, and that the resulting map 2.10.1 is an isomorphism. □

The functor of Thom–Sullivan cochains.

2.11. Let $\mathrm{dga}(\mathcal{O})$ denote the category of commutative differential $\mathbb{N}$–graded $\mathcal{O}$–algebras. That is, the category of $\mathbb{N}$–graded $\mathcal{O}$–algebras $A = \oplus_p A_p$ with a map $d : A \to A$ of degree 1 for which the formulas

$$x \cdot y = (-1)^{pq} y \cdot x, \quad d(x \cdot y) = dx \cdot y + (-1)^p x \cdot dy \tag{2.11.1}$$

hold for $x \in A_p$ and $y \in A_q$.

Theorem 2.12 ([H-S, 4.1])**.** *There is a functor*

$$T : \mathrm{dga}(\mathcal{O})^\Delta \longrightarrow \mathrm{dga}(\mathcal{O}), \tag{2.12.1}$$

together with a natural transformation of functors

$$\int : (\mathrm{forget} \circ T) \longrightarrow (\mathrm{Tot} \circ \mathrm{forget}) \tag{2.12.2}$$

between the two composites

$$\mathrm{dga}(\mathcal{O})^\Delta \xrightarrow{T} \mathrm{dga}(\mathcal{O}) \xrightarrow{\mathrm{forget}} C(\mathcal{O}) \tag{2.12.3}$$

$$\mathrm{dga}(\mathcal{O})^\Delta \xrightarrow{\mathrm{forget}} C(\mathcal{O})^\Delta \xrightarrow{\mathrm{Tot}} C(\mathcal{O}), \tag{2.12.4}$$

such that for every $A \in \mathrm{dga}(\mathcal{O})^\Delta$ the map 2.12.2 *applied to A is a quasi–isomorphism.*

The functor T is called the *functor of Thom–Sullivan cochains.*

2.13. The functor T is constructed as follows. Let R_p denote the $\mathcal{O}$–algebra

$$\mathcal{O}[t_0, \ldots, t_p]/(\sum t_i = 1), \tag{2.13.1}$$

and let $\nabla(p, \bullet) \in \mathrm{dga}(\mathcal{O})$ denote the de Rham–complex of R_p over $\mathcal{O}$. In other words, $\nabla(p, \bullet)$ is the free commutative differential graded algebra generated in degree 0 by variables $t_0, \ldots, t_p$ and in degree 1 by $dt_0, \ldots, dt_p$ subject to the relations

$$\sum t_i = 1, \quad \sum dt_i = 0. \tag{2.13.2}$$

The R_p form in a natural way a simplicial ring $R_\bullet$. The face and degeneracy maps are given by

$$d_i : R_p \longrightarrow R_{p-1}, \quad d_i t_m = \begin{cases} t_{m-1} & i < m \\ 0 & i = m \\ t_m & i > m \end{cases} \tag{2.13.3}$$

$$s_i : R_p \longrightarrow R_{p+1}, \quad s_i t_m = \begin{cases} t_{m+1} & i < m \\ t_m + t_{m+1} & i = m \\ t_m & i > m. \end{cases} \tag{2.13.4}$$

Since the formation of de Rham complex is functorial, the $\nabla(p, \bullet)$ define a simplicial differential graded algebra

$$\Omega : \Delta^o \longrightarrow \mathrm{dga}(\mathcal{O}), \quad [p] \mapsto \nabla(p, \bullet). \tag{2.13.5}$$

The functor T is defined to be the functor which sends $A \in \mathrm{dga}(\mathcal{O})^\Delta$ to $\Omega \otimes_\leftarrow A \in \mathrm{dga}(\mathcal{O})$, where the algebra structure is obtained from the fact that $\Omega \otimes_\leftarrow A$ is by construction an inverse limit of commutative differential graded algebras.

2.14. The transformation of functors 2.12.2 is obtained as follows. Let $\overline{\Omega} \in C(\mathcal{O})^{\Delta^o}$ denote the functor Ω composed with the forgetful functor $\mathrm{dga}(\mathcal{O}) \to C(\mathcal{O})$. For any $p \geq 0$ there is a well–defined map

$$\int_{|\Delta_p|} : \nabla(p, p) \longrightarrow \mathcal{O}. \tag{2.14.1}$$

To construct this map, it suffices to consider the case when $\mathcal{T}$ is the punctual topos and $\mathcal{O} = \mathbb{Q}$ (recall that $\mathcal{O}$ is a $\mathbb{Q}$-algebra). To construct the map in this case, it suffices to show that the usual integration over the standard simplex

$$\int_{|\Delta_p|} : \nabla(p, p) \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow \mathbb{R} \tag{2.14.2}$$

sends $\nabla(p, p)$ to $\mathbb{Q}$, which is immediate.

This integration gives rise to a morphism of functors

$$\int : \overline{\Omega} \longrightarrow Z^*. \tag{2.14.3}$$

If $\omega \in \nabla(p, q)$, define

$$\int \omega \in Z^*([p])_q = \mathrm{Hom}_{C(\mathcal{O})}(Z([p]), \mathcal{O})_q = \mathrm{Hom}_{\mathcal{O}}(Z^p_{-q}, \mathcal{O}) \tag{2.14.4}$$

to be the element induced by the description of Z^p_{-q} given in 2.5.5 and the map

$$\mathcal{O}^{\mathrm{Hom}([q],[p])} \longrightarrow \mathcal{O} \tag{2.14.5}$$

which sends $\mathbf{1}_{\alpha:[q]\to[p]}$ to

$$\int_{|\Delta_q|} R(\alpha)^*\omega \in \mathcal{O}, \tag{2.14.6}$$

where $R(\alpha)^*\omega$ denotes the pullback of the form ω via the map $R_p \to R_q$ induced by the map α and the simplicial structure on the R_p's.

Combining this with 2.9, we obtain the morphism 2.12.2.

To complete the sketch of the proof of 2.12, it remains only to see that if $A \in \mathrm{dga}(\mathcal{O})^\Delta$, then the induced map

$$\overline{\Omega} \otimes_\leftarrow A \longrightarrow Z^* \otimes_\leftarrow A \tag{2.14.7}$$

is a quasi-isomorphism. For this it suffices to consider the case when $\mathcal{T}$ is the punctual topos. Moreover, since $\otimes_\leftarrow A$ preserves homotopy equivalences [H-S, 4.3.1], it suffices to construct a morphism of functors $\tau : Z^* \to \overline{\Omega}$ and homotopies

$$\tau \circ \int \simeq \mathrm{id}, \quad \int \circ \tau \simeq \mathrm{id}. \tag{2.14.8}$$

This is done in [B-G, 2.4].

Remark 2.15. The functor T is functorial with respect to morphisms of topoi. That is, if $f : (\mathcal{T}', \mathcal{O}') \to (\mathcal{T}, \mathcal{O})$ is a morphism of ringed topoi, and if T' denotes the functor 2.12.1 for the ringed topos $(\mathcal{T}', \mathcal{O}')$, then for any $A \in \mathrm{dga}(\mathcal{O})^{\Delta^o}$ there is a natural map

$$f^*T(A) \longrightarrow T(f^*A) \tag{2.15.1}$$

compatible with the map 2.12.2.

Remark 2.16. The above construction can be applied to any object $M \in C(\mathcal{O})^{\Delta}$. More precisely, if $T(M) := \Omega \otimes_{\leftarrow} M$ then the above shows that there is a natural quasi–isomorphism $T(M) \to \mathrm{Tot}(M)$. Furthermore, if $A \in \mathrm{dga}(\mathcal{O})^{\Delta}$ and $M \to A$ is a morphism in $C(\mathcal{O})^{\Delta}$ then there is an induced map $T(M) \to T(A)$. Observe also that for $M, M' \in C(\mathcal{O})^{\Delta}$ there is a natural map $T(M) \otimes T(M') \to T(M \otimes M')$.

Remark 2.17. Since $T(A) \to \mathrm{Tot}(A)$ is a quasi–isomorphism, the induced map $H^*(T(A)) \to H^*(\mathrm{Tot}(A))$ is an isomorphism. By [H-S, 4.4.2] this isomorphism is compatible with the multiplicative structures.

Differential graded algebras and cosimplicial algebras.

2.18. It follows from a theorem of Quillen [Qu, Chapter II, §4, Theorem 4] that the category $C^{\geq 0}(\mathcal{O})$ has a model category structure in which a morphism $f : M \to N$ is a *fibration* (resp. *equivalence*) if it is a surjection with level–wise injective kernel (resp. quasi–isomorphism). By the Dold–Kan correspondence [G-J, III.2.3], the normalization functor 2.2 induces an equivalence of categories $\mathrm{Mod}_{\mathcal{O}}^{\Delta} \simeq C^{\geq 0}(\mathcal{O})$. Through this equivalence, the model category structure on $C^{\geq 0}(\mathcal{O})$ gives a model category structure on $\mathrm{Mod}_{\mathcal{O}}^{\Delta}$. This model category structure on $\mathrm{Mod}_{\mathcal{O}}^{\Delta}$ is by the theorem of Quillen [Qu, Chapter II, §4, Theorem 4] naturally a simplicial cofibrantly generated model category structure.

Lemma 2.19. *The model categories $C^{\geq 0}(\mathcal{O})$ and $\mathrm{Mod}_{\mathcal{O}}^{\Delta}$ are right proper in the sense of B.3.*

Proof. By the definition of the model category structure on $\mathrm{Mod}_{\mathcal{O}}^{\Delta}$, it suffices to prove the lemma for $C^{\geq 0}(\mathcal{O})$. In this case the assertion amounts to the statement that given a diagram in $C^{\geq 0}(\mathcal{O})$

$$\begin{array}{ccc} & & A^{\cdot} \\ & & \downarrow{\scriptstyle g} \\ S^{\cdot} & \xrightarrow{h} & R^{\cdot}, \end{array} \tag{2.19.1}$$

with h a quasi-isomorphism and each $g_n : A^n \to R^n$ surjective with injective kernel, the map

$$S^{\cdot} \times_{R^{\cdot}} A^{\cdot} \to A^{\cdot} \tag{2.19.2}$$

is a quasi-isomorphism. Let $I^{\cdot} \subset A^{\cdot}$ be the kernel of g. Then since g is surjective we have a morphism of exact sequences of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^{\cdot} & \longrightarrow & A^{\cdot} \times_{R^{\cdot}} S^{\cdot} & \longrightarrow & S^{\cdot} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & I^{\cdot} & \longrightarrow & A^{\cdot} & \longrightarrow & R^{\cdot} & \longrightarrow & 0. \end{array} \tag{2.19.3}$$

Since the map h is a quasi-isomorphism, it follows that the middle arrow pr_1 is a quasi-isomorphism as well. $\square$

Remark 2.20. Though we will not need it here, it is also true that the model categories $C^{\geq 0}(\mathcal{O})$ and $\mathrm{Mod}^{\Delta}_{\mathcal{O}}$ are left proper.

2.21. The model category structure on $\mathrm{Mod}^{\Delta}_{\mathcal{O}}$ enables one to define model category structures on $\mathrm{Alg}^{\Delta}_{\mathcal{O}}$ and $\mathrm{dga}_{\mathcal{O}}$ as follows (see [KPT, 1.3.2] for details). A morphism $f : A \to B$ in $\mathrm{dga}_{\mathcal{O}}$ is a *fibration* (resp. *equivalence*) if the underlying morphism in $C^{\geq 0}(\mathcal{O})$ is a fibration (resp. equivalence). Similarly, a morphism $g : C \to D$ in $\mathrm{Alg}^{\Delta}_{\mathcal{O}}$ is a fibration (resp. equivalence) if and only if the induced morphism on normalized complexes $N(C) \to N(D)$ is a fibration (resp. equivalence) in $C^{\geq 0}(\mathcal{O})$. A map $f : A \to B$ in $\mathrm{dga}_{\mathcal{O}}$ (resp. $\mathrm{Alg}^{\Delta}_{\mathcal{O}}$) is a cofibration if for all $n \geq 1$ the map $A_n \to B_n$ (resp. $N(A)_n \to N(B)_n$) is injective.

Since the forgetful functors

$$\mathrm{Alg}^{\Delta}_{\mathcal{O}} \to \mathrm{Mod}^{\Delta}_{\mathcal{O}}, \quad \mathrm{dga}_{\mathcal{O}} \to C^{\geq 0}(\mathcal{O}) \tag{2.21.1}$$

commute with fiber products, it follows that $\mathrm{Alg}^{\Delta}_{\mathcal{O}}$ and $\mathrm{dga}_{\mathcal{O}}$ are right proper model categories.

If $A \in \mathrm{dga}_{\mathcal{O}}$, then the "shuffle product" [Ma, 8.8] defines on the denormalization $D(A)$ a structure of an object in $\mathrm{Alg}^{\Delta}_{\mathcal{O}}$. Since D preserves equivalences and fibrations, it induces a functor

$$D : \mathrm{Ho}(\mathrm{dga}_{\mathcal{O}}) \longrightarrow \mathrm{Ho}(\mathrm{Alg}^{\Delta}_{\mathcal{O}}). \tag{2.21.2}$$

This functor is an equivalence with inverse provided by the functor of Thom–Sullivan cochains.

2.22. If $f : (\mathcal{T}', \mathcal{O}') \to (\mathcal{T}, \mathcal{O})$ is a morphism of ringed topoi with $\mathcal{O}'$ and $\mathcal{O}$ commutative $\mathbb{Q}$–algebras and $f^{-1}\mathcal{O} \to \mathcal{O}'$ flat, then the functor f^* is exact and hence its right adjoint f_* takes injectives to injectives. It follows that the functors

$$f_* : \mathrm{dga}_{\mathcal{O}'} \to \mathrm{dga}_{\mathcal{O}}, \quad f_* : \mathrm{Alg}^{\Delta}_{\mathcal{O}'} \to \mathrm{Alg}^{\Delta}_{\mathcal{O}} \tag{2.22.1}$$

preserve fibrations and trivial fibrations and induce derived functors $\mathbb{R}f_*$ such that the diagram

$$\begin{array}{ccc} \mathrm{Ho}(\mathrm{dga}_{\mathcal{O}'}) & \xrightarrow{D} & \mathrm{Ho}(\mathrm{Alg}^{\Delta}_{\mathcal{O}'}) \\ {\scriptstyle \mathbb{R}f_*}\downarrow & & \downarrow{\scriptstyle \mathbb{R}f_*} \\ \mathrm{Ho}(\mathrm{dga}_{\mathcal{O}}) & \xrightarrow{D} & \mathrm{Ho}(\mathrm{Alg}^{\Delta}_{\mathcal{O}}) \end{array} \tag{2.22.2}$$

commutes. Observe that by definition of the model category structures, if $A \in \mathrm{dga}_{\mathcal{O}'}$ (resp. $B \in \mathrm{Alg}^{\Delta}_{\mathcal{O}'}$) then the underlying complex (resp. normalization) of $\mathbb{R}f_*A$ (resp. $\mathbb{R}f_*B$) is isomorphic in the derived category to the usual derived functors of the complex underlying A (resp. the normalized complex of B).

If $(\mathcal{T}, \mathcal{O})$ is the punctual topos with $\mathcal{O} = \mathbb{Q}$ we write $\mathbb{R}\Gamma$ instead of $\mathbb{R}f_*$.

2.23. In what follows we will also consider an equivariant situation. Let K be a field of characteristic 0 and let G/K be an affine group scheme over K. Let $(\mathcal{T}, \mathcal{O})$ be a ringed topos with $\mathcal{O}$ a K–algebra. We obtain G–equivariant versions of the above results as follows.

Let $\mathrm{Rep}_K(G)$ denote the category of algebraic representations of the group scheme G on (possibly infinite dimensional) K–vector spaces. A sheaf M on $\mathcal{T}$ taking values in $\mathrm{Rep}_K(G)$ is a functor

$$M : \mathcal{T}^o \to \mathrm{Rep}_K(G) \tag{2.23.1}$$

such that the composite

$$\mathcal{T}^o \xrightarrow{M} \mathrm{Rep}_K(G) \xrightarrow{\text{forget}} \mathrm{Vec}_K \tag{2.23.2}$$

is representable by an object of $\mathcal{T}$. The category of such sheaves M is naturally a K–linear tensor category and so it makes sense to talk about $\mathcal{O}$–module objects in this category. We call the resulting objects *G–equivariant $\mathcal{O}$–modules* and write $G - \mathrm{Mod}_{\mathcal{O}}$ for the category of G–equivariant $\mathcal{O}$–modules. The category $G - \mathrm{Mod}_{\mathcal{O}}$ is naturally an $\mathcal{O}$–linear tensor category and so we can define categories $G - \mathrm{dga}_{\mathcal{O}}$ and $G - \mathrm{Alg}_{\mathcal{O}}^{\Delta}$ of G–equivariant differential graded algebras and G–equivariant cosimplicial algebras. By the same reasoning as above there are natural closed model category structures on these categories.

The above categories can be described more concretely as follows. Let $\mathcal{O}_G$ denote the coordinate ring of G. If V is a vector space, then to give an algebraic action of G on V is equivalent to a comodule structure on V [Sa, I.6.2.2]. That is a map

$$\rho : V \to V \otimes_K \mathcal{O}_G \tag{2.23.3}$$

such that

$$(1 \otimes \epsilon) \circ \rho = \mathrm{id}, \quad (1 \otimes \Delta) \circ \rho = (\rho \otimes 1) \circ \rho, \tag{2.23.4}$$

where $\delta : \mathcal{O}_G \to \mathcal{O}_G \otimes \mathcal{O}_G$ is the map giving the multiplication and $\epsilon : \mathcal{O}_G \to K$ is the unit. To give a G–equivariant $\mathcal{O}$–module is equivalent to giving a sheaf of $\mathcal{O}$–modules M together with maps of sheaves of $\mathcal{O}$-modules $\rho : M \to M \otimes_K \mathcal{O}_G$ such that the conditions 2.23.4 hold.

3. Review of the convergent topos

In this section we review for the convenience of the reader some aspects of the convergent topos. The references for the convergent topos is [Og2] and in the logarithmic context [Sh2].

3.1. Let k be a perfect field of characteristic $p > 0$, and let V be a complete discrete valuation ring of mixed characteristic with residue field k. Let $\pi \in V$ denote a uniformizer. In what follows we often view $\mathrm{Spec}(V)$ (resp. $\mathrm{Spf}(V)$) as a log scheme (resp. log formal scheme in the sense of [Sh1, Chap.2]) with the trivial log structure (and hence we omit the log structure from the notation).

If $T \to \mathrm{Spf}(V)$ is a morphism of formal schemes, we write $T_1 \subset T$ for the closed formal subscheme defined by $\pi\mathcal{O}_T$, and $T_0 \subset T_1$ for the largest reduced formal subscheme of T_1. We write $\mathcal{K}_T$ for the sheaf associated to the presheaf of rings on T_{et} given by

$$U \mapsto \Gamma(U, \mathcal{O}_U \otimes \mathbb{Q}). \tag{3.1.1}$$

3.2. Let

$$f : (X, M_X) \to \operatorname{Spec}(k) \tag{3.2.1}$$

be a finite type morphism of fine log schemes. A *pre-widening* is a commutative diagram

$$\begin{tikzcd} (Z, M_Z) \arrow[r, hook, "i"] \arrow[d, "z"] & (T, M_T) \arrow[dd] \\ (X, M_X) \arrow[d, "f"] & \\ \operatorname{Spec}(k) \arrow[r, hook] & \operatorname{Spf}(V), \end{tikzcd} \tag{3.2.2}$$

where

(i) $z : (Z, M_Z) \to (X, M_X)$ is a morphism of fine log schemes over k;
(ii) $(T, M_T) \to \operatorname{Spf}(V)$ is a morphism essentially of finite type of fine log formal schemes (but T does *not* necessarily have the π-adic topology);
(iii) i is an exact closed immersion.

Pre-widenings form a category in the obvious way. We often denote a pre-widening simply by $((T, M_T), (Z, M_Z), z)$, or even just T if no confusion seems likely to arise.

Definition 3.3. Let $((T, M_T), (Z, M_Z), z)$ be a pre-widening.

(i) $((T, M_T), (Z, M_Z), z)$ is a *widening* if $i : Z \hookrightarrow T$ is a subscheme of definition.

(ii) $((T, M_T), (Z, M_Z), z)$ is an *enlargement* if it is a widening, T/V is flat, and if Z contains T_0.

Morphisms of widenings or enlargements are morphisms of pre-widenings. We say that a pre-widening $((T, M_T), (Z, M_Z), z)$ is *affine* if T (and hence also Z) is an affine formal scheme.

Remark 3.4. This definition differs from [Sh2, 2.1.9] as we require i to be exact. In Shiho's terminology the above would be called 'exact pre-widenings' and 'exact widenings.'

Remark 3.5. We have automatically $Z \subset T_1$ since Z is a k-scheme, and therefore the condition that a widening $((T, M_T), (Z, M_Z), z)$ is an enlargement is equivalent to saying that T has the π-adic topology.

Remark 3.6. Products exist in the category of widenings. If

$$(Z_i, M_{Z_i}) \hookrightarrow (T_i, M_{T_i}), \quad i = 1, 2,$$

are two widenings, let (Z, M_Z) denote the fiber product $(Z_1, M_{Z_1}) \times_{(X, M_X)} (Z_2, M_{Z_2})$ in the category of fine log schemes. We then have a closed immersion

$$(Z, M_Z) \hookrightarrow (T_1, M_{T_1}) \widehat{\times} (T_2, M_{T_2}), \tag{3.6.1}$$

where the right side denotes the completion along Z of the product of (T_1, M_{T_1}) and (T_2, M_{T_2}) in the category of formal V-schemes. The product in the category of widenings is then given by the exactification in the sense of A.18 of 3.6.1.

3.7. Let $\mathrm{Enl}((X, M_X)/V)$ denote the category of enlargements. This category has a topology in which a family of morphisms

$$\{g_\lambda : ((T_\lambda, M_{T_\lambda}), (Z_\lambda, M_{Z_\lambda}), z_\lambda) \to ((T, M_T), (Z, M_Z), z)\} \tag{3.7.1}$$

is a covering if the following conditions hold:

(i) Each morphism $(T_\lambda, M_{T_\lambda}) \to (T, M_T)$ is strict;
(ii) The collection of maps $\{T_\lambda \to T\}$ is an étale covering of the formal scheme T;
(iii) For every λ the natural map $Z_\lambda \to Z \times_T T_\lambda$ is an isomorphism.

The resulting topos is denoted $((X, M_X)/V)_{\mathrm{conv}}$ (the *convergent topos*). There is a sheaf of rings $\mathcal{K}_{(X,M_X)/V}$ (or sometimes written just $\mathcal{K}$ if no confusion seems likely to arise) which to any object $((T, M_T), (Z, M_Z), z) \in \mathrm{Enl}((X, M_X)/V)$ associates $\Gamma(T, \mathcal{K}_T)$.

If E is a sheaf of $\mathcal{K}$-modules in $((X, M_X)/V)_{\mathrm{conv}}$ and $((T, M_T), (Z, M_Z), z) \in \mathrm{Enl}((X, M_X)/V)$, then we denote by E_T the sheaf of $\mathcal{K}_T$-modules on T_{et} defined by

$$(U \to T) \mapsto E((U, M_T|_U), (Z, M_Z) \times_{(T,M_T)} (U, M_T|_U), z|_{(U,M_T|_U)}). \tag{3.7.2}$$

If

$$g : ((T', M_{T'}), (Z', M_{Z'}), z') \to ((T, M_T), (Z, M_Z), z) \tag{3.7.3}$$

is a morphism in $\mathrm{Enl}((X, M_X)/V)$, then there is a canonical map

$$g^* E_T := g^{-1} E_T \otimes_{g^{-1}(\mathcal{K}_T)} (\mathcal{K}_{T'}) \to E_{T'}. \tag{3.7.4}$$

The sheaf E is called an *isocrystal* if the following hold:

(i) For every object $((T, M_T), (Z, M_Z), z) \in \mathrm{Enl}((X, M_X)/V)$ the sheaf E_T is isocoherent (see [Sh2, p. 8 (2)]).
(ii) For every morphism g as above, the map 3.7.4 is an isomorphism.

3.8. There is a morphism of topoi

$$u : ((X, M_X)/V)_{\mathrm{conv}} \to X_{\mathrm{et}}. \tag{3.8.1}$$

If $F \in X_{\mathrm{et}}$ then $u^* F$ is the sheaf

$$((T, M_T), (Z, M_Z), z) \mapsto \Gamma(Z, z^* F), \tag{3.8.2}$$

and if $E \in ((X, M_X)/V)_{\mathrm{conv}}$ then

$$u_* E(U) = \Gamma(((U, M_X|_U)/V)_{\mathrm{conv}}, E). \tag{3.8.3}$$

3.9. Let $T = ((T, M_T), (Z, M_Z), z)$ be a widening. Then T defines a sheaf h_T in $((X, M_X)/V)_{\mathrm{conv}}$ by associating to any $T' \in \mathrm{Enl}((X, M_X)/V)$ the set of morphisms of widenings $T' \to T$. As explained in [Sh2, 2.1.22 and 2.1.23], there is associated to T a canonical inductive system of widenings $\{T_n\}_{n \in \mathbb{N}}$ with compatible morphisms of widenings (equivalently, sections of h_T)

$$T_n \to T \tag{3.9.1}$$

such that the induced morphism of sheaves

$$\varinjlim h_{T_n} \to h_T \tag{3.9.2}$$

is an isomorphism.

3.10. As explained in [Sh2, p. 51], if T is affine then each T_n is also affine. In fact if $T = \mathrm{Spf}(A)$ and $I = (g_1, \dots, g_r) \subset A$ is the ideal defining Z, and $B_n = \Gamma(T_n, \mathcal{O}_{T_n}) \otimes_V K$, then

$$B_n = K \otimes_V (A[t_1, \dots, t_r]/(\pi t_1 - g_1^n, \dots, \pi t_r - g_r^n) + (\pi - \text{torsion}))^\wedge, \tag{3.10.1}$$

where $(-)^\wedge$ denotes π-adic completion. The transition maps

$$B_{n+1} \to B_n \tag{3.10.2}$$

are given by sending t_i to $g_i t_i$.

Lemma 3.11. *Let $j : M \hookrightarrow N$ be an inclusion of π-torsion free V-modules such that N/M is annihilated by π. Then the map on π-adic completions $\hat{j} : M^\wedge \to N^\wedge$ is injective.*

Proof. Let $\hat{\gamma} \in M^\wedge$ be an element with $\hat{j}(\hat{\gamma}) = 0$, and fix an integer r. We show that $\hat{\gamma}$ maps to zero in $M^\wedge/\pi^{r-1}M^\wedge = M/\pi^{r-1}M$.

For this choose a sequence of elements $\gamma_s \in M$ such that γ_s and $\hat{\gamma}$ have the same image in $M/\pi^s M$, and such that $\gamma_s = \gamma_r$ for $s \leq r$. Write $\gamma_{s+1} = \gamma_s + \pi^s \epsilon_s$ with $\epsilon_s \in M$ (and uniquely determined since M is π-torsion free). Since $\hat{\gamma}$ is in the kernel of $\hat{j}$, there exists for every s an element $b_s \in N$ such that $j(\gamma_s) = \pi^s b_s$. Since N/M is π-torsion and j is injective, this implies that $\gamma_s \in \pi^{s-1}M$ for all s. Since $\gamma_s = \gamma_r$ for $s \leq r$ this implies that $\gamma_s \in \pi^{r-1}M$ for all s. Write $\gamma_s = \pi^{r-1}\gamma'_s$ for some γ'_s. Then

$$\gamma'_{s+1} = \gamma'_s + \pi^{s-r+1}\epsilon_s \tag{3.11.1}$$

for $s \geq r$ since M is π-torsion free, and therefore the elements $\{\gamma'_s\}$ define an element $\hat{\gamma}'$ such that $\pi^{r-1}\hat{\gamma}' = \hat{\gamma}$. Since r was arbitrary this implies that $\hat{\gamma} = 0$. $\square$

3.12. Set

$$M_n := A[t_1, \dots, t_r]/(\pi t_1 - g_1^n, \dots, \pi t_r - g_r^n) + (\pi - \text{torsion}). \tag{3.12.1}$$

Then M_n is a flat V-module, and $M_n \otimes_V K = A$. In particular, the transition maps $j_n : M_{n+1} \to M_n$ are injective. Also observe that the cokernel of j_n is annihilated by π. By 3.11 this implies that the map on π-adic completions $M_{n+1}^\wedge \to M_n^\wedge$ is injective. Tensoring with K we obtain the following.

Corollary 3.13. *For every n, the transition map $B_{n+1} \to B_n$ is injective.*

3.14. Associated to the widening T is a topos $\vec{T}$ defined as follows. For $n \in \mathbb{N}$ write

$$T_n = ((T_n, M_{T_n}), (Z_n, M_{Z_n}), z_n) \tag{3.14.1}$$

so we have a commutative diagram

(3.14.2)
$$\begin{tikzcd}
(Z_n, M_{Z_n}) \arrow[r, hook] \arrow[d] & (T_n, M_{T_n}) \arrow[d, "\gamma_n"] \\
(Z, M_Z) \arrow[r, hook] \arrow[d, "z"] & (T, M_T) \arrow[dd] \\
(X, M_X) \arrow[d, "f"] & \\
\mathrm{Spec}(k) \arrow[r, hook] & \mathrm{Spf}(V).
\end{tikzcd}$$

The topos $\overrightarrow{T}$ is the topos associated to the following site:

Objects: pairs (n, U), where $n \in \mathbb{N}$ and $U \to T_n$ is étale.

Morphisms: the set $\mathrm{Hom}((n, U), (m, V))$ is the empty set unless $n \leq m$ in which case it is the set of commutative diagrams

(3.14.3)
$$\begin{tikzcd}
U \arrow[r] \arrow[d] & V \arrow[d] \\
T_n \arrow[r] & T_m.
\end{tikzcd}$$

Coverings: a collection of maps $\{(n_\lambda, U_\lambda) \to (n, U)\}$ is a *covering* if $n_\lambda = n$ for all λ and the set of maps $\{U_\lambda \to U\}$ is an étale covering.

There is a sheaf of rings $\mathcal{K}_{\overrightarrow{T}}$ in $\overrightarrow{T}$ given by

(3.14.4)
$$(n, U) \mapsto \Gamma(U, \mathcal{K}_U).$$

Giving a sheaf F in $\overrightarrow{T}$ is equivalent to giving a collection of sheaves $\{F_n \in T_{n,\mathrm{et}}\}$ and transition morphisms $\rho_\psi : \psi^{-1}F_m \to F_n$ for every morphism $\psi : T_n \to T_m$ satisfying the usual cocycle condition.

If F is a sheaf of $\mathcal{K}_{\overrightarrow{T}}$-modules, then F is called *crystalline* (see [Sh2, 2.1.30]) if for every morphism $g : T_n \to T_m$ the transition morphism

(3.14.5)
$$g^*F_m := g^{-1}F_m \otimes_{g^{-1}\mathcal{K}_{T_m}} \mathcal{K}_{T_n} \to F_n$$

is an isomorphism.

There is a morphism of ringed topoi

(3.14.6)
$$\gamma : (\overrightarrow{T}, \mathcal{K}_{\overrightarrow{T}}) \to (T_{\mathrm{et}}, \mathcal{K}_T).$$

The pushforward functor γ_* sends a sheaf F to the sheaf

(3.14.7)
$$\gamma_* F = \varprojlim \gamma_{n*} F_n,$$

where $\gamma_n : T_n \to T$ is the projection. The functor γ^* sends a sheaf H on T_{et} to the collection of sheaves whose T_n-component is $\gamma_n^* H$.

Lemma 3.15. *Let $F = \{F_n\}$ be a crystalline sheaf of $\mathcal{K}_{\vec{T}}$-modules in $\vec{T}$ such that each F_n is a flat isocoherent sheaf of $\mathcal{K}_{T_n}$-modules. Then for all n, the transition map*

$$\gamma_{n+1*}F_{n+1} \to \gamma_{n*}F_n \tag{3.15.1}$$

is injective.

Proof. Note first that since F_{n+1} is isocoherent, we have

$$\gamma_{n*}(g^*F_{n+1}) \simeq \gamma_{n+1*}F_{n+1} \otimes_{\gamma_{n+1*}\mathcal{K}_{T_{n+1}}} \gamma_{n*}\mathcal{K}_{T_n}, \tag{3.15.2}$$

where $g : T_n \to T_{n+1}$ is the transition morphism. Using this, and the isomorphism 3.14.5

$$\gamma_{n*}(g^*F_{n+1}) \simeq \gamma_{n*}(F_n) \tag{3.15.3}$$

we see that the map 3.15.1 can be identified with the map

$$\gamma_{n+1*}F_{n+1} \to (\gamma_{n+1*}F_{n+1}) \otimes_{\gamma_{n+1*}\mathcal{K}_{T_{n+1}}} \gamma_{n*}\mathcal{K}_{T_n}. \tag{3.15.4}$$

Now the map $\gamma_{n+1*}\mathcal{K}_{T_{n+1}} \to \gamma_{n*}\mathcal{K}_{T_n}$ is injective, as this can be verified locally and the affine case follows from 3.13. Since F_{n+1} is flat over $\mathcal{K}_{T_{n+1}}$ this implies that 3.15.4 is injective. $\square$

Lemma 3.16. *Let M be a K-vector space (possibly infinite dimensional), and let $F = \{F_n\}$ be a sheaf of $\mathcal{K}_{\vec{T}}$-modules in $\vec{T}$ such that for every n the transition map 3.15.1 is injective. Then the natural map*

$$(\gamma_*F) \otimes_K M \to \gamma_*(F \otimes_K M) \tag{3.16.1}$$

is an isomorphism.

Proof. To ease notation write (abusively) just F_n for $\gamma_{n*}F_n$. We then need to show that the natural map

$$(\varprojlim F_n) \otimes_K M \to \varprojlim(F_n \otimes_K M) \tag{3.16.2}$$

is an isomorphism. Let $\{e_i\}_{i\in I}$ be a basis for M, and let G denote $\varprojlim F_n$. Then we need to show that the natural map

$$\oplus_{i\in I}G \to \prod_{i\in I}{}' G \tag{3.16.3}$$

is an isomorphism, where $\prod'_{i\in I} G \subset \prod_{i\in I} G$ denotes the subsheaf of local sections $v \in \prod_{i\in I} G$ such that for every n the image of v in $\prod_{i\in I} F_n$ is contained in $\oplus_{i\in} F_n$. This follows from noting that since $G \to F_n$ is injective for all n by assumption, the square

$$\begin{tikzcd} \oplus_{i\in I}G \arrow[r] \arrow[d] & \oplus_{i\in I}F_n \arrow[d] \\ \prod_{i\in I} G \arrow[r] & \prod_{i\in I} F_n \end{tikzcd} \tag{3.16.4}$$

is cartesian for every n. $\square$

3.17. Let $((X, M_X)/V)_{\mathrm{conv}}|_T$ denote the localized topos. Then there is also a functor

$$\phi_{\vec{T}*} : ((X, M_X)/V)_{\mathrm{conv}}|_T \to \vec{T} \tag{3.17.1}$$

sending a sheaf $F \in ((X, M_X)/V)_{\mathrm{conv}}|_T$ to the sheaf

$$(n, U \to T_n) \mapsto F(U \to T_n \to T). \tag{3.17.2}$$

Recall that $\phi_{\vec{T}*}$ is not part of a morphism of topoi, but still by [Sh2, 2.3.1] the functor $\phi_{\vec{T}*}$ sends injective sheaves to flasque sheaves. Also if F is a crystalline sheaf of $\mathcal{K}_{\vec{T}}$-modules then one can define $\phi^*_{\vec{T}}F$.

Let

$$j_T : ((X, M_X)/V)_{\mathrm{conv}}|_T \to ((X, M_X)/V)_{\mathrm{conv}} \tag{3.17.3}$$

be the localization morphism. If E is an isocrystal in $((X, M_X)/V)_{\mathrm{conv}}$ we write

$$E_T := \gamma_* \phi_{\vec{T}*} j_T^* E \simeq \varprojlim \gamma_{n*} E_{T_n}. \tag{3.17.4}$$

Then there is a commutative diagram of functors

$$\begin{tikzcd} ((X, M_X)/V)_{\mathrm{conv}}|_T \arrow[r, "\phi_{\vec{T}*}"] \arrow[d, "j_{T*}"] & \vec{T} \arrow[r, "\gamma_*"] & T_{\mathrm{et}} \arrow[d, "\simeq"] \\ ((X, M_X)/V)_{\mathrm{conv}} \arrow[r, "u_*"] & X_{\mathrm{et}} & Z_{\mathrm{et}}. \arrow[l, "z_*"'] \end{tikzcd} \tag{3.17.5}$$

Lemma 3.18. *Assume that $z : Z \to X$ is quasi-compact. Let $E \in ((X, M_X)/V)_{\mathrm{conv}}|_T$ be a sheaf of $\mathcal{K}$-modules such that for every morphism $T' \to T$ of enlargements $E_{T'}$ is isocoherent on T' and such that for every morphism $g : T'' \to T'$ of enlargements over T the map $g^*E_{T'} \to E_{T''}$ is an isomorphism. Then for any widening W and K-vector space M the natural map*

$$(j_{T*}E)_W \otimes_K M \to (j_{T*}(E \otimes_K M))_W \tag{3.18.1}$$

is an isomorphism.

Proof. Write $W = ((W, M_W), (Y, M_Y), y)$, and let $W_n \times T$ and $W \times T$ denote the products in the category of widenings (see 3.6). Let

$$h : Y \times_X Z \to Y \tag{3.18.2}$$

be the projection.

Let

$$j_{T|_W} : ((X, M_X)/V)_{\mathrm{conv}}|_{W\times T} \to ((X, M_X)/V)_{\mathrm{conv}}|_W \tag{3.18.3}$$

be the morphism of topoi obtained from the localization morphism

$$(((X, M_X)/V)_{\mathrm{conv}}|_W)|_{W\times T\to W} \to ((X, M_X)/V)_{\mathrm{conv}}|_W \tag{3.18.4}$$

and the canonical isomorphism of topoi

$$(((X, M_X)/V)_{\mathrm{conv}}|_W)|_{W\times T\to W} \simeq ((X, M_X)/V)_{\mathrm{conv}}|_{W\times T}. \tag{3.18.5}$$

Then there is a commutative diagram of topoi

$$\begin{tikzcd} ((X,M_X)/V)_{\mathrm{conv}}|_{W\times T} \arrow[r, "j_{W|_T}"] \arrow[d, "j_{T|_W}"] & ((X,M_X)/V)_{\mathrm{conv}}|_T \arrow[d, "j_T"] \\ ((X,M_X)/V)_{\mathrm{conv}}|_W \arrow[r, "j_W"] & ((X,M_X)/V)_{\mathrm{conv}}. \end{tikzcd} \tag{3.18.6}$$

Furthermore, one verifies immediately that the adjunction map

$$j_W^* \circ j_{T*} \to j_{T|_W*} \circ j_{W|_T}^* \tag{3.18.7}$$

is an isomorphism. It follows from the definitions that we also have a commutative diagram of functors

$$\begin{tikzcd} (Y\times_X Z)_{\mathrm{et}} \arrow[d, "h_*"] & \overset{\rightarrow}{W\times T} \arrow[l, "\gamma_{W\times T*}"'] & ((X,M_X)/V)_{\mathrm{conv}}|_{W\times T} \arrow[l, "\phi_{\overset{\rightarrow}{W\times T}*}"'] \arrow[d, "j_{T|_W*}"] \\ Y_{\mathrm{et}} & \vec{W} \arrow[l, "\gamma_{W*}"'] & ((X,M_X)/V)_{\mathrm{conv}}|_W. \arrow[l, "\phi_{\vec{W}*}"'] \end{tikzcd} \tag{3.18.8}$$

For any sheaf $F\in ((X,M_X)/V)_{\mathrm{conv}}|_T$ we therefore have

$$\begin{aligned} (j_{T*}F)_W &= \gamma_{W*}\phi_{\vec{W}*}j_W^*j_{T*}F \quad \text{(definition)} \\ &= \gamma_{W*}\phi_{\vec{W}*}j_{T|_W*}j^*_{W|_T}F \quad (3.18.7) \\ &= h_*\gamma_{W\times T*}\phi_{\overset{\rightarrow}{W\times T}*}j^*_{W|_T}F \quad \text{(commutativity of 3.18.8)} \\ &= h_*\varprojlim_n F_{(W\times T)_n}, \end{aligned}$$

where we abusively write $F_{(W\times T)_n}$ for the pushforward of $F_{(W\times T)_n}$ to $(W\times T)_{\mathrm{et}}$.

Now let E be as in the lemma. Then each of the maps

$$E_{(W\times T)_{n+1}} \to E_{(W\times T)_n} \tag{3.18.9}$$

is injective by 3.15 (here we continue with the slightly abusive notation of viewing this as a map of sheaves on $(W\times T)_{\mathrm{et}}$). Since h_* is left exact, we conclude by the same argument used in the proof of 3.16 that the natural map

$$(\varprojlim_n h_*E_{(W\times T)_n})\otimes_K M \to \varprojlim_n((h_*E_{(W\times T)_n})\otimes_K M) \tag{3.18.10}$$

is an isomorphism. On the other hand, since h is quasi-compact we also have

$$(h_*E_{(W\times T)_n})\otimes_K M \simeq h_*(E_{(W\times T)_n}\otimes M) \tag{3.18.11}$$

and therefore the natural map

$$(j_{T*}E)_W\otimes M = (\varprojlim_n h_*E_{(W\times T)_n})\otimes_K M \to \varprojlim_n h_*(E_{(W\times T)_n}\otimes M) = j_{T*}(E\otimes_K M)_W \tag{3.18.12}$$

is an isomorphism. ∎

3.19. Suppose given a commutative diagram of fine log schemes

$$\begin{tikzcd}(X, M_X) \arrow[r, hook, "i"] \arrow[d, "f"] & (P, M_P) \arrow[d, "g"] \\ \mathrm{Spec}(k) \arrow[r, hook] & \mathrm{Spec}(V),\end{tikzcd} \tag{3.19.1}$$

where g is log smooth and i is an exact closed immersion. Let $(\widehat{P}, M_{\widehat{P}})$ denote the formal completion of P along X with $M_{\widehat{P}}$ defined to be the pullback of M_P. We then have a commutative diagram

$$\begin{tikzcd}(X, M_X) \arrow[r, hook, "i"] \arrow[d, "f"] & (\widehat{P}, M_{\widehat{P}}) \arrow[r] \arrow[d, "\hat{g}"] & (P, M_P) \arrow[d, "g"] \\ \mathrm{Spec}(k) \arrow[r, hook] & \mathrm{Spf}(V) \arrow[r] & \mathrm{Spec}(V),\end{tikzcd} \tag{3.19.2}$$

where $(X, M_X) \hookrightarrow (\widehat{P}, M_{\widehat{P}})$ is a widening.

Let E be an isocrystal in $((X, M_X)/V)_{\mathrm{conv}}$, and let $E_{\widehat{P}}$ be the induced sheaf of $\mathcal{K}_{\widehat{P}}$-modules. Then as explained in [Sh2, 2.2.9], there is a canonical integrable connection

$$\nabla : E_{\widehat{P}} \to E_{\widehat{P}} \otimes_{\mathcal{O}_P} \Omega^1_{(P,M_P)/V} \tag{3.19.3}$$

whose associated de Rham-complex we denote by $E_{\widehat{P}} \otimes \Omega^{\cdot}_{(P,M_P)/V}$. This is a complex of K-vector spaces on $\widehat{P}_{\mathrm{et}} \simeq X_{\mathrm{et}}$.

For $i \geq 0$, define

$$\omega^i_{\widehat{P}}(E) := j_{\widehat{P}*}(j^*_{\widehat{P}} E \otimes_{\mathcal{O}_{X/V}} \phi^*_{\widehat{P}} \gamma^* \Omega^i_{(P,M_P)/V}|_{\widehat{P}}). \tag{3.19.4}$$

Then [Sh2, 2.3.3 and 2.3.5] implies that for every $s > 0$ we have $R^s u_* \omega^i_{\widehat{P}}(E) = 0$ and the natural map (induced by the commutativity of 3.19.2)

$$E_{\widehat{P}} \otimes \Omega^i_{(P,M_P)/V} \to u_* \omega^i_{\widehat{P}}(E) \tag{3.19.5}$$

is an isomorphism.

In fact, by the same argument proving [Sh2, 2.3.5] there is a canonical structure of a complex

$$\omega^{\cdot}_{\widehat{P}}(E) : \omega^0_{\widehat{P}}(E) \xrightarrow{d} \omega^1_{\widehat{P}}(E) \longrightarrow \cdots \tag{3.19.6}$$

such that the natural map $E \to \omega^0_{\widehat{P}}(E)$ induces a quasi-isomorphism, and such that the isomorphisms 3.19.5 extend to an isomorphism of complexes

$$E_{\widehat{P}} \otimes \Omega^{\cdot}_{(P,M_P)/V} \to u_* \omega^{\cdot}_{\widehat{P}}(E). \tag{3.19.7}$$

In particular, since the $\omega^i_{\widehat{P}}(E)$ are acyclic for u_* this gives an isomorphism in the derived category

$$E_{\widehat{P}} \otimes \Omega^{\cdot}_{(P,M_P)/V} \simeq Ru_* E. \tag{3.19.8}$$

Remark 3.20. By the same argument used in [Og2, 0.3.7], if $\widehat{P}$ is affine, then each of the terms $E_{\widehat{P}} \otimes \Omega^i_{(P,M_P)/V}$ is acyclic for the global section functor. In this case we have an isomorphism in the derived category

$$R\Gamma(((X, M_X)/V)_{\mathrm{conv}}, E) \simeq (\Gamma(X, E_{\widehat{P}}) \to \Gamma(X, E_{\widehat{P}} \otimes \Omega^1_{(P,M_P)/V}) \to \cdots). \tag{3.20.1}$$

3.21. Let G/K be an affine group scheme. As in 2.23, we can then consider G-equivariant sheaves of $\mathcal{K}$-modules E in $((X, M_X)/V)_{\mathrm{conv}}$. Recall from 2.23 that such a sheaf consists of a sheaf E in the usual sense, together with morphisms of $\mathcal{K}$-modules in $((X, M_X)/V)_{\mathrm{conv}}$

$$\rho : E \to E \otimes_K \mathcal{O}_G \tag{3.21.1}$$

such that the equalities 2.23.4 hold.

If E is a flat isocrystal, then it follows from 3.16 that for every widening W, the sheaf of $\mathcal{K}_W$-modules E_W on W_{et} has a natural $\mathcal{O}_G$-comodule structure, and hence is a G-equivariant sheaf in W_{et}. Similarly (using 3.18) each of the sheaves $\omega^i_{\widehat{P}}(E)_W$ has an induced structure of a G-equivariant sheaf in W_{et}.

3.22. If we furthermore fix a diagram 3.19.1, and E is a G-equivariant flat isocrystal, then it follows from 3.18 that each of the sheaves $\omega^i_{\widehat{P}}(E)$ has a natural structure of a G-equivariant sheaf in $((X, M_X)/V)_{\mathrm{conv}}$, which induces for every widening W a G-equivariant structure on the sheaf $\omega^i_{\widehat{P}}(E)_W$. Moreover, by functoriality of the construction of the complexes $\omega^\cdot_{\widehat{P}}(E)$ each of the squares

$$\begin{tikzcd} \omega^i_{\widehat{P}}(E) \arrow[r, "d"] \arrow[d, "\text{coaction}"] & \omega^{i+1}_{\widehat{P}}(E) \arrow[d, "\text{coaction}"] \\ \omega^i_{\widehat{P}}(E) \otimes_K \mathcal{O}_G \arrow[r, "d"] & \omega^{i+1}_{\widehat{P}}(E) \otimes_K \mathcal{O}_G \end{tikzcd} \tag{3.22.1}$$

commutes, as well as the square

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\text{coaction}"] & \omega^0_{\widehat{P}}(E) \arrow[d, "\text{coaction}"] \\ E \otimes_K \mathcal{O}_G \arrow[r] & \omega^0_{\widehat{P}}(E) \otimes_K \mathcal{O}_G. \end{tikzcd} \tag{3.22.2}$$

Therefore the complex $\omega^\cdot_{\widehat{P}}(E)$ is a complex of G-equivariant sheaves, as is the de Rham complex $E_{\widehat{P}} \otimes_{\mathcal{O}_P} \Omega^\cdot_{(P,M_P)/V}$.

4. Simplicial presheaves associated to isocrystals

Review of simplicial presheaves [Bl, H-S, Ja, To1].

4.1. If $\mathcal{S}$ is a site, we denote by $\mathrm{SPr}(\mathcal{S})$ the category of simplicial presheaves on $\mathcal{S}$. That is, the category of functors (recall that Δ^o denotes the opposite category of Δ)

$$F : \Delta^o \longrightarrow \widehat{\mathcal{S}}, \tag{4.1.1}$$

where $\widehat{\mathcal{S}}$ denotes the category of presheaves on $\mathcal{S}$. We write $\mathrm{SPr}_*(\mathcal{S})$ for the category of pointed objects in $\mathrm{SPr}(\mathcal{S})$. An object $* \to F \in \mathrm{SPr}_*(\mathcal{S})$ is called *connected* if the sheaf associated to the presheaf $R \mapsto \pi_0(|F(R)|)$ is isomorphic to $*$, where $|F(R)|$ denotes the geometric realization of the simplicial set $F(R)$. For $(F, * \rightarrow F) \in \mathrm{SPr}_*(\mathcal{S})$ and $i \geq 0$, we write $\pi_i(F, *)$ for the sheaf on $\mathcal{S}$ associated to the presheaf sending $U \in \mathcal{S}$ to $\pi_i(|F(U)|, *)$. We will view $\mathrm{SPr}(\mathcal{S})$ and $\mathrm{SPr}_*(\mathcal{S})$ as model categories using the model category structure defined in [To1, 1.1.1]. Recall that a morphism $(F, * \rightarrow F) \rightarrow (F', * \rightarrow F')$ of connected objects in $\mathrm{SPr}_*(\mathcal{S})$ is an equivalence if and only if the induced map

$$\pi_i(F, *) \longrightarrow \pi_i(F', *) \tag{4.1.2}$$

is an equivalence for every $i > 0$. This implies in particular that one can define π_i for an object in the homotopy category $\mathrm{Ho}(\mathrm{SPr}_*(\mathcal{S}))$. We refer to the elements of $\mathrm{Ho}(\mathrm{SPr}(\mathcal{S}))$ (resp. $\mathrm{Ho}(\mathrm{SPr}_*(\mathcal{S}))$) as *stacks* (resp. *pointed stacks*).

In what follows, the site $\mathcal{S}$ will usually be the category Aff_B of affine B-schemes with the fpqc topology for some $\mathbb{Q}$–algebra B, and we write $\mathrm{SPr}(B)$ (resp. $\mathrm{SPr}_*(B)$) instead of $\mathrm{SPr}(\mathrm{Aff}_B)$ (resp. $\mathrm{SPr}_*(\mathrm{Aff}_B)$).

Remark 4.2. If $\mathcal{S}$ has enough points (which will always be the case in this paper), then a map $F \rightarrow F'$ in $\mathrm{SPr}(\mathcal{S})$ is an equivalence if and only if for every point x of the corresponding topos $\tilde{\mathcal{S}}$ the induced map on stalks $F \rightarrow F'$ is an equivalence.

Remark 4.3. Recall (see the discussion in [To1, 1.1.1]) that there is another model category structure on $\mathrm{SPr}(\mathcal{S})$, called the *strong model category structure*, in which a morphism $F \rightarrow F'$ is an equivalence (resp. fibration) if for every $X \in \mathcal{S}$ the map of simplicial sets $F(X) \rightarrow F'(X)$ is an equivalence (resp. fibration). We refer to equivalences, cofibrations, and fibrations with respect to this model structure as *strong equivalences*, *strong cofibrations*, and *strong fibrations*.

The model category structure on $\mathrm{SPr}(\mathcal{S})$ used in 4.1 (and the rest of the paper) is then characterized by the definition of weak equivalences in [To1, 1.1.1] and by declaring cofibrations to be strong cofibrations (and then fibrations are defined using the right lifting property). The model category structure on $\mathrm{SPr}_*(\mathcal{S})$ is obtained from the model category structure on $\mathrm{SPr}(\mathcal{S})$ as in [Ho, 1.1.8].

The strong model category structure on $\mathrm{SPr}(\mathcal{S})$ is clearly a proper model category structure in the sense of B.5, as the category of simplicial sets with the usual model category structure is proper. The model category structure on $\mathrm{SPr}(\mathcal{S})$ defined in [To1, 1.1.1] is therefore also left proper (as it is a left localization of a proper model category structure [Hi, 4.1.1]). If $\mathcal{S}$ has enough points then this model category is also right proper, as this can be verified on stalks by 4.2.

Remark 4.4. If $U \in \mathcal{S}$ is an object of the site $\mathcal{S}$, then we can consider the category $\mathrm{SPr}(\mathcal{S}|_U)$ of simplicial presheaves on the category of objects over U. The category $\mathcal{S}|_U$ has a natural Grothendieck topology induced by that on $\mathcal{S}$. There is a functor $r^* : \mathrm{SPr}(\mathcal{S}) \rightarrow \mathrm{SPr}(\mathcal{S}|_U)$ sending F to $(V \rightarrow U) \mapsto F(V)$. The functor r^* has a left adjoint $r_!$ sending $G \in \mathrm{SPr}(\mathcal{S}|_U)$ to the presheaf

$$r_! G(V) = \coprod_{s \in \mathrm{Hom}(V, U)} G(V \stackrel{s}{\rightarrow} U). \tag{4.4.1}$$

In particular, if $* \to r_! G(V)$ is a point mapping to the component corresponding to $s : V \to U$, then the sheaf $\pi_1(r_! G, *)$ on $\mathcal{S}|_V$ is isomorphic to $\pi_1(G|_{s:V\to U}, *)$. This implies that $r_!$ takes weak equivalences to weak equivalences.

Note also that the functor r^* takes strong equivalences (resp. strong fibrations) to strong equivalences (resp. strong fibrations), and therefore $r_!$ preserves strong cofibrations (=cofibrations). Therefore $r_!$ preserves both cofibrations and trivial cofibrations, which implies that $(r_!, r^*)$ is a Quillen adjunction.

In particular there is an induced functor

$$\mathbb{R}r^* : \mathrm{Ho}(\mathrm{SPr}(\mathcal{S})) \longrightarrow \mathrm{Ho}(\mathrm{SPr}(\mathcal{S}|_U)). \tag{4.4.2}$$

Since r^* preserves arbitrary equivalences, the functor r^* derives trivially (that is, if $F \in \mathrm{SPr}(\mathcal{S})$ is a not necessary fibrant object representing an object $F^h \in \mathrm{Ho}(\mathrm{SPr}(\mathcal{S}))$ then $\mathbb{R}r^* F^h$ is represented by $r^* F$).

Equivariant cosimplicial algebras and pointed stacks.

4.5. Let R be a $\mathbb{Q}$–algebra and G/R an affine flat group scheme. Let $G - \mathrm{Alg}_R$ denote the category of R–algebras with right G–action. That is, $G - \mathrm{Alg}_R$ is the category of R–algebras in the tensor category of right G–representations $\mathrm{Rep}_R(G)$. We denote by $G - \mathrm{Alg}_R^\Delta$ the category of cosimplicial objects in $G - \mathrm{Alg}_R$. Let $G - \mathrm{dga}_R$ denote the category of commutative differential graded algebras in the category of right G–representations. We view $G - \mathrm{Alg}_R^\Delta$ and $G - \mathrm{dga}_R$ as closed model categories using 2.21. As discussed in 2.21, the Dold–Kan correspondence induces an equivalence of categories

$$\mathrm{Ho}(G - \mathrm{dga}_R) \simeq \mathrm{Ho}(G - \mathrm{Alg}_R^\Delta). \tag{4.5.1}$$

4.6. Denote by $G - \mathrm{SPr}(R)$ (resp. $G - \mathrm{SPr}_*(R)$) the category of objects in $\mathrm{SPr}(R)$ (resp. $\mathrm{SPr}_*(R)$) equipped with a left action of G (viewed as a sheaf via the Yoneda embedding). As discussed in [KPT, §1.2] there is a model category structure on $G - \mathrm{SPr}(R)$ in which a morphism $X \to Y$ is an equivalence (resp. fibration) if the morphism in $G - \mathrm{SPr}(R)$ obtained by forgetting the G-action is an equivalence (resp. fibration). Similarly for $G - \mathrm{SPr}_*(R)$.

For any $A \in G - \mathrm{Alg}_R^\Delta$, define $\mathrm{Spec}_G(A)$ to be the simplicial presheaf sending a R–algebra D to

$$[n] \mapsto \mathrm{Hom}_R(A_n, D). \tag{4.6.1}$$

The right action of G on A_n induces a left action of G on $\mathrm{Spec}_G(A)$ and hence we obtain a functor

$$\mathrm{Spec}_G : G - \mathrm{Alg}_R^\Delta \longrightarrow G - \mathrm{SPr}(R). \tag{4.6.2}$$

If G is the trivial group and $A \in \mathrm{Alg}_R^\Delta$, we write simply $\mathrm{Spec}(A) \in \mathrm{SPr}(R)$ for the associated simplicial presheaf.

As explained in [KPT, p. 16], the functor Spec_G is right Quillen. We denote by

$$\mathbb{R}\mathrm{Spec}_G : \mathrm{Ho}(G - \mathrm{Alg}_R^\Delta)^{\mathrm{op}} \longrightarrow \mathrm{Ho}(G - \mathrm{SPr}(R)) \tag{4.6.3}$$

the resulting derived functor.

4.7. Let EG denote the simplicicial presheaf which is the nerve of the morphism $G \to *$. So we have

$$(EG)_m = G^{m+1} \tag{4.7.1}$$

with the face and degeneracy maps given by the projections and diagonals. The group G acts on EG through left translation, and we write BG for the quotient. The identity section of G defines a map $* \to EG$, which is an equivalence. Therefore EG and BG are naturally pointed simplicial presheaves.

By [KPT, 1.2.1] there are natural equivalences of categories

$$\mathrm{Ho}(G-\mathrm{SPr}(R)) \simeq \mathrm{Ho}(\mathrm{SPr}(R)|_{BG}), \quad \mathrm{Ho}(G-\mathrm{SPr}_*(R)) \simeq \mathrm{Ho}(\mathrm{SPr}_*(R)|_{BG}). \tag{4.7.2}$$

For $F \in \mathrm{Ho}(G-\mathrm{SPr}(R))$, we write $[F/G]$ for the corresponding object of $\mathrm{Ho}(\mathrm{SPr}(R)|_{BG})$. The equivalences 4.7.2 are induced by a Quillen adjunction (De, Mo) between $\mathrm{SPr}(R)|_{BG}$ and $G-\mathrm{SPr}(R)$. The functor

$$De : G-\mathrm{SPr}(R) \to \mathrm{SPr}(R)|_{BG} \tag{4.7.3}$$

sends $F \in G-\mathrm{SPr}(R)$ to $(EG \times F)/G$, where G acts diagonally on $EG \times F$. This functor then induces an equivalence

$$\mathbb{L}De : \mathrm{Ho}(G-\mathrm{SPr}(R)) \to \mathrm{Ho}(\mathrm{SPr}(R)|_{BG}). \tag{4.7.4}$$

4.8. If $A \in G-\mathrm{Alg}_R^{\Delta}$, then any augmentation $A \to R$ (not necessarily compatible with the action of G) gives $[\mathbb{R}\mathrm{Spec}_G(A)/G]$ a natural structure of an object of $\mathrm{Ho}(\mathrm{SPr}_*(R))$. For this note that the forgetful functor $\mathrm{Rep}(G) \to \mathrm{Mod}_R$ has a right adjoint $M \mapsto M \otimes \mathcal{O}_G$. Hence giving an augmentation $A \to R$ is equivalent to giving an equivariant map $A \to \mathcal{O}_G$.

As discussed in appendix B, the category $G-\mathrm{Alg}^{\Delta}_{R,/\mathcal{O}_G}$ of objects over $\mathcal{O}_G$ has a natural model category structure induced by the model category structure on $G-\mathrm{Alg}_R^{\Delta}$. A morphism $f : A \to A'$ in $G-\mathrm{Alg}^{\Delta}_{R,/\mathcal{O}_G}$ is an equivalence (resp. cofibration, fibration) of the underlying morphism $A \to A'$ in $G-\mathrm{Alg}_R^{\Delta}$ is an equivalence (resp. cofibration, fibration). The functor $\mathrm{Spec}_G(-)$ induces a functor

$$(G-\mathrm{Alg}^{\Delta}_{R,/\mathcal{O}_G})^o \to G-\mathrm{SPr}(R)|_{\backslash \mathrm{Spec}_G(\mathcal{O}_G)} \tag{4.8.1}$$

which we again denote by $\mathrm{Spec}_G(-)$. Here $G-\mathrm{SPr}(R)|_{\backslash \mathrm{Spec}_G(\mathcal{O}_G)}$ denotes the category of objects of $G-\mathrm{SPr}(R)$ under $\mathrm{Spec}_G(\mathcal{O}_G)$. We have a commutative diagrams

$$\begin{tikzcd} (G-\mathrm{Alg}^{\Delta}_{R,/\mathcal{O}_G})^o \arrow[r, "\mathrm{Spec}_G(-)"] \arrow[d, "\mathrm{forget}"'] & G-\mathrm{SPr}(R)|_{\backslash \mathrm{Spec}_G(\mathcal{O}_G)} \arrow[d, "\mathrm{forget}"] \\ (G-\mathrm{Alg}^{\Delta}_R)^o \arrow[r, "\mathrm{Spec}_G(-)"] & G-\mathrm{SPr}(R), \end{tikzcd} \tag{4.8.2}$$

and

$$\begin{tikzcd} G-\mathrm{SPr}(R)|_{\backslash \mathrm{Spec}_G(\mathcal{O}_G)} \arrow[r, "De"] \arrow[d, "\mathrm{forget}"] & \mathrm{SPr}(R)|_{BG,\backslash EG} \arrow[d, "\mathrm{forget}"] \\ G-\mathrm{SPr}(R) \arrow[r, "De"] & \mathrm{SPr}(R)|_{BG}, \end{tikzcd} \tag{4.8.3}$$

where $\mathrm{SPr}(R)|_{BG,\backslash EG}$ denotes the category of objects in $\mathrm{SPr}(R)|_{BG}$ under EG. As explained in B.7 diagram 4.8.2 is a commutative diagram of right Quillen functors. Passing to the associated homotopy categories we therefore obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Ho}(G - \mathrm{Alg}^{\Delta}_{R,/\mathcal{O}_G})^o \arrow[r, "\mathbb{R}\mathrm{Spec}_G(-)"] \arrow[d, "\mathrm{forget}"'] & \mathrm{Ho}(G - \mathrm{SPr}(R)|_{\backslash \mathrm{Spec}_G(\mathcal{O}_G)}) \arrow[d, "\mathrm{forget}"] \\ \mathrm{Ho}(G - \mathrm{Alg}^{\Delta}_{R})^o \arrow[r, "\mathbb{R}\mathrm{Spec}_G(-)"] & \mathrm{Ho}(G - \mathrm{SPr}(R)). \end{tikzcd} \tag{4.8.4}$$

Lemma 4.9. *The G-equivariant presheaf $G = \mathrm{Spec}_G(\mathcal{O}_G)$ is cofibrant in $G - \mathrm{SPr}(R)$.*

Proof. Let $H : G - \mathrm{SPr}(R) \to \mathrm{SPr}(R)$ be the functor forgetting the G-action. Then for any $F \in G - \mathrm{SPr}(R)$ we have

$$\mathrm{Hom}_{G-\mathrm{SPr}(R)}(G, F) = \mathrm{Hom}_{\mathrm{SPr}(R)}(*, H(F)). \tag{4.9.1}$$

Since H takes fibrations (resp. equivalences) to fibrations (resp. equivalences), the statement that G is cofibrant follows from the fact that $*$ is cofibrant in $\mathrm{SPr}(R)$ (since $*$ is obviously strongly cofibrant). $\square$

By B.9, the diagram 4.8.3 therefore induces a commutative diagram of derived functors

$$\begin{tikzcd} \mathrm{Ho}(G - \mathrm{SPr}(R)|_{\backslash \mathrm{Spec}_G(\mathcal{O}_G)}) \arrow[r, "\mathbb{L}De"] \arrow[d, "\mathrm{forget}"] & \mathrm{Ho}(\mathrm{SPr}(R)|_{BG,\backslash EG}) \arrow[d, "\mathrm{forget}"] \\ \mathrm{Ho}(G - \mathrm{SPr}(R)) \arrow[r, "\mathbb{L}De"] & \mathrm{Ho}(\mathrm{SPr}(R)|_{BG}). \end{tikzcd} \tag{4.9.2}$$

By forgetting the map to BG we also obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Ho}(\mathrm{SPr}(R)|_{BG,\backslash EG}) \arrow[rr] \arrow[dr, "\mathrm{forget}"'] & & \mathrm{Ho}(\mathrm{SPr}(R)_{\backslash EG}) \arrow[dl, "\mathrm{forget}"] \\ & \mathrm{Ho}(\mathrm{SPr}(R)). & \end{tikzcd} \tag{4.9.3}$$

Now by B.5, the point $* \to EG$ induces an equivalence

$$\mathrm{Ho}(\mathrm{SPr}(R)_{\backslash EG}) \to \mathrm{Ho}(\mathrm{SPr}_*(R)) \tag{4.9.4}$$

sending

$$(EG \to F) \mapsto (* \to EG \to F). \tag{4.9.5}$$

Therefore the diagram

$$\begin{tikzcd} \mathrm{Ho}(\mathrm{SPr}(R)_{\backslash EG}) \arrow[rr] \arrow[dr, "\mathrm{forget}"'] & & \mathrm{Ho}(\mathrm{SPr}_*(R)) \arrow[dl, "\mathrm{forget}"] \\ & \mathrm{Ho}(\mathrm{SPr}(R)) & \end{tikzcd} \tag{4.9.6}$$

commutes. Combining 4.8.4, 4.9.2, 4.9.3, and 4.9.6 we obtain a commutative diagram
(4.9.7)

$$\begin{array}{ccccccc}
\mathrm{Ho}(G-\mathrm{Alg}^{\Delta}_{R,/\mathcal{O}_G})^o & \xrightarrow{\mathbb{R}\mathrm{Spec}_G} & \mathrm{Ho}(G-\mathrm{SPr}(R)_{\backslash \mathrm{Spec}_G(\mathcal{O}_G)}) & \xrightarrow{\mathbb{L}De} & \mathrm{Ho}(\mathrm{SPr}(R)|_{BG,\backslash EG}) & \longrightarrow & \mathrm{Ho}(\mathrm{SPr}_*(R)) \\
\downarrow{\scriptstyle \mathrm{forget}} & & & & & & \downarrow{\scriptstyle \mathrm{forget}} \\
\mathrm{Ho}(G-\mathrm{SPr}(R))^o & & & \xrightarrow{[\mathbb{R}\mathrm{Spec}_G(-)/G]} & & & \mathrm{Ho}(\mathrm{SPr}(R)).
\end{array}$$

We denote the top horizontal composite again by

$$(4.9.8) \qquad [\mathbb{R}\mathrm{Spec}_G(-)/G] : \mathrm{Ho}(G-\mathrm{Alg}^{\Delta}_{R,/\mathcal{O}_G})^o \to \mathrm{Ho}(\mathrm{SPr}_*(R)).$$

4.10. In what follows it will also be useful to be able to replace $\mathcal{O}_G$ by an equivalent algebra. If $S \in G-\mathrm{Alg}^{\Delta}_R$ is any object with an equivalence $\mathcal{O}_G \to S$, then by B.5 there is a natural equivalence $\mathrm{Ho}(G-\mathrm{Alg}^{\Delta}_{R,/\mathcal{O}_G}) \simeq \mathrm{Ho}(G-\mathrm{Alg}^{\Delta}_{R,/S})$. Composing this equivalence with 4.9.8 we obtain a functor

$$(4.10.1) \qquad [\mathbb{R}\mathrm{Spec}_G(-)/G] : \mathrm{Ho}(G-\mathrm{Alg}^{\Delta}_{R,/S})^o \longrightarrow \mathrm{Ho}(\mathrm{SPr}_*(R)).$$

4.11. Let $R \to R'$ be a flat morphism of $\mathbb{Q}$–algebras. Then the forgetful functor $f : G_{R'} - \mathrm{Alg}^{\Delta}_{R'} \to G_R - \mathrm{Alg}^{\Delta}_R$ has an exact left adjoint $A \mapsto A \otimes_R R'$ and hence f preserves fibrations and trivial fibrations. Therefore the pair $(\otimes_R R', f)$ is a Quillen adjunction.

If $r^* : \mathrm{SPr}(R) \to \mathrm{SPr}(R')$ denotes the restriction functor 4.4, then the diagram

$$(4.11.1) \qquad \begin{array}{ccc}
(G-\mathrm{Alg}^{\Delta}_R)^{\mathrm{op}} & \xrightarrow{\mathrm{Spec}_G} & G-\mathrm{SPr}(R) \\
{\scriptstyle \otimes_R R'}\downarrow & & \downarrow{\scriptstyle r^*} \\
(G_{R'}-\mathrm{Alg}^{\Delta}_{R'})^{\mathrm{op}} & \xrightarrow{\mathrm{Spec}_{G_{R'}}} & G_{R'}-\mathrm{SPr}(R')
\end{array}$$

commutes. It follows that if $A \in G-\mathrm{Alg}^{\Delta}_R$, then $\mathbb{R}r^*\mathbb{R}\mathrm{Spec}_G(A) \simeq \mathbb{R}\mathrm{Spec}_{G_{R'}}(A \otimes_R R')$.

Stacks associated to isocrystals.

4.12. Let k be a perfect field, V its ring of Witt vectors, K the field of fractions of V, and $\sigma : K \to K$ the automorphism induced by the canonical lift of Frobenius to V. Let X/V be a smooth proper scheme, $D \subset X$ a divisor with normal crossings relative to V, and $x : \mathrm{Spec}(V) \to X^o$ a section. We view X as a log scheme (X, M_X) in the sense of Fontaine and Illusie [Ka] with log structure M_X defined by the divisor D. Denote by $(Y, M_Y)/k$ the reduction of (X, M_X), and by $\mathrm{Isoc}((Y, M_Y)/K)$ the category of log isocrystals on the convergent site of (Y, M_Y) (see section 3).

Let $\mathrm{Isoc}^{\mathrm{lf}}((Y, M_Y)/K) \subset \mathrm{Isoc}((Y, M_Y)/K)$ denote the full subcategory of locally free objects. Since X/V is proper, the category of $\mathrm{Isoc}^{\mathrm{lf}}((Y, M_Y)/K)$ is naturally identified with a full subcategory of the category $MIC((X_K, M_{X_K})/K)$ of coherent sheaves on the generic fiber (X_K, M_{X_K}) with integrable logarithmic connection (combine [Sh1, proof of 5.2.9, 5.2.10, and 3.2.16]). In particular, there is a natural restriction functor

$$(4.12.1) \qquad \mathrm{Isoc}^{\mathrm{lf}}((Y, M_Y)/K) \longrightarrow MIC(X^o_K/K),$$

where $X^o := X - D$. By [Sa, VI.1.2.2], the category $MIC(X^o_K/K)$ is Tannakian and every object is locally free.

Remark 4.13. As in the case without log structures, the category $\mathrm{Isoc}^{\mathrm{lf}}((Y, M_Y)/K)$ can also be described as the category of locally free isocrystals on the log crystalline site of $(Y, M_Y)/V$ [Sh2, 3.1]. If $E \in \mathrm{Isoc}^{\mathrm{lf}}((Y, M_Y)/K)$ we can therefore compute the cohomology of E in either the convergent topos or the crystalline topos. By [Sh2, 3.1.1] these two different cohomology groups are canonically isomorphic. We will therefore simply write $H^*_{\mathrm{cris}}((Y, M_Y), E)$ for these groups. From this and the comparison between log crystalline and log de Rham cohomology [Ka, 6.4] it follows that if $(\mathcal{E}, \nabla) \in MIC(X_K, M_{X_K}/K)$ denotes the module with integrable connection corresponding to E, then there is a canonical isomorphism

$$H^*_{\mathrm{cris}}((Y, M_Y), E) \simeq H^*_{\text{log-dR}}((X_K, M_{X_K}), (\mathcal{E}, \nabla)). \tag{4.13.1}$$

4.14. Let R denote the cokernel of the map $\Omega^1_{X_K/K} \to \Omega^1_{(X_K, M_{X_K})/K}$ (the *sheaf of residues*), and observe that if $(\mathcal{E}, \nabla) \in MIC((X_K, M_{X_K})/K)$ then the composite

$$\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1_{(X_K, M_{X_K})/K} \longrightarrow \mathcal{E} \otimes R \tag{4.14.1}$$

is $\mathcal{O}_{X_K}$–linear. In particular, for every point $y \in X_K$ we obtain a map

$$R(y)^* \longrightarrow \mathrm{End}_{k(y)}(\mathcal{E}(y)). \tag{4.14.2}$$

We say that $(\mathcal{E}, \nabla)$ has *unipotent local monodromy* if for every $y \in X_K$ the image of 4.14.2 consists of nilpotent endomorphisms (see also [K-N] for this notion in the analytic context).

In local coordinates this condition can be described as follows. Etale locally around y, there exists an étale morphism

$$X \longrightarrow \mathrm{Spec}(V[T_1, \ldots, T_r, T^{\pm}_{r+1}, \ldots, T^{\pm}_n]) \tag{4.14.3}$$

with D given by the equation $T_1 \cdots T_r = 0$ and $y \in \{T_1 = \cdots = T_r = 0\}$. The choice of such a morphism identifies R with the sheaf associated to the module

$$\oplus^r_{i=1}(\mathcal{O}_X/(T_i)) \cdot d\log(T_i), \tag{4.14.4}$$

so $R(y)$ has a basis given by $d\log(T_i)$ $(1 \leq i \leq r)$ and the corresponding endomorphisms D_i of $\mathrm{End}_{k(y)}(\mathcal{E}(y))$ all commute. Therefore $(\mathcal{E}, \nabla)$ has unipotent local monodromy if and only if the endomorphisms D_i are nilpotent. Observe also that it suffices to verify the nilpotence at closed points $y \in X_K$.

We denote the full subcategory of $MIC((X_K, M_{X_K})/K)$ of vector bundles with integrable logarithmic connection with unipotent local monodromy by $V_{\mathrm{nilp}}(X_K, M_{X_K})$.

Remark 4.15. Modules with integrable connection of "geometric origin" often have unipotent local monodromy [Il1].

Lemma 4.16. *If $(\mathcal{E}, \nabla)$ and $(\mathcal{F}, \Upsilon)$ are in $V_{\mathrm{nilp}}(X_K, M_{X_K})$, then $(\mathcal{E} \otimes \mathcal{F}, \nabla \otimes \Upsilon)$ is also in $V_{\mathrm{nilp}}(X_K, M_{X_K})$. The category $V_{\mathrm{nilp}}(X_K, M_{X_K})$ has internal homs, duals, and is closed under extensions in $MIC((X_K, M_{X_K})/K)$.*

Proof. Let $y \in X$ be a point, $\tau \in R(y)^*$ and element and let N (resp. M) be the corresponding endomorphism of $\mathcal{E}(y)$ (resp. $\mathcal{F}(y)$). The endomorphism of $\mathcal{E}(y) \otimes \mathcal{F}(y) = (\mathcal{E} \otimes \mathcal{F})(y)$ corresponding to the connection $\nabla \otimes \Upsilon$ is then $N \otimes 1 + 1 \otimes M$. Since the endomorphisms

$N \otimes 1$ and $1 \otimes M$ of $\mathcal{E}(y) \otimes \mathcal{F}(y)$ commute, we have

$$(N \otimes 1 + 1 \otimes M)^r = \sum_{i=0}^{r} \binom{r}{i} N^i \otimes M^{r-i}. \tag{4.16.1}$$

This implies that $(\mathcal{E} \otimes \mathcal{F}, \nabla \otimes \Upsilon) \in V_{\mathrm{nilp}}(X_K, M_{X_K})$. The statement about duals follows from the fact that the transpose of a nilpotent endomorphism is nilpotent. From this we get internal homs by the formula

$$\underline{\mathrm{Hom}}((\mathcal{E}, \nabla), (\mathcal{F}, \Upsilon)) = (\mathcal{E}, \nabla)^* \otimes (\mathcal{F}, \Upsilon). \tag{4.16.2}$$

The statement about extensions follows from the observation that if (V, N) is a vector space with an endomorphism, and if V admits a N–stable filtration Fil such that the induced endomorphism of $\mathrm{gr}^{\cdot}_{\mathrm{Fil}}(V)$ is nilpotent, then N is nilpotent. □

Lemma 4.17. (i) *For any $(\mathcal{E}, \nabla) \in V_{\mathrm{nilp}}(X_K, M_{X_K})$ the natural map on cohomology*

$$H^*_{\text{log-dR}}((X_K, M_{X_K}), (\mathcal{E}, \nabla)) \longrightarrow H^*_{\mathrm{dR}}(X^o_K, (\mathcal{E}^o, \nabla^o)) \tag{4.17.1}$$

is an isomorphism.

(ii) *The restriction functor*

$$V_{\mathrm{nilp}}(X_K, M_{X_K}) \longrightarrow MIC(X^o_K/K) \tag{4.17.2}$$

is fully faithful with essential image closed under the operations of direct sums, tensor products, duals, internal hom, subquotients, and extensions.

(iii) *The category $V_{\mathrm{nilp}}(X_K, M_{X_K})$ is Tannakian with fiber functor given by*

$$\omega_{x_K} : V_{\mathrm{nilp}}(X_K, M_{X_K}) \longrightarrow \mathrm{Vec}_K, \quad (\mathcal{E}, \nabla) \mapsto \mathcal{E}(x). \tag{4.17.3}$$

Proof. To see (i), it suffices by a standard reduction to consider the case when $K = \mathbb{C}$. Then $(\mathcal{E}, \nabla) \in V_{\mathrm{nilp}}(X_K, M_{X_K})$ is the canonical extension of $(\mathcal{E}^o, \nabla^o)$ in the sense of Deligne [De1, 5.2]. From this and [De1, 3.14 and 5.2 (d)] (i) follows.

All the statements of (ii) except for the statement about subquotients follow immediately from (i) and the natural isomorphisms

$$\mathrm{Ext}^i_{V_{\mathrm{nilp}}(X_K, M_{X_K})}((\mathcal{E}, \nabla), (\mathcal{F}, \Upsilon)) \simeq H^i_{\text{log-dR}}((X_K, M_{X_K}), (\mathcal{E}, \nabla)^* \otimes (\mathcal{F}, \Upsilon)), \tag{4.17.4}$$

$$\mathrm{Ext}^i_{MIC(X^o_K/K)}((\mathcal{E}^o, \nabla^o), (\mathcal{F}^o, \Upsilon^o)) \simeq H^i_{\mathrm{dR}}(X^o_K, (\mathcal{E}^o, \nabla^o)^* \otimes (\mathcal{F}^o, \Upsilon^o)). \tag{4.17.5}$$

To verify that the essential image is closed under subquotients, note that by the uniqueness of the canonical extension in [De1, 5.2] it suffices to consider the case when $K = \mathbb{C}$ in which case the result follows from (loc. cit.).

Statement (iii) follows from (ii) and the fact that $MIC(X^o_K/K)$ is Tannakian. □

4.18. Let $V^{\mathrm{cris}}_{\mathrm{nilp}}((Y, M_Y)/K) \subset \mathrm{Isoc}((Y, M_Y)/K)$ denote the full subcategory of objects whose image in $MIC((X_K, M_{X_K})/K)$ lies in $V_{\mathrm{nilp}}(X_K, M_{X_K})$. Since the inclusion functor

$$\mathrm{Isoc}((Y, M_Y)/K) \subset MIC((X_K, M_{X_K})/K) \tag{4.18.1}$$

has essential image closed under sub–quotients, direct sums, duals, tensor products, and extensions (see for example [De2, 11.4]), the category $V^{\mathrm{cris}}_{\mathrm{nilp}}((Y, M_Y)/K)$ is identified with a

Tannakian subcategory of $V_{\mathrm{nilp}}(X_K, M_{X_K})$. Moreover, the restriction of the functor 4.17.3 to $V^{\mathrm{cris}}_{\mathrm{nilp}}((Y, M_Y)/K)$ is naturally identified with the functor

$$V^{\mathrm{cris}}_{\mathrm{nilp}}((Y, M_Y)/K) \longrightarrow \mathrm{Isoc}(k/K) \simeq \mathrm{Vec}_K, \quad E \mapsto x^*E. \tag{4.18.2}$$

Lemma 4.19. *For any $\mathcal{V} \in V^{\mathrm{cris}}_{\mathrm{nilp}}((Y, M_Y)/K)$, the pullback by Frobenius $F^*\mathcal{V}$ is again in $V^{\mathrm{cris}}_{\mathrm{nilp}}((Y, M_Y)/K)$.*

Proof. Because X/V is proper, the scheme X_K is covered by maps $\mathrm{Spec}(R^\wedge \otimes_V K) \to X_K$, where $R^\wedge$ is the p–adic completion of an affine étale $\mathrm{Spec}(R) \to X$ which admits an étale morphism

$$\mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(V[T_1, \dots, T_r, T^{\pm}_{r+1}, \dots, T^{\pm}_n]) \tag{4.19.1}$$

as in 4.14.3. For such an R, there exists a lift of Frobenius

$$\tilde{F} : R^\wedge \longrightarrow R^\wedge \tag{4.19.2}$$

sending T_i to T_i^p. In this case the pullback of $F^*\mathcal{V}$ to $\mathrm{Spec}(R^\wedge_K)$ is the module $\mathcal{V} \otimes_{R^\wedge, \tilde{F}} R^\wedge$ with connection $F^*(\nabla)$ such that

$$\langle F^*(\nabla)(v \otimes 1)), T_i \frac{\partial}{\partial T_i}\rangle = p\langle \nabla(v), T_i \frac{\partial}{\partial T_i}\rangle \otimes 1. \tag{4.19.3}$$

Hence if the operator $\langle \nabla(-), T_i \frac{\partial}{\partial T_i}\rangle$ is nilpotent so is the operator $\langle F^*(\nabla)(-), T_i \frac{\partial}{\partial T_i}\rangle$. This implies the lemma. ☐

Lemma 4.20 ([LS-E, 2.1])**.** *For any $\mathcal{V} \in \mathrm{Isoc}((Y, M_Y)/K)$, the natural map*

$$F^* : H^*_{\mathrm{cris}}((Y, M_Y), \mathcal{V}) \longrightarrow H^*_{\mathrm{cris}}((Y, M_Y), F^*\mathcal{V}) \tag{4.20.1}$$

is an isomorphism.

Proof. Let $(Y', M_{Y'})$ denote the fiber product $(Y, M_Y) \otimes_{k, \mathrm{Frob}_k} k$ and let $F_{(Y,M_Y)/k} : (Y, M_Y) \to (Y', M_{Y'})$ be the natural map induced by Frobenius on (Y, M_Y). Let $\mathcal{V}'$ be the pullback of $\mathcal{V}$ to $(Y', M_{Y'})$.

Note first that

$$H^*_{\mathrm{cris}}((Y, M_Y), F^*\mathcal{V}) \simeq H^*_{\mathrm{cris}}((Y', M_{Y'}), F_{(Y,M_Y)/k*}F^*_{(Y,M_Y)/k}\mathcal{V}') \tag{4.20.2}$$

and $F_{(Y,M_Y)/k*}F^*_{(Y,M_Y)/k}\mathcal{V}' \simeq (F_{(Y,M_Y)/k*}\mathcal{O}_{Y/K}) \otimes_{\mathcal{O}_{Y'/K}} \mathcal{V}'$ (projection formula).

Sub-Lemma 4.21. *The sheaf $F_{(Y,M_Y)/k*}\mathcal{O}_{Y/K}$ is a locally free sheaf of $\mathcal{O}_{Y'/K}$–modules of finite rank.*

Proof. This can be verified locally, so we may assume that there exists an étale morphism

$$X \to \mathrm{Spec}(V[T_1, \dots, T_r, T^{\pm}_{r+1}, \dots, T^{\pm}_n]) \tag{4.21.1}$$

as in 4.14.3 and let $(\widehat{X}, M_{\widehat{X}})$ denote the p–adic completion of (X, M_X). Let $(\widehat{X}', M_{\widehat{X}'})$ denote the base change of $(\widehat{X}, M_{\widehat{X}})$ via the map $\mathrm{Spec}(V) \to \mathrm{Spec}(V)$ induced by the lift of Frobenius, and let $F : (\widehat{X}, M_{\widehat{X}}) \to (\widehat{X}', M_{\widehat{X}'})$ be the map induced by $T_i \mapsto T_i^p$. The morphism F is finite and flat and the natural map $F^*\Omega^1_{(\widehat{X}', M_{\widehat{X}'})/V} \to \Omega^1_{(\widehat{X}, M_{\widehat{X}})/V}$ becomes an isomorphism after tensoring with $\mathbb{Q}$ ($F \otimes \mathbb{Q}$ is log étale). It follows that $(F_*\mathcal{O}_{\widehat{X}}) \otimes_V K$ has a natural integrable connection which defines a locally free crystal $\mathcal{A}$ of $\mathcal{O}_{(Y',M_{Y'})/K}$–algebras. We leave to the reader the task of verifying that this crystal $\mathcal{A}$ is isomorphic to $F_{(Y,M_Y)/k*}\mathcal{O}_{Y/K}$. $\square$

It follows that there is a trace map $\mathrm{tr} : F_{(Y,M_Y)/k*}\mathcal{O}_{Y/K} \to \mathcal{O}_{Y'/K}$ such that the composite $\mathcal{O}_{Y'/K} \to F_{(Y,M_Y)/k*}\mathcal{O}_{Y/K} \to \mathcal{O}_{Y'/K}$ is multiplication by some non–zero integer. This implies that the map 4.20.1 is injective.

To see that 4.20.1 is surjective, choose a totally ramified finite extension $V \to \widetilde{V}$ with fraction field extension $K \to \widetilde{K}$ such that $\widetilde{K}$ contains the p–th roots of 1, and note that there are natural isomorphisms

$$H^*_{\mathrm{cris}}((Y, M_Y), F^*\mathcal{V}) \otimes_K \widetilde{K} \simeq H^*(((Y, M_Y)/\widetilde{K})_{\mathrm{conv}}, F^*\mathcal{V}), \tag{4.21.2}$$

$$H^*_{\mathrm{cris}}((Y', M_{Y'}), \mathcal{V}') \otimes_K \widetilde{K} \simeq H^*(((Y', M_{Y'})/\widetilde{K})_{\mathrm{conv}}, \mathcal{V}'). \tag{4.21.3}$$

It therefore suffices to show that the natural map

$$H^*(((Y', M_{Y'})/\widetilde{K})_{\mathrm{conv}}, \mathcal{V}') \longrightarrow H^*(((Y, M_Y)/\widetilde{K})_{\mathrm{conv}}, F^*\mathcal{V}) \tag{4.21.4}$$

is an isomorphism.

To see that 4.20.1 is surjective, it suffices to show the stronger statement that the map

$$F^* \circ \mathrm{tr} : \mathbb{R}u_*\mathcal{V}' \to \mathbb{R}u_*[(F_{(Y,M_Y)/k*}\mathcal{O}_{Y/\widetilde{K}}) \otimes_{\mathcal{O}_{Y'/\widetilde{K}}} \mathcal{V}'] \tag{4.21.5}$$

is an isomorphism, where u_* denotes the projection to the étale topos of Y'. This is a local assertion so it suffices to verify it in the case when there is an étale morphism as in 4.14.3. In this case, the map $X \to X' := X \otimes_{V,\sigma} V$ induced by $T_i \mapsto T_i^p$ is a lift of Frobenius and the resulting map $X \otimes_V \widetilde{V} \to X' \otimes_V \widetilde{V}$ identifies $X' \otimes_V \widetilde{V}$ with the quotient of $X \otimes_V \widetilde{V}$ by the natural action of the group $\mu_p^n \simeq (\mathbb{Z}/p\mathbb{Z})^n$ (since $\tilde{V}$ contains the p–th roots of 1). It follows that the map $F^* \circ \mathrm{tr}$ is equal to $\sum_{\sigma\in\mu_p^n} \sigma$. Since the reduction of each σ is the identity, it follows that this is simply multiplication by p^n and so 4.21.5 is a quasi–isomorphism. $\square$

4.22. Let (E, φ_E) be an F–isocrystal on $(Y, M_Y)/K$ with $E \in V^{\mathrm{cris}}_{\mathrm{nilp}}((Y, M_Y)/K)$. Denote by $\langle E\rangle_\otimes \subset V^{\mathrm{cris}}_{\mathrm{nilp}}((Y, M_Y)/K)$ the Tannakian subcategory generated by E, and let $\mathcal{C} \subset V^{\mathrm{cris}}_{\mathrm{nilp}}((Y, M_Y)/K)$ be the smallest Tannakian subcategory closed under extensions, and containing E. The category $\mathcal{C}$ consists of objects $\mathcal{V} \in V^{\mathrm{cris}}_{\mathrm{nilp}}((Y, M_Y)/K)$ which admit a filtration $0 = \mathcal{V}_0 \subset \mathcal{V}_2 \subset \cdots \mathcal{V}_n = \mathcal{V}$ whose successive quotients are objects of $\langle E\rangle_\otimes$.

Observe that the restriction functor 4.17.2 identifies $\langle E\rangle_\otimes$ (resp. $\mathcal{C}$) with the Tannakian subcategory of $MIC(X^o_K/K)$ generated by $\mathcal{E}^o_K$ (resp. the smallest Tannakian subcategory of $MIC(X^o_K/K)$ closed under extensions and containing $\mathcal{E}^o_K$), where $\mathcal{E}^o_K$ denotes the module with integrable connection on X^o_K defined by E.

We make the following assumption:

Assumption 4.23. *The category $\langle E\rangle_\otimes$ is semi–simple.*

This assumption implies in particular that an isocrystal $\mathcal{V}$ is in $\langle E\rangle_\otimes$ if and only if $\mathcal{V}$ is isomorphic to a direct summand of $E^a \otimes (E^*)^b$ for some $a, b \in \mathbb{N}$.

Lemma 4.24. (i) *For any $\mathcal{V}$ in $\mathcal{C}$ (resp. $\langle E\rangle_\otimes$), the pullback by Frobenius $F^*\mathcal{V}$ is again in $\mathcal{C}$ (resp. $\langle E\rangle_\otimes$).*

(ii) *The induced functors*

$$F^*:\mathcal{C}\to\mathcal{C},\quad F^*:\langle E\rangle_\otimes\to\langle E\rangle_\otimes \tag{4.24.1}$$

are equivalences.

Proof. Statement (i) is immediate.

For (ii), note first that the essential image of $F^*:\mathcal{C}\to\mathcal{C}$ is closed under extensions. For given any two objects $\mathcal{V}_1,\mathcal{V}_2\in\mathcal{C}$ and an extension

$$0\longrightarrow F^*\mathcal{V}_1\longrightarrow\mathcal{V}\longrightarrow F^*\mathcal{V}_2\longrightarrow 0, \tag{4.24.2}$$

corresponding to a class $e\in H^1_{\mathrm{cris}}(((Y,M_Y),F^*(\mathcal{V}_1^*\otimes\mathcal{V}_2))$, there exists by 4.20 (i)) a class $e'\in H^1_{\mathrm{cris}}(((Y,M_Y),\mathcal{V}_1^*\otimes\mathcal{V}_2)$ with $F^*(e')=e$. The class e' then gives an extension of $\mathcal{V}_2$ by $\mathcal{V}_1$ inducing $\mathcal{V}$. Note also that 4.20 (i)) implies that F^* is fully faithful since for any $\mathcal{V}_1,\mathcal{V}_2\in\mathcal{C}$

$$\begin{array}{ccc} H^0_{\mathrm{cris}}(((Y,M_Y),\mathcal{V}_1^*\otimes\mathcal{V}_2) & \longrightarrow & H^0_{\mathrm{cris}}(((Y,M_Y),F^*(\mathcal{V}_1^*\otimes\mathcal{V}_2)) \\ \simeq\uparrow & & \simeq\downarrow \\ \mathrm{Hom}_{\mathcal{C}}(\mathcal{V}_1,\mathcal{V}_2) & & \mathrm{Hom}_{\mathcal{C}}(F^*\mathcal{V}_1,F^*\mathcal{V}_2) \end{array} \tag{4.24.3}$$

is bijective.

Hence it suffices to show that $F^*:\langle E\rangle_\otimes\to\langle E\rangle_\otimes$ is essentially surjective. For this it suffices to show that if $E'\in\langle E\rangle_\otimes$ is an object obtained from E by performing the operations of tensor product, direct sum, and dual and if $\mathcal{V}\subset F^*E'$ then there exists a sub–object $\mathcal{V}'\subset E'$ with $F^*\mathcal{V}'=\mathcal{V}$. Since $\langle E\rangle_\otimes$ is semi–simple, the subcrystal $\mathcal{V}\subset F^*E'$ is obtained by projection from an idempotent $e\in H^0_{\mathrm{cris}}((Y,M_Y),\underline{\mathrm{End}}(F^*E'))$. By 4.20 the map

$$F^*:H^0_{\mathrm{cris}}((Y,M_Y),\underline{\mathrm{End}}(E'))\longrightarrow H^0_{\mathrm{cris}}((Y,M_Y),F^*\underline{\mathrm{End}}(E')) \tag{4.24.4}$$

is an isomorphism, and since $F^*\underline{\mathrm{End}}(E')\simeq\underline{\mathrm{End}}(F^*E')$ there exists an idempotent $e'\in H^0_{\mathrm{cris}}((Y,M_Y),\underline{\mathrm{End}}(E'))$ with $F^*(e')=e$. The corresponding direct summand $\mathcal{V}'\subset E'$ then pulls back to $\mathcal{V}$. □

4.25. The method of [Ol1] reviewed below associates to $\mathcal{C}$ a stack $X_{\mathcal{C}}\in\mathrm{Ho}(\mathrm{SPr}_*(K))$ (the case when $D=\emptyset$ is treated in (loc. cit.) but the same method works in the present logarithmic situation). The stack $X_{\mathcal{C}}$ has the following properties 4.26–4.28.

4.26. There is a canonical isomorphism

$$\pi_1(X_{\mathcal{C}})\simeq\pi_1(\mathcal{C},\omega_x). \tag{4.26.1}$$

4.27. There is a natural isomorphism $\varphi_{X_{\mathcal{C}}}:X_{\mathcal{C}}\to X_{\mathcal{C}}\otimes_{K,\sigma}K$, where $X_{\mathcal{C}}\otimes_{K,\sigma}K$ denotes the object of $\mathrm{Ho}(\mathrm{SPr}_*(K))$ which to any $R\in\mathrm{Alg}_K$ associates $X_{\mathcal{C}}(R\otimes_{K,\sigma^{-1}}K)$. This isomorphism induces an isomorphism $\varphi_{\pi_1}:\pi_1(X_{\mathcal{C}})\to\pi_1(X_{\mathcal{C}})\otimes_{K,\sigma}K$ which under the isomorphism 4.26.1 corresponds to the isomorphism induced by Tannaka duality and the equivalence $\mathcal{C}\otimes_{K,\sigma}K\to\mathcal{C}$ induced by Frobenius pullback.

4.28. For any representation V of $\pi_1(X_{\mathcal{C}})$ corresponding to an isocrystal $\mathcal{V}$ on $(Y, M_Y)/K$, there is a canonical isomorphism

$$H^*(X_{\mathcal{C}}, V) \simeq H^*_{\text{cris}}((Y, M_Y), \mathcal{V}), \tag{4.28.1}$$

where $H^*(X_{\mathcal{C}}, V)$ denotes cohomology of the simplicial presheaf $X_{\mathcal{C}}$ as defined in [To1, 1.3]. If $\varphi_V : V \otimes_{K,\sigma} K \to V$ is an isomorphism such that the diagram

$$\begin{tikzcd} \pi_1(X_{\mathcal{C}}) \arrow[r] \arrow[d, "\varphi_{\pi_1}"'] & \mathrm{Aut}(V) \arrow[d, "\varphi_V"] \\ \pi_1(X_{\mathcal{C}}) \otimes_{K,\sigma} K \arrow[r] & \mathrm{Aut}(V^\sigma) \end{tikzcd} \tag{4.28.2}$$

commutes, and if $\varphi_{\mathcal{V}} : F^*\mathcal{V} \to \mathcal{V}$ denotes the associated F–isocrystal structure on $\mathcal{V}$, then the F–isocrystal structure induced by φ_V on $H^*(X_{\mathcal{C}}, V)$ agrees under the isomorphism 4.28.1 with the F–isocrystal structure on $H^*_{\text{cris}}((Y, M_Y), \mathcal{V})$ induced by $\varphi_{\mathcal{V}}$.

4.29. The description of the stack $X_{\mathcal{C}}$ in 4.35 below uses the following basic construction. Let G/K be an algebraic group, and let $\mathbb{L}$ be a differential graded algebra in the category of ind–isocrystals on $(Y, M)/K$ with G–action. To any such algebra $\mathbb{L}$ we can associate an algebra $\mathbb{R}\Gamma_{\text{cris}}(\mathbb{L}) \in \mathrm{Ho}(G - \mathrm{dga}_K)$ whose cohomology ring is $H^*_{\text{cris}}(\mathbb{L})$ with the natural G–action. If $\mathbb{L}$ is an equivariant algebra in the category of ind–F–isocrystals then there is an induced semi–linear automorphism $\mathbb{R}\Gamma_{\text{cris}}(\mathbb{L}) \to \mathbb{R}\Gamma_{\text{cris}}(\mathbb{L})$ in $\mathrm{Ho}(G - \mathrm{dga}_K)$.

To construct $\mathbb{R}\Gamma_{\text{cris}}(\mathbb{L})$ choose an affine étale cover $U \to Y$ and an embedding $(U, M_U) \hookrightarrow (Z, M_Z)$ of (U, M_U) into an affine log scheme (Z, M_Z) log smooth over V (with the trivial log structure). For each $n \geq 0$, let $t : (Z^*_n, M_{Z^*_n}) \to (Z, M_Z)^{(n+1)}$ denote the universal object over $(Z, M_Z)^{(n+1)}$ defined in A.8.

Recall that the log scheme $(Z^*_n, M_{Z^*_n})$ has the following universal property. For each i, let $\mathrm{pr}_i : (Z, M_Z)^{(n+1)} \to (Z, M_Z)$ be the projection to the i–th factor. Then the natural map $t^*\mathrm{pr}_i^* M_Z \to M_{Z^*_n}$ is an isomorphism and $(Z^*_n, M_{Z^*_n})$ is universal with this property. That is, if $g : (T, M_T) \to (Z, M_Z)^{(n+1)}$ is a morphism of log schemes such that for each i the map $g^*\mathrm{pr}_i^* M_Z \to M_T$ is an isomorphism, then g factors uniquely through $(Z^*_n, M_{Z^*_n})$. In particular, the diagonal

$$(Z, M_Z) \hookrightarrow (Z, M_Z)^{(n+1)} \tag{4.29.1}$$

factors canonically through $(Z^*_n, M_{Z^*_n})$. Recall also that by the construction of $(Z^*_n, M_{Z^*_n})$ in the proof of A.8, $(Z^*_n, M_{Z^*_n})$ is an affine log scheme.

Let (U_n, M_{U_n}) denote the $n+1$–st fold fiber product of (U, M_U) over (Y, M_Y). By the universal property of $(Z^*_n, M_{Z^*_n})$, the natural embedding $(U_n, M_{U_n}) \hookrightarrow (Z, M_Z)^{(n+1)}$ factors uniquely through $(Z^*_n, M_{Z^*_n})$. Denote by (Z_n, M_{Z_n}) the completion of $(Z^*_n, M^*_{Z_n})$ along (U_n, M_{U_n}). The (Z_n, M_{Z_n}) form in a natural way a simplicial log formal scheme denoted $(Z_\bullet, M_{Z_\bullet})$.

Let $(U_\bullet, M_{U_\bullet})$ denote the 0–coskeleton of the map $(U, M_U) \to (Y, M_Y)$ so that there is an embedding $(U_\bullet, M_{U_\bullet}) \hookrightarrow (Z_\bullet, M_{Z_\bullet})$. Each $(U_n, M_{U_n}) \hookrightarrow (Z_n, M_{Z_n})$ is a widening and hence we can evaluate $\mathbb{L}$ on each (Z_n, M_{Z_n}) (in the sense of 3.17) to get a module $\mathcal{L}_n$ with logarithmic connection $\nabla : \mathcal{L}_n \to \mathcal{L}_n \otimes \Omega^1_{(Z_n, M_{Z_n})/V}$. Forming the de Rham complex of this connection, we obtain a sheaf of differential graded algebras $\mathcal{L}_\bullet \otimes \Omega^\bullet_{(Z_\bullet, M_{Z_\bullet})}$ on $Z_\bullet$ with action of the group scheme G. From this we get a G–equivariant cosimplicial algebra

$$\Delta \rightarrow G - \mathrm{dga}_K, \quad [n] \mapsto \Gamma(Z_n, \mathcal{L}_n \otimes \Omega^\bullet_{(Z_n, M_{Z_n})}). \tag{4.29.2}$$

In what follows we will write $DR(\mathbb{L})_{U_\bullet}$ for this cosimplicial differential graded algebra. We define $\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L})$ (also sometimes written $\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L})_{U_\bullet}$ if we want the dependence on $U_\bullet$ to be clear) to be the object of $G - \mathrm{dga}_K$ obtained by applying the functor of Thom–Sullivan cochains to $DR(\mathbb{L})_{U_\bullet}$ (note that $DR(\mathbb{L})_{U_\bullet}$ and $\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L})_{U_\bullet}$ also depend on $(Z_\bullet, M_{Z_\bullet})$ but we omit this from the notation).

4.30. Since we chose the Z_n to be affine, the underlying complex of $\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L})$ is simply the usual complex used to compute convergent cohomology by 3.20. This implies that if $(U', M_{U'}) \hookrightarrow (Z', M_{Z'})$ is a second choice of étale cover and lifting and if we are given a commutative square

$$\begin{tikzcd}
(U', M_{U'}) \arrow[r] \arrow[d, "\rho"'] & (Z', M_{Z'}) \arrow[d] \\
(U, M_U) \arrow[r] & (Z, M_Z),
\end{tikzcd} \tag{4.30.1}$$

where ρ is a morphism over Y, giving rise to a commutative square

$$\begin{tikzcd}
(U'_\bullet, M_{U'_\bullet}) \arrow[r] \arrow[d, "\rho"'] & (Z'_\bullet, M_{Z'_\bullet}) \arrow[d] \\
(U_\bullet, M_{U_\bullet}) \arrow[r] & (Z_\bullet, M_{Z_\bullet}),
\end{tikzcd} \tag{4.30.2}$$

the induced map

$$DR(\mathbb{L})_{U_\bullet} \longrightarrow DR(\mathbb{L})_{U'_\bullet} \tag{4.30.3}$$

is an equivalence and the resulting map

$$\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L})_{U_\bullet} \longrightarrow \mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L})_{U'_\bullet} \tag{4.30.4}$$

in $\mathrm{Ho}(G - \mathrm{dga}_K)$ is independent of the choice of the morphism $U' \rightarrow U$. Furthermore, if we choose a lifting of Frobenius to (Z, M_Z) then the F–isocrystal structure on $\mathbb{L}$ induces a semi–linear equivalence

$$DR(\mathbb{L})_{U_\bullet} \longrightarrow DR(\mathbb{L})_{U_\bullet}. \tag{4.30.5}$$

As explained in [Ol1, 2.23], the induced equivalence

$$\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L}) \otimes_{K,\sigma} K \longrightarrow \mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L}) \tag{4.30.6}$$

in $\mathrm{Ho}(G - \mathrm{dga}_K)$ is independent of the choices.

In what follows it will be also be necessary to describe the G–equivariant algebra $\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L})$ in a slightly different manner.

Lemma 4.31. *Let $\mathbb{L}$ be a differential graded algebra in the category of ind–isocrystals with G–action on $(Y, M_Y)/K$. Associated to the lifting $(X, M_X)/V$ of (Y, M_Y) to V is a G–equivariant differential graded algebra $\mathbb{R}^\bullet$ with each $\mathbb{R}^i$ a sheaf on $(Y, M_Y)/K$ which is acyclic for the projection u_* to the étale topos of Y, and a map of G–equivariant algebras $\mathbb{L} \rightarrow \mathbb{R}^\bullet$ which is a quasi–isomorphism on the underlying complexes of sheaves.*

Remark 4.32. In the case when $\mathbb{L}$ is a sheaf (i.e. concentrated in degree 0), the complex $\mathbb{R}^\bullet$ was denoted $\omega^{\cdot}_{\widehat{X}}(\mathbb{L})$ in 3.19.

Proof. Let (T, M_T) denote the object of the convergent site of (Y, M_Y) given by $(Y, M_Y) \hookrightarrow (X, M_X)$. There is then a diagram of topoi

$$\begin{array}{ccc} ((Y, M_Y)/V)_{\mathrm{conv}}|_T & \xrightarrow{\phi_T} & Y_{\mathrm{et}} \\ {\scriptstyle j_T}\big\downarrow & & \\ ((Y, M_Y)/V)_{\mathrm{conv}}. & & \end{array} \tag{4.32.1}$$

For each integer t, there is by [Sh2, 2.3.6] a natural resolution $\mathbb{L}^t \to R^t_\bullet$ with $R^t_j := j_{T*}(\phi_T^*\Omega^j_{(X,M_X)/V} \otimes j_T^*\mathbb{L}^t)$ (see also 3.19). The sheaf R^t_j is acyclic for the projection u_*. Furthermore, the complex $R^t_\bullet$ is functorial so we obtain a double complex $R^\bullet_\bullet$ with an equivalence $\mathbb{L} \to R^\bullet_\bullet$. Let $\mathbb{R}^\bullet$ be the associated single complex.

By definition $\mathbb{R}^s = \oplus_{i+j=s} j_{T*}(\phi_T^*\Omega^j_{(X,M_X)/V} \otimes j_T^*\mathbb{L}^i)$ and the differential $d : \mathbb{R}^s \to \mathbb{R}^{s+1}$ is induced by the maps

$$\phi_T^*\Omega^j_{(X,M_X)/V} \otimes j_T^*\mathbb{L}^i \to \phi_T^*\Omega^j_{(X,M_X)/V} \otimes j_T^*\mathbb{L}^{i+1} \oplus \phi_T^*\Omega^{j+1}_{(X,M_X)/V} \otimes j_T^*\mathbb{L}^i \tag{4.32.2}$$

sending $\omega \otimes \ell$ to $(\delta(\ell) \otimes \omega, (-1)^i\delta(\omega \otimes \ell))$, where we have written δ for the differentials. We define an algebra structure on $\mathbb{R}^\bullet$ by the formula

$$(\omega \otimes \ell) \wedge (\omega' \otimes \ell') := (-1)^{i'j}(\omega \wedge \omega') \otimes (\ell \cdot \ell'), \tag{4.32.3}$$

where $\omega \otimes \ell \in j_{T*}(\phi_T^*\Omega^j_{(X,M_X)/V} \otimes j_T^*\mathbb{L}^i)$ and $\omega' \otimes \ell' \in \phi_T^*\Omega^{j'}_{(X,M_X)/V} \otimes j_T^*\mathbb{L}^{i'}$. $\square$

4.33. Let $\mathbb{L} \to \mathbb{R}^\bullet$ be the resolution corresponding to our lifting $(X, M_X)/V$. The sheaves $\mathbb{R}^i$ are not isocrystals, but still the value $\mathbb{R}^i(Z, M_Z)$ of $\mathbb{R}^i$ on any affine widening $(U, M_U) \hookrightarrow (Z, M_Z)$, with (Z, M_Z) formally smooth over V, has a canonical integrable connection

$$\mathbb{R}^i(Z, M_Z) \to \mathbb{R}^i(Z, M_Z) \otimes \Omega^1_{(Z,M_Z)/V}. \tag{4.33.1}$$

This follows from the construction of $\mathbb{R}^i$ (see the proof of [Sh2, 2.3.5] and without log structures [Og2, 0.5.4]). Moreover, by the proof of the convergent Poincaré lemma [Sh2, 2.3.5 (2)], the complex $\mathbb{R}^\bullet(Z, M_Z)$ is a resolution of $\mathbb{L}(Z, M_Z)$.

Let $(U_\bullet, M_{U_\bullet}) \hookrightarrow (Z_\bullet, M_{Z_\bullet})$ be as in 4.29. For each n as in 4.29, let $\mathbb{R}^\bullet(((U_n, M_{U_n})/K)_{\mathrm{cris}})$ denote the G–equivariant differential graded algebra

$$\Gamma((((U_n, M_{U_n})/K)_{\mathrm{conv}}), \mathbb{R}^\bullet). \tag{4.33.2}$$

For each i, let $\mathcal{R}^i_\bullet$ denote the module with connection on $(Z_\bullet, M_{Z_\bullet})$ obtained by evaluating $\mathbb{R}^i$ on $(Z_\bullet, M_{Z_\bullet})$, and let $\mathcal{R}^i_\bullet \otimes \Omega^\bullet_{(Z_\bullet,M_{Z_\bullet})}$ denote the corresponding de Rham complex of $\mathcal{R}^i$. Let $DR(\mathbb{R}^\bullet)_{(U_n,M_{U_n})}$ denote the differential graded algebra with

$$DR(\mathbb{R}^\bullet)^s_{(U_n,M_{U_n})} := \oplus_{i+j=s}\mathcal{R}^i_n \otimes \Omega^j_{(Z_n,M_{Z_n})}. \tag{4.33.3}$$

If δ denotes the differential of $\mathcal{R}^\bullet_n$, then the differential on $DR(\mathbb{R}^\bullet)_{(U_n,M_{U_n})}$ is defined by the maps

$$\mathcal{R}^i_n \otimes \Omega^j_{(Z_n,M_{Z_n})} \to \mathcal{R}^{i+1}_n \otimes \Omega^j_{(Z_n,M_{Z_n})} \oplus \mathcal{R}^i_n \otimes \Omega^{j+1}_{(Z_n,M_{Z_n})}, \qquad r\omega \mapsto \delta(r)\omega + (-1)^i\nabla(r\omega) \tag{4.33.4}$$

and as in the proof of 4.31 $DR(\mathbb{R}^\bullet)_{(U_n,M_{U_n})}$ has a differential graded algebra structure given by

$$(r\otimes\omega)\wedge(r'\otimes\omega') := (-1)^{i'j}(r\cdot r')\otimes(\omega\omega') \tag{4.33.5}$$

for $r\otimes\omega\in\mathcal{R}^i_n\otimes\Omega^j_{(Z_n,M_{Z_n})}$ and $r'\otimes\omega'\in\mathcal{R}^{i'}_n\otimes\Omega^{j'}_{(Z_n,M_{Z_n})}$.

We then obtain a diagram of G–equivariant algebras

$$\begin{tikzcd} \mathbb{R}^\bullet(((U_n,M_{U_n})/K)_{\mathrm{cris}}) \arrow[r] & DR(\mathbb{R}^\bullet)_{(U_n,M_{U_n})} \\ & DR(\mathbb{L})_{(U_n,M_{U_n})} \arrow[u] \end{tikzcd} \tag{4.33.6}$$

with all morphisms equivalences. This construction is functorial so the above constructions induce cosimplicial differential graded algebras $\mathbb{R}^\bullet(((U_\bullet,M_{U_\bullet})/K)_{\mathrm{cris}})$ and $DR(\mathbb{R}^\bullet)_{U_\bullet}$ sitting in a diagram of equivalences

$$\mathbb{R}^\bullet(((U_\bullet,M_{U_\bullet})/K)_{\mathrm{cris}}) \longrightarrow DR(\mathbb{R}^\bullet)_{U_\bullet} \longleftarrow DR(\mathbb{L})_{U_\bullet}. \tag{4.33.7}$$

Applying the functor of Thom–Sullivan cochains T we see that the G–equivariant algebra $\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L})$ is canonically isomorphic in $\mathrm{Ho}(G-\mathrm{dga}_K)$ to $T(\mathbb{R}^\bullet(((U_\bullet,M_{U_\bullet})/K)_{\mathrm{cris}}))$.

4.34. If $x\in X^o(V)$ is a point, then we can apply the preceding discussion with $X=\mathrm{Spec}(V)$, and $x^*\mathbb{L}$. If $U_{x,\bullet}$ denotes the simplicial scheme obtained by pulling back $U_\bullet$ to x, then we obtain a commutative diagram

$$\begin{tikzcd} \mathbb{R}^\bullet(((U_\bullet,M_{U_\bullet})/K)_{\mathrm{cris}}) \arrow[r] \arrow[d] & DR(\mathbb{R}^\bullet)_{U_\bullet} \arrow[d] & DR(\mathbb{L})_{U_\bullet} \arrow[l] \arrow[d] \\ \mathbb{L}(x)\otimes_K\mathcal{O}_{U_{x,\bullet}} \arrow[r, "\mathrm{id}"] & \mathbb{L}(x)\otimes_K\mathcal{O}_{U_{x,\bullet}} \arrow[r, "\mathrm{id}"] & \mathbb{L}(x)\otimes_K\mathcal{O}_{U_{x,\bullet}}. \end{tikzcd} \tag{4.34.1}$$

Here the vertical arrows are induced by the functoriality of the formation of the resolution $\mathbb{L}\to\mathbb{R}^\bullet$, and the functoriality of the de Rham complex.

If furthermore, we are given an augmentation $\mathbb{L}(x)\to K$ we see that the algebras in 4.33.7 admit natural augmentations to $\mathcal{O}_{U_{x,\bullet}}$. Since the map $K\to\mathcal{O}_{U_{x,\bullet}}$ is an equivalence the discussion of 4.8 applies, and there is an isomorphism of pointed stacks

$$[\mathbb{R}\mathrm{Spec}_G(\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L}))/G] \simeq [\mathbb{R}\mathrm{Spec}_G T(\mathbb{R}^\bullet(((U_\bullet,M_{U_\bullet})/K)_{\mathrm{cris}}))/G]. \tag{4.34.2}$$

4.35. The stack $X_{\mathcal{C}}$ is obtained using construction 4.29 as follows. Let G denote the pro–reductive completion of $\pi_1(\mathcal{C},\omega_x)$, and let $\mathcal{O}_G$ be the coordinate ring of G. Right translation induces a left action of G on $\mathcal{O}_G$ which by Tannaka duality corresponds to an ind–isocrystal $\mathbb{L}(\mathcal{O}_G)\in V^{\mathrm{cris}}_{\mathrm{nilp}}((Y,M_Y)/K)$. Left translation induces a right action of G on $\mathcal{O}_G$ which commutes with the left action and hence induces an action of G on $\mathbb{L}(\mathcal{O}_G)$. Furthermore the identity section induces a map $x^*\mathbb{L}(\mathcal{O}_G)\simeq\mathcal{O}_G\to K$. By definition

$$X_{\mathcal{C}} \simeq [\mathbb{R}\mathrm{Spec}_G(\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_G)))/G]. \tag{4.35.1}$$

The ind–isocrystal $\mathbb{L}(\mathcal{O}_G)$ has a natural F–isocrystal structure $\psi : F^*\mathbb{L}(\mathcal{O}_G)\to\mathbb{L}(\mathcal{O}_G)$ which induces a Frobenius automorphism on $\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_G))$ giving the Frobenius structure $\varphi_{X_{\mathcal{C}}}$.

4.36. Let us also remark that by cohomological descent we can carry out the above construction using any diagram

$$\begin{array}{ccc} (U_\bullet, M_{U_\bullet}) & \longrightarrow & (Z_\bullet, M_{Z_\bullet}) \\ \big\downarrow & & \\ (Y, M_Y), & & \end{array} \tag{4.36.1}$$

where each $(U_n, M_{U_n}) \hookrightarrow (Z_n, M_{Z_n})$ is a widening of affine formal schemes, (Z_n, M_{Z_n}) is formally smooth over V, and $(U_\bullet, M_{U_\bullet}) \to (Y, M_Y)$ is an étale hypercover. In what follows, we will use the same notation as in the preceding paragraphs in this more general setting as well. In particular, there are diagram of equivalences

$$\mathbb{R}^\bullet(((U_\bullet, M_{U_\bullet})/K)_{\text{cris}}) \longrightarrow DR(\mathbb{R}^\bullet)_{U_\bullet} \longleftarrow DR(\mathbb{L})_{U_\bullet}. \tag{4.36.2}$$

as in 4.33.7, and $\mathbb{R}\Gamma_{\text{cris}}(\mathbb{L})$ can be computed by applying the functor of Thom–Sullivan cochains to any of these algebras.

Remark 4.37. If $\omega : \mathcal{C} \to \text{Mod}_R$ is any fiber functor to the category of R–modules, for some K–algebra R, the preceding constructions can also be carried out with ω instead of ω_x. If G/R denotes the group scheme of tensor automorphisms of ω, then left and right translation on $\mathcal{O}_G$ induce an ind–isocrystal $\mathbb{L}_\omega$ with structure of an R–module and right G–action. We can then apply the constructions of 4.29–4.33 to $\mathbb{L}_\omega$.

Precisely, for any diagram as in 4.36.1 we obtain a diagram of differential graded R–algebras with right G–action

$$\mathbb{R}^\bullet_\omega(((U_\bullet, M_{U_\bullet})/K)_{\text{cris}}) \longrightarrow DR(\mathbb{R}^\bullet_\omega)_{U_\bullet} \longleftarrow DR(\mathbb{L}_\omega)_{U_\bullet} \tag{4.37.1}$$

where all morphisms are equivalences. Moreover, this diagram is functorial in the sense that for any extension $R \to R'$ and fiber functor $\omega' : \mathcal{C} \to R'$ with an isomorphism $\sigma : \omega' \simeq \omega \otimes_R R'$ there is a natural commutative square of differential graded algebras with right G–action

$$\begin{array}{ccccc} \mathbb{R}^\bullet_\omega(((U_\bullet, M_{U_\bullet})/K)_{\text{cris}}) & \longrightarrow & DR(\mathbb{R}^\bullet_\omega)_{U_\bullet} & \longleftarrow & DR(\mathbb{L}_\omega)_{U_\bullet} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathbb{R}^\bullet_{\omega'}(((U_\bullet, M_{U_\bullet})/K)_{\text{cris}}) & \longrightarrow & DR(\mathbb{R}^\bullet_{\omega'})_{U_\bullet} & \longleftarrow & DR(\mathbb{L}_{\omega'})_{U_\bullet} \end{array} \tag{4.37.2}$$

If $R \to R'$ is flat the vertical arrows induce equivalences after tensoring the top row with R'.

5. Simplicial presheaves associated to smooth sheaves

5.1. Let $Y/\overline{K}$ be a smooth connected scheme over an algebraically closed field $\overline{K}$. Denote by $\text{FEt}(Y)$ the site whose objects are finite étale morphisms $U \to Y$ and whose coverings are surjective morphisms. The inclusion $\text{FEt}(Y) \hookrightarrow \text{Et}(Y)$ induces a morphism of the associated topoi

$$\rho : Y_{\text{et}} \longrightarrow Y_{\text{fet}}. \tag{5.1.1}$$

If $\bar{y} \to Y$ is a geometric point, then the category Y_{fet} is equivalent to the category of ind-objects in the category of sets with continuous $\pi_1(Y, \bar{y})$–action [SGA4, IV.2.7]. The pullback functor ρ^* takes such a set F to the corresponding sheaf on Y_{et}. Recall [Fa2, Chapter II] that Y is called a $K(\pi,1)$ if for any abelian sheaf $A \in Y_{\text{fet}}$ the natural map

$$H^*(Y_{\text{fet}}, A) \longrightarrow H^*(Y_{\text{et}}, \rho^* A) \tag{5.1.2}$$

is an isomorphism.

Following [Fa2], if Y/K is a smooth scheme over an arbitrary field K, we will also call Y a $K(\pi,1)$ if each connected component of the geometric fiber $Y_{\overline{K}}$ is a $K(\pi,1)$. By [Fa2, Chapter II, 2.1] (see also [Ol3, 5.4]), any point $y \in Y$ admits an open neighborhood $U \subset Y$ which is a $K(\pi,1)$.

5.2. If Y is a connected normal scheme, and $\bar\eta : \text{Spec}(\Omega) \to Y$ is a geometric generic point, there is a natural isomorphism $\pi_1(Y,\bar\eta) \simeq \text{Gal}(\Omega_Y/k(Y))$, where $\Omega_Y \subset \Omega$ denotes the maximal extension of $k(Y)$ in Ω which is unramified over all of Y. It will be useful to have a base point free description of the category of continuous $\mathbb{Q}_p$–representations of $\pi_1(Y,\bar\eta)$.

Let $\mathcal{G}_Y$ denote the category whose objects are geometric generic points $\text{Spec}(\Omega) \to Y$ with Ω a separable closure of $k(Y)$ and whose morphisms are morphisms of schemes over Y. Consider the category $\widetilde{\mathcal{D}}$ whose objects are collections $((V_{\bar\eta})_{\bar\eta\in\mathcal{G}_Y}, \{\iota_s\})$ where each $V_{\bar\eta}$ is a continuous $\mathbb{Q}_p$–representation of $\text{Gal}(k(\bar\eta)_Y/k(Y))$ and for every morphism $s : \bar\eta' \to \bar\eta$ in $\mathcal{G}_Y$ we are given an isomorphism of $\text{Gal}(k(\bar\eta')_Y/k(Y))$–representations $\iota_s : s^*V_{\bar\eta} \to V_{\bar\eta'}$. The isomorphisms ι_s are also required to satisfy the usual cocycle condition for compositions.

Fix an objet $((V_{\bar\eta})_{\bar\eta\in\mathcal{G}_Y}, \{\iota_s\}) \in \widetilde{\mathcal{D}}$ and $\bar\eta \in \mathcal{G}_Y$. Let $\rho_{\bar\eta} : \text{Gal}(k(\bar\eta)_Y/k(Y)) \to \text{Aut}(V_{\bar\eta})$ be the given action. For each element $g \in \text{Gal}(k(\bar\eta)/k(Y))$, the pullback of $V_{\bar\eta}$ via the morphism in $\mathcal{G}_Y$ induced by $g : k(\bar\eta) \to k(\bar\eta)$ is the representation ρ^g of $\text{Gal}(k(\bar\eta)_Y/k(Y))$ given by the composite

$$\text{Gal}(k(\bar\eta)_Y/k(Y)) \xrightarrow{h\mapsto g^{-1}hg} \text{Gal}(k(\bar\eta)_Y/k(Y)) \xrightarrow{\rho} \text{Aut}(V_{\bar\eta}). \tag{5.2.1}$$

By definition of $\widetilde{\mathcal{D}}$, we are given an isomorphism $\iota_g : (V_{\bar\eta}, \rho^g) \to (V_{\bar\eta}, \rho)$. Denote by $\mathcal{D} \subset \widetilde{\mathcal{D}}$ the full subcategory of objects $((V_{\bar\eta})_{\bar\eta\in\mathcal{G}_Y}, \{\iota_s\})$ such that for every $V_{\bar\eta}$ and g as above the isomorphism ι_g is given by $\rho(g) : V_{\bar\eta} \to V_{\bar\eta}$.

We will refer to the category $\mathcal{D}$ as the category of *Galois modules* on Y. If Y is not connected we define a Galois module on Y to be the data of a Galois module on each connected component. This terminology is justified by the following lemma:

Lemma 5.3. *For any geometric generic point $\bar\eta_0 : \text{Spec}(\Omega) \to Y$, the functor $(\{V_{\bar\eta}\}, \{\iota_s\}) \mapsto V_{\bar\eta_0}$ defines an equivalence between the category $\mathcal{D}$ and the category of continuous representations of* $\text{Gal}(\Omega_Y/k(Y))$.

Proof. Let F denote the functor $(\{V_{\bar\eta}\}, \{\iota_s\}) \mapsto V_{\bar\eta_0}$. For each $\bar\eta \to Y$, choose an isomorphism $\sigma_{\bar\eta} : \bar\eta \to \bar\eta_0$ over Y. Then any morphism $f : (\{V_{\bar\eta}\}, \{\iota_s\}) \to (\{V'_{\bar\eta}\}, \{\iota'_s\})$ is determined by the induced morphism $f_{\bar\eta_0} : V_{\bar\eta_0} \to V'_{\bar\eta_0}$ since the diagram

$$\begin{tikzcd} V_{\bar\eta_0} \arrow[r, "f_{\bar\eta_0}"] \arrow[d, "\sigma^*_{\bar\eta}"'] & V'_{\bar\eta_0} \arrow[d, "\sigma^*_{\bar\eta}"] \\ V_{\bar\eta} \arrow[r, "f_{\bar\eta}"] & V'_{\bar\eta} \end{tikzcd} \tag{5.3.1}$$

commutes. Therefore F is faithful. In fact given $f_{\bar\eta_0}$, use 5.3.1 to define $f_{\bar\eta}$. Then $f_{\bar\eta}$ is independent of the choice of $\sigma_{\bar\eta} : \bar\eta \to \bar\eta_0$. For if $\sigma'_{\bar\eta} : \bar\eta \to \bar\eta_0$ is a second isomorphism, there exists a unique element $g \in \mathrm{Gal}(k(\bar\eta)/k(Y))$ such that $\sigma'_{\bar\eta} = \sigma_{\bar\eta} \circ g^*$. If $f'_{\bar\eta}$ denotes the morphism obtained from $\sigma'_{\bar\eta}$, the fact that $(\{V_{\bar\eta}\}, \{\iota_s\})$ is in $\mathcal{D}$ implies that $f'_{\bar\eta}$ is equal to the composite

$$V_{\bar\eta} \xrightarrow{\rho(g^{-1})} V_{\bar\eta} \xrightarrow{f_{\bar\eta}} V'_{\bar\eta} \xrightarrow{\rho'(g)} V'_{\bar\eta}. \tag{5.3.2}$$

Since $f_{\bar\eta}$ commutes with the action of $\mathrm{Gal}(k(\bar\eta)/k(Y))$, it follows that $f'_{\bar\eta} = f_{\bar\eta}$. This implies that the functor F is fully faithful.

To see that F is essentially surjective, fix a representation $V_{\bar\eta_0}$ and choose isomorphisms $\sigma_{\bar\eta} : \bar\eta \to \bar\eta_0$ as above. For each $\bar\eta$, set $V_{\bar\eta} := \sigma^*_{\bar\eta} V_{\bar\eta_0}$. For any morphism $s : \bar\eta \to \bar\eta'$, there exists a unique element $g \in \mathrm{Gal}(k(\bar\eta')/k(Y))$ such that $\sigma_{\bar\eta'} = \sigma_{\bar\eta} \circ s \circ g^*$. Define ι_s to be the isomorphism

$$s^*\sigma^*_{\bar\eta} V_{\bar\eta_0} \xrightarrow{\rho(g^{-1})} g^* s^* \sigma^*_{\bar\eta} V_{\bar\eta_0}, \tag{5.3.3}$$

where ρ denotes the action of $\mathrm{Gal}(k(\bar\eta')/k(Y))$. The value of F on the resulting object $(\{V_{\bar\eta}\}, \{\iota_s\}) \in \mathcal{D}$ is then equal to $V_{\bar\eta_0}$. □

Remark 5.4. The category $\mathcal{D}$ is Tannakian with fiber functor $(\{V_{\bar\eta}\}, \{\iota_s\}) \mapsto V_{\bar\eta_0}$.

5.5. Let $Y/\overline{K}$ be a normal connected scheme of finite type over an algebraically closed field $\overline{K}$, and let L be a differential graded algebra in the ind–category of Galois modules on Y. We define Galois cohomology of L as follows. Choose a geometric point $\bar\eta \to Y$ mapping to the generic point and write π for the group $\pi_1(Y, \bar\eta)$. In this paragraph we view L as a representation of π and denote it simply by L. For each $n \geq 0$, let $\mathcal{C}^n(\pi, L)$ denote the differential graded algebra with

$$\mathcal{C}^n(\pi, L)_r := \{\text{continuous } \pi\text{–equivariant maps } \pi^{n+1} \to L_r\}, \tag{5.5.1}$$

where π acts on the left of π^{n+1} via the diagonal action. If $\delta : [m] \to [n]$ is a morphism in the simplicial category Δ, then there is an induced π–equivariant map

$$\pi^{n+1} \longrightarrow \pi^{m+1}, \quad (a_0, \dots, a_n) \mapsto (a_{\delta(0)}, \dots, a_{\delta(m)}), \tag{5.5.2}$$

which induces a morphism of differential graded algebras

$$\delta^* : \mathcal{C}^m(\pi, L) \longrightarrow \mathcal{C}^n(\pi, L). \tag{5.5.3}$$

These maps are compatible with composition in Δ, and hence we obtain a cosimplicial differential graded algebra $\mathcal{C}^\bullet(\pi, L) \in \mathrm{dga}^{\Delta}_{\mathbb{Q}_p}$. Applying the functor of Thom–Sullivan cochains we obtain a differential graded algebra $T(\mathcal{C}^\bullet(\pi, L)) \in \mathrm{dga}_{\mathbb{Q}_p}$. This construction is functorial in L, so in particular if the sheaf L comes equipped with an action of a group scheme $G/\mathbb{Q}_p$ then the resulting algebra $T(\mathcal{C}^\bullet(\pi, L))$ is naturally an object of $G - \mathrm{dga}^{\Delta}_{\mathbb{Q}_p}$.

Remark 5.6. In the definition of $\mathcal{C}^n(\pi, L)_r$ above, when L_r is infinite dimensional the set of continuous maps $\pi^{n+1} \to L_r$ should be interpreted as the set of continuous maps from π^{n+1} to finite dimensional subspaces of L_r.

5.7. For technical reasons, it will be important to have a base point free description of this Galois cohomology. For this let $Y/\overline{K}$ be as above and let $E \to Y$ be a finite disjoint union of geometric points mapping to the generic point of Y. For simplicity in what follows we will always assume that a geometric generic point $\mathrm{Spec}(\Omega) \to Y$ has the property that $k(Y) \to \Omega$ is a separable closure of $k(Y)$ (i.e. no transcendental part). For a smooth sheaf L on Y we define a cosimplicial module denoted $\mathcal{C}^\bullet(Y, E, L)$ as follows.

Note first that the scheme E is isomorphic to $\coprod_{p\in E} \mathrm{Spec}(k(p))$ where $k(p)$ is a separable closure of $k(Y)$. Define $k_Y(p) \subset k(p)$ to be the maximal extension of $k(Y)$ in $k(p)$ which is unramified over Y, and set $E_Y := \coprod_{p\in E} \mathrm{Spec}(k_Y(p))$. Let $E_Y^\bullet$ be the 0–coskeleton of the morphism $E_Y \to Y$. Note that each E_Y^n is non–canonically isomorphic to a disjoint union of copies of $\mathrm{Spec}(k_Y(p))$. In particular, for each point $q \in E_Y^n$ we can form the stalk L_q. Define $\mathcal{C}^n(Y, E, L) \subset \prod_{q\in E_Y^n} L_q$ as follows.

Fix a separable closure Ω of $k(Y)$ and let $\Omega_Y \subset \Omega$ be the maximal subextension unramified over Y. Choose an isomorphism $\iota : E \simeq \coprod_{i=0}^m \mathrm{Spec}(\Omega)$. This choice of isomorphism induces an isomorphism

$$E_Y^n \simeq \coprod_{\mathrm{Fun}([n],[m])} \mathrm{Spec}(\Omega_Y^{\otimes(n+1)}), \tag{5.7.1}$$

where $\Omega_Y^{\otimes(n+1)}$ denotes the $(n+1)$–fold tensor product of Ω_Y over $k(Y)$ and $\mathrm{Fun}([n], [m])$ denotes the set of all functions $[n] \to [m]$ (not necessarily order preserving). Let π denote the Galois group $\mathrm{Gal}(\Omega_Y/k(Y))$. There is a natural map

$$\coprod_{\pi^{n+1}} \mathrm{Spec}(\Omega_Y) \longrightarrow \mathrm{Spec}(\Omega_Y^{\otimes(n+1)}) \tag{5.7.2}$$

which on the $(g_0, \ldots, g_n)$–th component is given by

$$\Omega_Y \otimes_{k(Y)} \Omega_Y \cdots \Omega_Y \otimes_{k(Y)} \Omega_Y \xrightarrow{(g_0,\ldots,g_n)} \Omega_Y. \tag{5.7.3}$$

There is an action of π on $\coprod_{\pi^{n+1}} \mathrm{Spec}(\Omega_Y)$ for which $\gamma \in \pi$ sends the $(g_0, \ldots, g_n)$-component to the $(\gamma g_0, \ldots, \gamma g_n)$-component via the map

$$\gamma : \Omega_Y \to \Omega_Y. \tag{5.7.4}$$

Note that the diagram

$$\begin{tikzcd} \Omega_Y^{\otimes(n+1)} \arrow[r, "(g_0,\ldots,g_n)"] \arrow[dr, "(\gamma g_0,\ldots,\gamma g_n)"'] & \Omega_Y \arrow[d, "\gamma"] \\ & \Omega_Y \end{tikzcd} \tag{5.7.5}$$

commutes, so we have an action of π on $\coprod_{\pi^{n+1}} \mathrm{Spec}(\Omega_Y)$ over $\mathrm{Spec}(\Omega_Y^{\otimes(n+1)})$. Furthermore the map 5.7.3 induces an isomorphism

$$[\coprod_{\pi^{n+1}} \mathrm{Spec}(\Omega_Y)/\pi] \simeq \mathrm{Spec}(\Omega_Y^{\otimes(n+1)}). \tag{5.7.6}$$

It follows that ι induces an isomorphism

$$E_Y^n \simeq \coprod_{\mathrm{Fun}([n],[m])} [\coprod_{\pi^{n+1}} \mathrm{Spec}(\Omega_Y)/\pi]. \tag{5.7.7}$$

With these identifications, the set $\prod_{q\in E_Y^n} L_q$ becomes identified with the set

$$\prod_{\mathrm{Fun}([n],[m])} \mathrm{Hom}_{\pi-\mathrm{equivariant}}(\pi^{n+1}, L_\Omega), \tag{5.7.8}$$

and we define $\mathcal{C}^n(Y, E, L)$ to be the subset

$$\prod_{\mathrm{Fun}([n],[m])} \mathrm{Hom}^{\mathrm{cts}}_{\pi-\mathrm{equivariant}}(\pi^{n+1}, L_\Omega). \tag{5.7.9}$$

Lemma 5.8. *The subset $\mathcal{C}^n(Y, E, L) \subset \prod_{q\in E_Y^n} L_q$ defined above is independent of the choice of ι.*

Proof. Any other isomorphism $\iota' : E \simeq \coprod_{i=0}^m \mathrm{Spec}(\Omega)$ is obtained from ι by composing with an automorphism α of $\coprod_{i=0}^m \mathrm{Spec}(\Omega)$ over $k(Y)$. It therefore suffices to verify that $\mathcal{C}^n(Y, E, L)$ is invariant under automorphisms of $\prod_{q\in E_Y^n} L_q$ induced by automorphisms of $\coprod_{i=0}^m \mathrm{Spec}(\Omega)$. Any such automorphism is the composite of a permutation of $[m]$ with the automorphism induced by a sequence $(h_0, \ldots, h_m) \in \pi^{m+1}$ acting on the i–th component $\mathrm{Spec}(\Omega)$ by h_i^*. Hence it suffices to consider each of these two kinds of automorphisms in turn.

The action of a permutation τ of $[m]$ is simply that induced by composing a function $[n] \to [m]$ with τ. Hence the automorphisms obtained in this way preserve $\mathcal{C}^n(Y, E, L)$.

The action of $(h_0, \ldots, h_m) \in \pi^{m+1}$ is induced by the action on

$$\coprod_{(f,g_0,\ldots,g_n)\in \mathrm{Fun}([n],[m])\times\pi^{n+1}} \mathrm{Spec}(\Omega_Y) \tag{5.8.1}$$

sending the $(f, g_0, \ldots, g_n)$–th component to the $(f, g_0h_{f(0)}, \ldots, g_nh_{f(n)})$–th component. From this description it follows that $\mathcal{C}^n(Y, E, L)$ is preserved. $\square$

5.9. Observe that for any morphism $E' \to E$, there is a natural induced map $\mathcal{C}^\bullet(Y, E, L) \to \mathcal{C}^\bullet(Y, E', L)$. We claim that this map is an equivalence. For this it suffices to consider the case when $E' = \mathrm{Spec}(\Omega)$ is a single geometric generic point and $E = \coprod_{i=0}^m \mathrm{Spec}(\Omega)$. But in this case it follows from the construction that

$$\mathcal{C}^n(Y, E, L) = \mathcal{C}^n(Y, E', L)^{\mathrm{Fun}([n],[m])}. \tag{5.9.1}$$

Thus the result follows from the following lemma:

Lemma 5.10. *Let $A_\bullet$ be a cosimplicial differential graded $\mathbb{Q}$–algebra and fix $m \geq 0$ and a map $h : [0] \to [m]$. Denote by $A_\bullet^{\mathrm{Fun}(\bullet,[m])}$ the cosimplicial algebra $[n] \mapsto A_n^{\mathrm{Fun}([n],[m])}$. Then the map*

$$A_\bullet \simeq A_\bullet^{\mathrm{Fun}(\bullet,[0])} \xrightarrow{h^*} A_\bullet^{\mathrm{Fun}(\bullet,[m])} \tag{5.10.1}$$

induces a quasi–isomorphism on the associated normalized complexes.

Proof. If the result holds for $A_\bullet$ equal to the constant cosimplicial ring $\mathbb{Q}$, then the result holds in general by tensoring with $A_\bullet$. So it suffices to consider $A_\bullet = \mathbb{Q}$. In this case, the algebra $[n] \mapsto \mathbb{Q}^{\mathrm{Fun}([n],[m])}$ is simply the cosimplicial algebra computing the cohomology of the constant sheaf $\mathbb{Q}$ on the punctual topos with respect to the 0-coskeleton of the covering

$$\coprod_{i \in [m]} * \longrightarrow *. \tag{5.10.2}$$

□

Remark 5.11. If $E = \mathrm{Spec}(\Omega)$ consists just of a single geometric generic point, then it follows from the construction that the above $\mathcal{C}^\bullet(Y, E, L)$ is canonically isomorphic to the cosimplicial module $\mathcal{C}^\bullet(\pi, L)$ defined in 5.5.

5.12. If Y is not connected and $Y = \coprod_i Y_i$ is the decomposition into connected components, we modify the above definition of $\mathcal{C}^\bullet(Y, E, L)$ as follows. We consider a morphism $h : E \to Y$, where E is a disjoint union of geometric points mapping to generic points of Y such that for every $e \in E$ the field extension $k(h(e)) \to k(e)$ is a separable closure of $k(h(e))$. Let $E_i \subset E$ be the subscheme of points mapping to Y_i so that $E = \coprod_i E_i$, and set $\mathcal{C}^\bullet(Y, E, L) := \oplus_i \mathcal{C}^\bullet(Y_i, E_i, L|_{Y_i})$.

5.13. In order to deal with base points, we also need functoriality of the above construction with respect to the inclusion of a point $y \in Y(\overline{K})$ for $Y/\overline{K}$ connected and normal. For a family of geometric generic points $E \to Y$, let $\widetilde{Y}_E \to Y$ be the normalization of Y in E_Y (notation as in 5.7), and let $\widetilde{Y}_{E,y}$ be the pullback to $y = \mathrm{Spec}(\overline{K})$. There is a natural decomposition $\widetilde{Y}_E = \coprod_{p \in E} \widetilde{Y}_{k(p)}$, where $\widetilde{Y}_{k(p)}$ denotes the normalization of Y in $\mathrm{Spec}(k(p)) \to Y$, and hence also a decomposition $\widetilde{Y}_{E,y} = \coprod_{p \in E} \widetilde{Y}_{k(p),y}$. The projection to $\mathrm{Spec}(\overline{K})$ therefore factors through a morphism $\widetilde{Y}_{E,y} \to \coprod_{p \in E} \mathrm{Spec}(\overline{K})$. Define *specialization data for E relative to y* to be a section s of this map.

The choice of specialization data s determines for each $p \in E$ an isomorphism of stalks $L_p \simeq L_y$ for any smooth $\mathbb{Q}_p$–sheaf L on Y. For this write $L = (\varprojlim L_n) \otimes \mathbb{Q}$ for some locally constant sheaves L_n of $\mathbb{Z}/(p^n)$–modules. To obtain the above isomorphism it suffices to construct a canonical isomorphism of stalks $L_{n,p} \simeq L_{n,y}$ for each n. Since L_n is representable by a finite étale morphism $U_n \to Y$, this in turn amounts to showing that if $U \to Y$ is a finite étale morphism, then the specialization data s determines a canonical bijection $U_p \simeq U_y$. For this note that there is a canonical isomorphism

$$U \times_Y \widetilde{Y}_{k(p)} \simeq \coprod_{t \in U_p} \widetilde{Y}_{k(p)}, \tag{5.13.1}$$

and hence pulling back via the map $s : \mathrm{Spec}(\overline{K}) \to \widetilde{Y}_{k(p)}$ we obtain an isomorphism

$$U_y \simeq \coprod_{t \in U_y} \mathrm{Spec}(\overline{K}) \to \coprod_{t \in U_p} \mathrm{Spec}(\overline{K}). \tag{5.13.2}$$

This gives the desired isomorphism of stalks.

Remark 5.14. Note that in order to obtain the isomorphism of stalks $L_p \simeq L_y$ we can replace E_Y in the above by the disjoint union of spectra of subfields $\Omega'_p \subset k(p)$ unramified over Y such that the action of $\pi_1(Y, p)$ on L_p factors through the Galois group of Ω'_p.

5.15. If L is a differential graded algebra (possibly G–equivariant for some pro–algebraic group scheme G) in the category of ind–smooth $\mathbb{Q}_p$–sheaves, and s is a choice of specialization data for $E \to Y$ relative to a point $y \in Y(\overline{K})$, then the cosimplicial differential graded algebra $\mathcal{C}^\bullet(Y, E, L)$ admits a natural augmentation to an algebra equivalent to L_y. For this let $|E|$ denote the underlying set of E and for any $[m] \in \Delta$ let $\mathrm{Fun}([m], |E|)$ be the set of functions $[m] \to |E|$. We construct an augmentation $\mathcal{C}^\bullet(Y, E, L) \to L_y^{\mathrm{Fun}(\bullet, |E|)}$ as follows.

Let E^n_Y be as in 5.7, so that $\mathcal{C}^n(Y, E, L)$ is a certain subset of $\prod_{p \in E^n_Y} L_p$. It suffices to construct a natural map

$$\prod_{p \in E^n_Y} L_p \longrightarrow \prod_{\mathrm{Fun}([n], |E|)} L_y. \tag{5.15.1}$$

For this note that the projection $\widetilde{Y}_{E,y} \to \coprod_{|E|} \mathrm{Spec}(\overline{K})$ induces a projection

$$\widetilde{Y}^n_{E,y} \longrightarrow \coprod_{\mathrm{Fun}([n], |E|)} \mathrm{Spec}(\overline{K}), \tag{5.15.2}$$

and the specialization data s induces a section s_n of this projection. On the other hand, $\widetilde{Y}_E \to Y$ is a projective limit of finite étale morphisms. From this and [SGAI, I.10.5] it follows that $\widetilde{Y}^n_E$ is isomorphic to the normalization of Y in E^n_Y. In particular, $\widetilde{Y}^n_E$ is a disjoint union of connected components indexed by $|E^n_Y|$. We therefore obtain a map $\lambda : \mathrm{Fun}([n], |E|) \to |E^n_Y|$, and the discussion in 5.13 furnishes for every $p \in \mathrm{Fun}([n], |E|)$ an isomorphism $L_{\lambda(p)} \simeq L_p$. In this way we obtain 5.15.1.

Remark 5.16. Observe that any two choices s and s' of specialization data relative to y differ by the choice for each $p \in E$ of an automorphism of $\mathrm{Spec}(k(p))$ over X. This implies in particular that the augmentation of the preceding paragraph is canonical up to the automorphism of $\mathcal{C}^\bullet(Y, E)$ obtained from an automorphism of E over Y.

5.17. Let X/K be a smooth quasi–compact scheme over a field K, and let $K \hookrightarrow \overline{K}$ be an algebraic closure. Let L be a differential graded algebra in the category of ind–smooth sheaves on X_{et}, and let $L_{\overline{K}}$ denotes its restriction to $X_{\overline{K}}$. The above discussion enables us to compute the étale cohomology of $L_{\overline{K}}$ using group cohomology as follows.

Lemma 5.18. *Let $\mathcal{U} \subset \mathrm{Et}(X)$ be a full subcategory closed under products and fiber products of the category of étale X–schemes such that every étale $U \to X$ admits a covering by an object of $\mathcal{U}$. Then for any étale hypercover $U_\bullet \to X$, there exists a hypercover $U'_\bullet \to X$ with each $U'_n \in \mathcal{U}$ and a morphism $U'_\bullet \to U_\bullet$ over X.*

Proof. This is a standard application of the construction of simplicial spaces in [SGA4, $\mathrm{V}^{\mathrm{bis}}$, §5]. □

5.19. Fix a finite set of geometric generic points $E \to X_{\overline{K}}$ whose image meets every connected component of $X_{\overline{K}}$. If $U \to X$ is an étale scheme there is a natural family of geometric generic points $E_U \to U_{\overline{K}}$ over $E \to X_{\overline{K}}$ given by

$$\coprod_{\mathrm{Hom}_X(E,U)} E \to U_{\overline{K}}. \tag{5.19.1}$$

Moreover, for any morphism $U' \to U$ over X there is a natural commutative diagram

$$\begin{array}{ccc} \coprod_{\mathrm{Hom}_X(E,U')} E & \longrightarrow & U'_{\overline{K}} \\ \downarrow & & \downarrow \\ \coprod_{\mathrm{Hom}_X(E,U)} E & \longrightarrow & U_{\overline{K}} \\ \downarrow & & \downarrow \\ E & \longrightarrow & X. \end{array} \tag{5.19.2}$$

Remark 5.20. Specialization data s for $E \to X_{\overline{K}}$ induces for each point $u \in U(\overline{K})$ lying over x and $q \in E_U$ an isomorphism $L_q \simeq L_u$.

5.21. By [Fa2, Chapter II, 2.1] (see also [Ol3, 5.4]) and 5.18, there exists an étale hypercover $U_\bullet \to X$ with each U_n a $K(\pi, 1)$. For each n, let $GC(U_{n,\overline{K}}, E, L)$ denote the differential graded algebra obtained from $\mathcal{C}^\bullet(U_{n,\overline{K}}, E_U, L_{\overline{K}})$ by applying the functor of Thom–Sullivan cochains. By functoriality, the algebras $GC(U_{n,\overline{K}}, E, L)$ form in a natural way a cosimplicial differential graded algebra $GC(U_{\bullet,\overline{K}}, E, L)$

$$[n] \mapsto GC(U_{n,\overline{K}}, E, L). \tag{5.21.1}$$

Denote by $GC(L, E)_{U_\bullet} \in \mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}_p})$ the algebra obtained from $GC(U_{\bullet,\overline{K}}, E, L)$ by applying the functor of Thom–Sullivan cochains. If L is a G–equivariant algebra for some affine group scheme $G/\mathbb{Q}_p$ then $GC(L, E)_{U_\bullet}$ is naturally an object of $\mathrm{Ho}(G - \mathrm{dga}_{\mathbb{Q}_p})$.

If furthermore $x \in X(\overline{K})$ is a point and $L_x \to \mathbb{Q}_p$ is a map in $\mathrm{dga}_{\mathbb{Q}_p}$ then the construction of 5.15 and the observation 5.20 gives a map of objects of $\mathrm{dga}_{\mathbb{Q}_p}^{\Delta\times\Delta}$

$$([n] \mapsto \mathcal{C}^\bullet(U_{n,\overline{K}}, E, L)) \to ([n] \mapsto \prod_{u \in U_{n,x}} L_u^{\mathrm{Fun}(\bullet, |E_{U_n}|)}). \tag{5.21.2}$$

For each n and $u \in U_{n,x}$ the map $L_u \to L_u^{\mathrm{Fun}(\bullet,|E_{U_n}|)}$ is an equivalence by 5.10, and hence the right hand side of 5.21.2) is equivalent to the algebra $[n] \mapsto \prod_{u\in U_{n,x}} L_u$. This algebra is in turn equivalent to the algebra $[n] \mapsto L_x^{|U_{n,x}|}$ which since $U_{\bullet,x} \to x$ is a hypercovering is equivalent to the constant algebra L_x. By the reasoning of 4.34 the augmentation $L_x \to \mathbb{Q}_p$ therefore gives $GC(L, E)_{U_\bullet}$ the structure of an object in the homotopy category $\mathrm{Ho}(G - \mathrm{dga}_{\mathbb{Q}_p,/\mathcal{O}_G})$ of algebras over $\mathcal{O}_G$.

5.22. If $\sigma : U'_\bullet \to U_\bullet$ is a morphism of hypercovers of X with each U_n and U'_n a $K(\pi, 1)$, then there is a natural induced map

$$\sigma^* : GC(L, E)_{U_\bullet} \longrightarrow GC(L, E)_{U'_\bullet}. \tag{5.22.1}$$

Since the cohomology groups of both sides are equal to the étale cohomology of L, it follows that this morphism is an equivalence. In particular, for any two choices of hypercovers $U_\bullet$ and $U'_\bullet$ we obtain a canonical isomorphism $GC(L, E)_{U_\bullet} \simeq GC(L, E)_{U'_\bullet}$ in $\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}_p})$ from the diagram of equivalences

$$GC(L, E)_{U_\bullet} \xrightarrow{\mathrm{pr}_1^*} GC(L, E)_{U_\bullet\times_X U'_\bullet} \xleftarrow{\mathrm{pr}_2^*} GC(L, E)_{U'_\bullet}. \tag{5.22.2}$$

If there exists a morphism of hypercovers $\sigma : U'_\bullet \to U_\bullet$, then the morphism $(\mathrm{pr}_2^*)^{-1} \circ \mathrm{pr}_1^*$ in $\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}})$ is equal to σ^*. For this consider the diagonal $\Gamma_\sigma : U'_\bullet \to U_\bullet \times_X U'_\bullet$ so that

$\sigma^* = \Gamma_\sigma^* \circ \mathrm{pr}_1^*$. Since $\mathrm{pr}_2 \circ \Gamma_\sigma = \mathrm{id}$, it follows that $\Gamma_\sigma^* = (\mathrm{pr}_2^*)^{-1}$ and hence $\sigma^* = \Gamma_\sigma^* \circ \mathrm{pr}_1^* = (\mathrm{pr}_2^*)^{-1} \circ \mathrm{pr}_1^*$.

Since $GC(L,E)_{U_\bullet}$ is up to canonical isomorphism independent of the choice of $U_\bullet$, in what follows we will usually simply write $GC(L,E)$ for $GC(L,E)_{U_\bullet}$.

5.23. The algebra $GC(L,E)$ is also functorial in E. For any morphism $E' \to E$ of geometric generic points over $X_{\overline{K}}$ there is an induced equivalence

$$GC(L,E) \longrightarrow GC(L,E'). \tag{5.23.1}$$

This implies that $GC(L,E)$ and $GC(L,E')$ are canonically isomorphic by the isomorphism in $\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}_p})$ (or $\mathrm{Ho}(G-\mathrm{dga}_{\mathbb{Q}_p})$) obtained from

$$GC(L,E) \longrightarrow GC(L,E\times_{X_{\overline{K}}} E') \longleftarrow GC(L,E'). \tag{5.23.2}$$

Here we abuse notation as $E\times_{X_{\overline{K}}} E'$ is an infinite disjoint union of geometric generic points. To deal with this write $E\times_{X_{\overline{K}}} E' = \varprojlim \widetilde{E}_i$ with each $\widetilde{E}_i$ a finite set collection of geometric generic points and define $GC(L,E\times_{X_{\overline{K}}} E')$ to be the direct limit of the $GC(L,\widetilde{E}_i)$ (note that by the construction everything here can be represented by actual complexes).

By the same reasoning as in 5.22, when there exists a morphism $E' \to E$ over $X_{\overline{K}}$, then the morphism 5.23.1 is equal to the morphism obtained from 5.23.2 in $\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}_p})$ (or $\mathrm{Ho}(G-\mathrm{dga}_{\mathbb{Q}_p})$ if L is a G–equivariant sheaf).

5.24. If $\sigma : E \to E$ is an automorphism over $X_{\overline{K}}$, then the induced morphism

$$\sigma^* : GC(L,E) \longrightarrow GC(L,E) \tag{5.24.1}$$

in $\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}_p})$ (or $\mathrm{Ho}(G-\mathrm{dga}_{\mathbb{Q}_p})$ if L is a G–equivariant algebra) is the identity. To see this, choose a quasi–compact subscheme $E' \subset E\times_X E$ containing the diagonal $\Delta \subset E\times_X E$ and the graph Γ_σ of σ. We then have maps

$$\Delta, \Gamma_\sigma : E \to E', \quad \mathrm{pr}_1, \mathrm{pr}_2 : E' \to E \tag{5.24.2}$$

such that $\mathrm{pr}_i \circ \Delta = \mathrm{id}$, $\mathrm{pr}_1 \circ \Gamma_\sigma = \mathrm{id}$, and $\mathrm{pr}_2 \circ \Gamma_\sigma = \sigma$. Since the induced maps

$$\Delta^*, \Gamma_\sigma^* : GC(L,E') \to GC(L,E), \quad \mathrm{pr}_1^*, \mathrm{pr}_2^* : GC(L,E) \to GC(L,E') \tag{5.24.3}$$

are equivalences and $\mathrm{pr}_i \circ \Delta = \mathrm{id}$ for $i = 1,2$ it follows that the maps pr_i^* induce the same map in $\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}_p})$ (namely the inverse of Δ^*). From this we deduce that

$$\sigma^* = \Gamma_\sigma^* \circ \mathrm{pr}_2^* = \Gamma_\sigma^* \circ \mathrm{pr}_1^* = \mathrm{id}^*. \tag{5.24.4}$$

5.25. The same reasoning combined with 5.16 shows that when X has a point $x \in X(K)$ and L_x has an augmentation $L_x \to \mathbb{Q}_p$, then the structure on $GC(L,E)$ of an object in $\mathrm{Ho}(G-\mathrm{dga}_{\mathbb{Q}_p,/\mathcal{O}_G})$ constructed in 5.21 is independent of all the choices.

5.26. If $E = \mathrm{Spec}(\Omega)$ consists of just a single point (so in particular X is geometrically connected), then the preceding discussion implies that there is a natural action of $\mathrm{Gal}(\overline{K}/K)$ on $GC(L,E)$. For this observe that since X is geometrically connected, the structure morphism $X \to \mathrm{Spec}(K)$ induces a surjection $\mathrm{Aut}(\mathrm{Spec}(\Omega)/X) \to \mathrm{Gal}(\overline{K}/K)$ whose kernel is

$\mathrm{Aut}(\mathrm{Spec}(\Omega)/X_{\overline{K}})$. Any $\sigma \in \mathrm{Aut}(\mathrm{Spec}(\Omega)/X)$ with image $\bar{\sigma} \in \mathrm{Gal}(\overline{K}/K)$ induces a commutative square

$$\begin{array}{ccc} \mathrm{Spec}(\Omega) & \xrightarrow{\sigma} & \mathrm{Spec}(\Omega) \\ \downarrow & & \downarrow \\ X_{\overline{K}} & \xrightarrow{\bar{\sigma}} & X_{\overline{K}}, \end{array} \tag{5.26.1}$$

and hence an automorphism $\sigma^* : GC(L,E) \to GC(L,E)$ in $\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}_p})$. This gives an action of $\mathrm{Aut}(\mathrm{Spec}(\Omega)/X)$ on $GC(L,E)$ which by 5.24 factors through $\mathrm{Gal}(\overline{K}/K)$.

Remark 5.27. In the above we have written $\mathrm{Aut}(\mathrm{Spec}(\Omega)/X)$ instead of $\mathrm{Gal}(\Omega/k(X))$ so that the preceding discussion also applies to Deligne–Mumford stacks.

5.28. We can use this construction to associate a pointed stack to a smooth sheaf L on a smooth geometrically connected pointed scheme $(X, x \in X(K))$ over a field K (see also [To1, 3.5.3]). Let $\widetilde{\langle L_{\overline{K}}\rangle}_{\otimes}$ denote the smallest Tannakian subcategory of the category of smooth sheaves on $X_{\overline{K}}$ which is closed under extensions and contains $L_{\overline{K}}$.

The point x defines a point $\bar{x} = \mathrm{Spec}(\overline{K}) \to X_{\overline{K}}$ which defines a fiber functor $H \mapsto H_{\bar{x}}$ for the category $\widetilde{\langle L_{\overline{K}}\rangle}_{\otimes}$. Let $\langle L_{\overline{K}}\rangle_{\otimes}$ denotes the Tannakian subcategory of the category of smooth sheaves on $X_{\overline{K}}$ generated by $L_{\overline{K}}$.

Assumption 5.29. *Assume that the group $G := \pi_1(\langle L_{\overline{K}}\rangle_{\otimes}, \bar{x})$ is reductive.*

Remark 5.30. One can often reduce to the case when G is reductive as follows. Let $\mathcal{U} \subset \pi_1(\langle L_{\overline{K}}\rangle_{\otimes}, \bar{x})$ be the unipotent radical. The action of $\mathcal{U}$ induces a canonical filtration $\mathrm{Fil}^{\cdot}$ on $L_{\overline{K}}$ such that for every i the group $\pi_1(\langle \mathrm{gr}^i_{\mathrm{Fil}}(L_{\overline{K}})\rangle_{\otimes}, \bar{x})$ is reductive. The point $\mathrm{Spec}(K) \to X$ induces a section of $\pi_1(X_K, \bar{x}) \to \mathrm{Gal}(\overline{K}/K)$ which identifies $\pi_1(X_K, \bar{x})$ with the semi–direct product $\pi_1(X_{\overline{K}}, \bar{x}) \rtimes \mathrm{Gal}(\overline{K}/K)$. By the uniqueness of the unipotent radical, the action of $\mathrm{Gal}(\overline{K}/K)$ preserves $\mathcal{U}$ and hence the filtration descends to a filtration on L. Let L' denote the associated graded of this filtration. Then L' satisfies 5.29 and $\widetilde{\langle L_{\overline{K}}\rangle}_{\otimes} = \widetilde{\langle L'_{\overline{K}}\rangle}_{\otimes}$.

5.31. Right translation induces a left action of $\pi_1(\widetilde{\langle L_{\overline{K}}\rangle}_{\otimes}, \bar{x})$ on the coordinate ring $\mathcal{O}_G$ which by Tannaka duality corresponds to a differential graded algebra $\mathbb{V}(\mathcal{O}_G)$ in the category of ind–smooth sheaves on $X_{\overline{K}}$. Furthermore, left translation induces a right action of G on $\mathcal{O}_G$ which commutes with the left action and hence induces a right action of G on $\mathbb{V}(\mathcal{O}_G)$.

Applying the construction of 5.17, we obtain a G–equivariant differential graded algebra $\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_G)) \in G - \mathrm{dga}_{\mathbb{Q}_p}$. Since $\mathbb{V}(\mathcal{O}_G)_{\bar{x}} \simeq \mathcal{O}_G$ has a natural map to $\mathbb{Q}_p$, $\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_G))$ is naturally an object in $\mathrm{Ho}(G - \mathrm{dga}_{\mathbb{Q}_p,/\mathcal{O}_G})$. We define

$$X_{\widetilde{\langle L_{\overline{K}}\rangle}_{\otimes}} := [\mathbb{R}\mathrm{Spec}(\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_G)))/G] \in \mathrm{Ho}(\mathrm{SPr}_*(\mathbb{Q}_p)). \tag{5.31.1}$$

Because L is defined over K and not $\overline{K}$, the functoriality of the above construction induces a natural $\mathrm{Gal}(\overline{K}/K)$–action on $X_{\widetilde{\langle L_{\overline{K}}\rangle}_{\otimes}}$.

Another way to view this action is to note that the sheaf $\mathbb{V}(\mathcal{O}_G)$ is naturally an algebra in the category of ind–objects of smooth sheaves on X_K. For this observe that the section $x : \mathrm{Spec}(K) \to X$ induces a section of the natural projection $\pi_1(X_K, \bar{x}) \to \mathrm{Gal}(\overline{K}/K)$. It follows that there is an isomorphism $\pi_1(X_K, \bar{x}) \simeq \pi_1(X_{\overline{K}}, \bar{x}) \rtimes \mathrm{Gal}(\overline{K}/K)$, where $\mathrm{Gal}(\overline{K}/K)$ acts on $\pi_1(X_{\overline{K}}, \bar{x})$ by the natural action on $X_{\overline{K}}$. For any $\sigma \in \mathrm{Gal}(\overline{K}/K)$, there is a natural commutative square

$$\begin{array}{ccc} \mathrm{Spec}(\overline{K}) & \xrightarrow{\sigma^*} & \mathrm{Spec}(\overline{K}) \\ {\scriptstyle \bar{x}}\downarrow & & \downarrow{\scriptstyle \bar{x}} \\ X_{\overline{K}} & \xrightarrow{\sigma^*} & X_{\overline{K}}. \end{array} \tag{5.31.2}$$

Since $\sigma^* L_{\overline{K}} \simeq L_{\overline{K}}$, pullback by σ induces an auto–equivalence $\sigma^* : \langle L_{\overline{K}}\rangle_\otimes \to \langle L_{\overline{K}}\rangle_\otimes$ and hence also an automorphism $\sigma^* : G \to G$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\sigma^*} & G \\ \downarrow & & \downarrow \\ \mathrm{Aut}(L_{\bar{x}}) & \xrightarrow{\tau_\sigma} & \mathrm{Aut}(L_{\bar{x}}) \end{array} \tag{5.31.3}$$

commutes, where τ_σ denotes conjugation by the action of σ on $L_{\bar{x}}$.

Let λ be the action of $\pi_1(X_K, \bar{x}) \simeq \pi_1(X_{\overline{K}}, \bar{x}) \rtimes \mathrm{Gal}(\overline{K}/K)$ on $\mathrm{Aut}(L_{\bar{x}})$ in which an element (g, σ) sends $A \in \mathrm{Aut}(L_{\bar{x}})$ to $\tau_{\sigma^{-1}}(A\rho(g))$. Then it follows from the above that λ induces a right action of $\pi_1(X_K, \bar{x})$ on G, and hence also on $\mathcal{O}_G$, whose restriction to $\pi_1(X_{\overline{K}}, \bar{x})$ is that induced by right translation. This action then induces a model for $\mathbb{V}(\mathcal{O}_G)$ over $X_{K,\mathrm{et}}$.

More explicitly, let $\mathcal{S}_{\mathrm{et}}$ denote the $\pi_1(X_K, \bar{x})$–module $(\mathrm{Sym}^\bullet L_{\bar{x}} \otimes L_{\bar{x}}^*)_{\det}$, where $\pi_1(X_K, \bar{x})$ acts on $L_{\bar{x}}$ via the natural identification of smooth sheaves on X_K with $\pi_1(X_K, \bar{x})$–modules and on $L_{\bar{x}}^*$ by viewing $L_{\bar{x}}$ as the stalk of the smooth sheaf x^*L on $\mathrm{Spec}(K)$ and using the surjection $\pi_1(X_K, \bar{x}) \to \mathrm{Gal}(\overline{K}/K)$. It follows from the above discussion that the kernel of the surjection $\mathcal{S}_{\mathrm{et}} \to \mathcal{O}_G$ induced by the inclusion $G \subset \mathrm{Aut}(L_{\bar{x}})$ is stable under the action of $\pi_1(X_K, \bar{x})$ and hence we obtain an induced action of $\pi_1(X_K, \bar{x})$ on $\mathcal{O}_G$. If $\mathbb{V}(\mathcal{O}_G)'$ denotes the model for $\mathbb{V}(\mathcal{O}_G)$ on $X_{K,\mathrm{et}}$ constructed above, then there is a surjection of sheaves on X_K

$$(\mathrm{Sym}^\bullet L \otimes L_x^*)_{\det} \longrightarrow \mathbb{V}(\mathcal{O}_G)', \tag{5.31.4}$$

where L_x denotes the pullback of L to $\mathrm{Spec}(K)$. This shows in particular that $\mathbb{V}(\mathcal{O}_G)'$ is a direct limit of smooth sheaves on X_K.

Theorem 5.32. (i) *There is a natural isomorphism*

$$\pi_1(\widetilde{\langle L\rangle}_\otimes, \bar{x}) \simeq \pi_1(X_{\widetilde{\langle L_{\overline{K}}\rangle}_\otimes}) \tag{5.32.1}$$

compatible with the $\mathrm{Gal}(\overline{K}/K)$*–actions.*

(ii) *For any representation M of $\pi_1(X_{\widetilde{\langle L_{\overline{K}}\rangle}_\otimes})$ corresponding via* 5.32.1 *to a smooth sheaf $\mathcal{M}$ on $X_{\overline{K}}$, there is a natural isomorphism*

$$H^*(X_{\widetilde{\langle L\rangle}_\otimes}, M) \simeq H^*(X_{\overline{K},\mathrm{et}}, \mathcal{M}). \tag{5.32.2}$$

The proof of 5.32 will be in several steps 5.33-5.37 following the outline of [Ol1] in the crystalline case.

5.33. Let $\widetilde{G}$ denote $\pi_1(\widetilde{\langle L_{\overline{K}}\rangle}_{\otimes}, \bar{x})$. By repeating the constructions of 5.31 replacing G by $\widetilde{G}$, we obtain an ind–smooth sheaf $\mathbb{V}(\mathcal{O}_{\widetilde{G}})$ of algebras with right $\widetilde{G}$–action on $X_{\overline{K}}$ and a pointed stack

$$\widetilde{X} := [\mathbb{R}\mathrm{Spec}_{\widetilde{G}}\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{\widetilde{G}}))/\widetilde{G}] \in \mathrm{Ho}(\mathrm{SPr}_*(\mathbb{Q}_p)). \tag{5.33.1}$$

There is a natural commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\pi}} & B\widetilde{G} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle r} \\ X_{\widetilde{\langle L_{\overline{K}}\rangle}_{\otimes}} & \xrightarrow{\pi} & BG. \end{array} \tag{5.33.2}$$

Lemma 5.34. *The map $\tilde{\pi}_* : \pi_1(\widetilde{X}) \to \pi_1(B\widetilde{G}) \simeq \widetilde{G}$ is an isomorphism.*

Proof. By the same reasoning as in [KPT, 1.3.10] the homotopy fiber of $\tilde{\pi}$ is isomorphic to $F := \mathbb{R}\mathrm{Spec}(\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{\widetilde{G}})))$ and by the long exact sequence of homotopy groups associated to $\tilde{\pi}$ it suffices to show that $\pi_1(F) = 0$. By [To1, 2.4.5] the homotopy groups of F are all pro–unipotent group schemes, so it suffices to show that $\mathrm{Hom}(\pi_1(F), \mathbb{G}_a)$ is 0. By [To1, 2.2.6] this group is isomorphic to

$$H^1(\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{\widetilde{G}}))) \simeq H^1(X_{\overline{K}}, \mathbb{V}(\mathcal{O}_{\widetilde{G}})). \tag{5.34.1}$$

Since first étale cohomology agrees with group cohomology of $\pi_1(X_{\overline{K}}, \bar{x})$, there is a natural isomorphism

$$H^1(X_{\overline{K}}, \mathbb{V}(\mathcal{O}_{\widetilde{G}})) \simeq H^1(\mathrm{Rep}(\pi_1(X_{\overline{X}}, \bar{x})), \mathcal{O}_{\widetilde{G}}). \tag{5.34.2}$$

On the other hand, since $\widetilde{\langle L_{\overline{K}}\rangle}_{\otimes}$ is closed under extensions in the category of all smooth sheaves, there are natural isomorphisms

(5.34.3)
$$H^1(\mathrm{Rep}(\pi_1(X_{\overline{X}}, \bar{x})), \mathcal{O}_{\widetilde{G}}) \simeq \mathrm{Ext}^1_{\pi_1(X_{\overline{X}}, \bar{x})}(\mathbb{Q}_p, \mathcal{O}_{\widetilde{G}}) \simeq \mathrm{Ext}^1_{\widetilde{G}}(\mathbb{Q}_p, \mathcal{O}_{\widetilde{G}}) \simeq H^1(\mathrm{Rep}(\widetilde{G}), \mathcal{O}_{\widetilde{G}}).$$

By [Ol1, 2.18 (i)] $\mathcal{O}_{\widetilde{G}}$ is injective in $\mathrm{Rep}(\widetilde{G})$ and hence these groups are zero. $\square$

Lemma 5.35. (i) *For any left representation V of $\widetilde{G}$ corresponding to a smooth sheaf $\mathcal{V}$ on $X_{\overline{K}}$ there is a natural isomorphism*

$$H^*(X_{\overline{K}}, \mathcal{V}) \to H^*(\widetilde{X}, V). \tag{5.35.1}$$

(ii) *For any representation V of G corresponding to a smooth sheaf $\mathcal{V}$ on $X_{\overline{K}}$ there is a natural isomorphism*

$$H^*(X_{\overline{K}}, \mathcal{V}) \to H^*(X_{\widetilde{\langle L_{\overline{K}}\rangle}_{\otimes}}, V). \tag{5.35.2}$$

Proof. We prove (i) leaving to the reader the task of rewriting the proof with G instead of $\widetilde{G}$ to get (ii).

By [Ol1, 2.33] the right hand side of 5.35.1 is isomorphic to

$$H^m(\mathrm{Rep}(\widetilde{G}), V^c \otimes \mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{\widetilde{G}}))), \tag{5.35.3}$$

where V^c denotes the contragredient representation. On the other hand, by [Ol1, 2.18 (ii)], there is a natural isomorphism of ind–smooth sheaves with right $\widetilde{G}$–action $V^c \otimes \mathbb{V}(\mathcal{O}_{\widetilde{G}}) \simeq \mathcal{V} \otimes \mathbb{V}(\mathcal{O}_{\widetilde{G}})$, where $\widetilde{G}$ acts trivially on $\mathcal{V}$. The natural map $\mathcal{V} \to \mathcal{V} \otimes \mathbb{V}(\mathcal{O}_{\widetilde{G}})$ induces the desired map 5.35.1.

To prove that 5.35.1 is an isomorphism proceed as follows. Note first that there is a commutative diagram of categories

$$
\begin{array}{ccc}
\mathrm{Rep}(\widetilde{G})^{\Delta} & \xrightarrow{s^{\bullet}} & \mathrm{Vec}_{\mathbb{Q}_p}^{\Delta} \\
{\scriptstyle \mathrm{Tot}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tot}} \\
\mathrm{Rep}(\widetilde{G}) & \xrightarrow{s} & \mathrm{Vec}_{\mathbb{Q}_p},
\end{array}
\tag{5.35.4}
$$

where s and $s^{\bullet}$ take $\widetilde{G}$–invariants. For each $[n] \in \Delta$ there is a natural restriction functor $j_n : \mathrm{Rep}(\widetilde{G})^{\Delta} \to \mathrm{Rep}(\widetilde{G})$ sending $V^{\bullet} \in \mathrm{Rep}(\widetilde{G})^{\Delta}$ to V^n which has an exact left adjoint $j_{n!}$ given by the formula

$$
j_{n!}F : \Delta \to \mathrm{Rep}(\widetilde{G}), \quad [k] \mapsto \oplus_{[n]\to[k]}F. \tag{5.35.5}
$$

In particular, the functor j_n takes injectives to injectives. It follows that if $I \in \mathrm{Rep}(\widetilde{G})^{\Delta}$ is an injective object, then $\mathrm{Tot}(I)$ is a complex of injectives in $\mathrm{Rep}(\widetilde{G})$. From this we deduce that if $U_{\bullet} \to X$ is a hypercover with each U_n a $K(\pi, 1)$ then to prove that the morphism 5.35.1 is an isomorphism it suffices to show for each n that the natural map

$$
GC(U_{n,\overline{K}}, E, \mathcal{V}) \to \mathbb{R}s^n GC(U_{n,\overline{K}}, E, \mathcal{V} \otimes \mathbb{V}(\mathcal{O}_{\widetilde{G}})) \tag{5.35.6}
$$

is an isomorphism (here the notation as as in 5.19). Furthermore, it suffices to prove this for each connected component W of $U_{n,\overline{K}}$. Choose a geometric generic point $\mathrm{Spec}(\Omega) \to W$ and let π denote $\pi_1(V, \Omega)$. By 5.9, it suffices to show that the natural map

$$
GC(W, \mathrm{Spec}(\Omega), \mathcal{V}) \to \mathbb{R}sGC(W, \mathrm{Spec}(\Omega), \mathcal{V} \otimes \mathbb{V}(\mathcal{O}_{\widetilde{G}})) \tag{5.35.7}
$$

is an isomorphism. By the description of group cohomology in [SGA3, I.5.3.1]

$$
\mathbb{R}sGC(W, \mathrm{Spec}(\Omega), \mathcal{V} \otimes \mathbb{V}(\mathcal{O}_{\widetilde{G}})) \tag{5.35.8}
$$

is isomorphic to the simple complex associated to the double complex $E^{\cdot,\cdot}$ with

$$
E^{p,q} = \mathrm{Hom}_{\pi}^{\mathrm{cts}}(\pi^{p+1}, (\mathcal{V} \otimes \mathbb{V}(\mathcal{O}_{\widetilde{G}}))_{\Omega} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\widetilde{G}}^{\otimes q}), \tag{5.35.9}
$$

where $\mathcal{O}_{\widetilde{G}}$ is viewed as a trivial π–module, $GC(W, \mathrm{Spec}(\Omega), \mathcal{V})$ is isomorphic to the complex with p–th term $\mathrm{Hom}_{\pi}^{\mathrm{cts}}(\pi^{p+1}, \mathcal{V}_{\Omega})$ and the map 5.35.1 is the one induced by the map $\mathcal{V}_{\Omega} \hookrightarrow \mathcal{V}_{\Omega} \otimes \mathbb{V}(\mathcal{O}_{\widetilde{G}})_{\Omega}$.

This double complex can also be viewed as the double complex computing the continuous cohomology of the complex of π–representations

$$
(\mathcal{V}\otimes\mathbb{V}(\mathcal{O}_{\widetilde{G}}))_{\Omega}\otimes\mathcal{O}_{\widetilde{G}}^{\otimes -} : \cdots \to (\mathcal{V}\otimes\mathbb{V}(\mathcal{O}_{\widetilde{G}}))_{\Omega}\otimes\mathcal{O}_{\widetilde{G}}^{\otimes q} \to (\mathcal{V}\otimes\mathbb{V}(\mathcal{O}_{\widetilde{G}}))_{\Omega}\otimes\mathcal{O}_{\widetilde{G}}^{\otimes q+1} \to \cdots, \tag{5.35.10}
$$

where the differential is as in [SGA3, I.5.3]. Thus it suffices to show that the natural map $\mathcal{V} \to (\mathcal{V}\otimes\mathbb{V}(\mathcal{O}_{\widetilde{G}}))_{\Omega}\otimes\mathcal{O}_{\widetilde{G}}^{\otimes -}$ is a quasi–isomorphism. Since $\mathcal{V}$ is a trivial right $\widetilde{G}$–representation, to prove this it suffices to show that the right $\widetilde{G}$–module $\mathbb{V}(\mathcal{O}_{\widetilde{G}})_{\Omega}$ is injective in the category of right $\widetilde{G}$–modules and that the $\widetilde{G}$–invariants of $\mathbb{V}(\mathcal{O}_{\widetilde{G}})_{\Omega}$ are equal to $\mathbb{Q}_p$. By [SGAI, V.5.7] any two stalks of a smooth sheaf are isomorphic, and hence $\mathbb{V}(\mathcal{O}_{\widetilde{G}})_\Omega$ is non–canonically isomorphic to $\mathbb{V}(\mathcal{O}_{\widetilde{G}})_{\bar{x}} \simeq \mathcal{O}_{\widetilde{G}}$. Thus the result follows from [Ol1, 2.18]. $\square$

Corollary 5.36. *For any representation V of $\pi_1(X_{\widetilde{\langle L_{\overline{K}}\rangle}_\otimes})$, the pullback map*

$$f^* : H^*(X_{\widetilde{\langle L_{\overline{K}}\rangle}_\otimes}, V) \to H^*(\widetilde{X}, f^*V) \tag{5.36.1}$$

is an isomorphism.

Proof. Since $\pi_1(X_{\widetilde{\langle L_{\overline{K}}\rangle}_\otimes}))$ has pro–reductive completion equal to G, every representation V admits a filtration whose graded pieces are obtained from representations of G. Using the long exact sequence of cohomology, this reduces the problem to the case when V is obtained from a representation of G. In this case, the corollary follows from 5.35 which identifies both sides with étale cohomology. $\square$

5.37 (Proof of 5.32)**.** By 5.34, $\pi_1(\widetilde{\langle L_{\overline{K}}\rangle}_\otimes, \bar{x}) \simeq \pi_1(\widetilde{X})$ so to prove 5.32 (i)) it suffices to show that the map $f_* : \pi_1(\widetilde{X}) \to \pi_1(X_{\widetilde{\langle L_{\overline{K}}\rangle}_\otimes})$ is an isomorphism. For this it suffices by Tannaka duality to show that the pullback functor

$$f^* : \mathrm{Rep}(\pi_1(X_{\widetilde{\langle L_{\overline{K}}\rangle}_\otimes})) \to \mathrm{Rep}(\pi_1(\widetilde{X})) \tag{5.37.1}$$

is an equivalence. Since the kernel of $\widetilde{G} \to G$ is pro–unipotent, every object of $\mathrm{Rep}(\pi_1(\widetilde{X}))$ admits a filtration whose graded pieces are obtained from representation of G and hence are in the essential image of f^*. Thus to prove that 5.37.1 is an equivalence it suffices to show that for two objects $V_1, V_2 \in \mathrm{Rep}(\pi_1(X_{\widetilde{\langle L_{\overline{K}}\rangle}_\otimes}))$ the pullback functor

$$\mathrm{Ext}^i_{\mathrm{Rep}(\pi_1(X_{\widetilde{\langle L_{\overline{K}}\rangle}_\otimes}))}(V_1, V_2) \to \mathrm{Ext}^i_{\mathrm{Rep}(\pi_1(\widetilde{X}))}(f^*V_1, f^*V_2) \tag{5.37.2}$$

is an isomorphism for $i = 0, 1$. Setting $M = V_1^* \otimes V_2$ this is equivalent to the map

$$H^i(\mathrm{Rep}(\pi_1(X_{\widetilde{\langle L_{\overline{K}}\rangle}_\otimes})), M) \to H^i(\mathrm{Rep}(\pi_1(\widetilde{X})), f^*M) \tag{5.37.3}$$

being an isomorphism for $i = 0, 1$. By definition of cohomology there are natural isomorphisms

$$H^i(\mathrm{Rep}(\pi_1(X_{\widetilde{\langle L_{\overline{K}}\rangle}_\otimes})), M) \simeq H^i(X_{\widetilde{\langle L_{\overline{K}}\rangle}_\otimes}, M), \tag{5.37.4}$$

$$H^i(\mathrm{Rep}(\pi_1(\widetilde{X})), f^*M) \simeq H^i(\widetilde{X}, f^*M) \tag{5.37.5}$$

for $i = 0, 1$. Thus 5.32 (i)) follows from 5.35.

Statement 5.32 (ii)) also follows from 5.35 and the identification in 5.34. This completes the proof of 5.32. $\square$

Remark 5.38. By the same argument used in [KPT, 1.3.11] the pointed stack $X_{\widetilde{\langle L_{\overline{K}}\rangle}_\otimes}$ is a schematic homotopy type in the sense of [To1].

Remark 5.39. By [To1, 3.3.2] the proof of 5.32 shows that the map $\widetilde{X} \to X_{\widetilde{\langle L_{\overline{K}}\rangle}_\otimes}$ is an isomorphism of pointed stacks.

5.40. There is a variant of the above constructions which will be used below. Let V be a complete discrete valuation ring of mixed characteristic $(0,p)$, field of fractions K, and let $V \to R$ be a smooth V–algebra with $\mathrm{Spec}(R/pR)$ connected. Assume given a divisor $D \subset \mathrm{Spec}(R)$ with normal crossings relative to V, and let $\mathrm{Spec}(R^o) \subset \mathrm{Spec}(R)$ be the complement of D. Denote by $R^\wedge$ the p–adic completion of R and set $R^{\wedge o} := R^\wedge \otimes_R R^o$. Since $\mathrm{Spec}(R/pR)$ is connected the scheme $\mathrm{Spec}(R^\wedge)$ is also connected. Let $E \to \mathrm{Spec}(R_{\overline{K}})$ be a finite set of geometric generic points meeting every connected component of $\mathrm{Spec}(R_{\overline{K}})$, and let $\widehat{E} \to \mathrm{Spec}(R^\wedge_{\overline{K}})$ denote the generic points obtained by completing the fields $k(p)$ $(p \in E)$ with respect to the p–adic topology induced by that on R (if $\mathrm{Spec}(R^\wedge)$ is empty define $\widehat{E}$ to be the empty set). For a smooth sheaf L on $\mathrm{Spec}(R^o_{\overline{K}})$ with pullback $L^\wedge$ to $\mathrm{Spec}(R^{\wedge o}_{\overline{K}})$ we can then apply the construction of 5.7 to $\mathrm{Spec}(R^{\wedge o}_{\overline{K}})$ to obtain a cosimplicial differential graded algebra

$$[n] \mapsto \mathcal{C}^n(R^{\wedge o}_{\overline{K}}, \widehat{E}, L^\wedge). \tag{5.40.1}$$

If $\mathrm{Spec}(R^\wedge)$ is empty we define $\mathcal{C}^n(R^{\wedge o}_{\overline{K}}, \widehat{E}, L^\wedge)$ to be zero. There is a natural map

$$\mathcal{C}^\bullet(R^o_{\overline{K}}, E, L) \to \mathcal{C}^\bullet(R^{\wedge o}_{\overline{K}}, \widehat{E}, L^\wedge). \tag{5.40.2}$$

Furthermore, for any morphism $E' \to E$ there is a natural commutative diagram

$$\begin{array}{ccc} \mathcal{C}^\bullet(R^o_{\overline{K}}, E, L) & \longrightarrow & \mathcal{C}^\bullet(R^{\wedge o}_{\overline{K}}, \widehat{E}, L^\wedge) \\ \downarrow & & \downarrow \\ \mathcal{C}^\bullet(R^o_{\overline{K}}, E', L) & \longrightarrow & \mathcal{C}^\bullet(R^{\wedge o}_{\overline{K}}, \widehat{E}', L^\wedge), \end{array} \tag{5.40.3}$$

and the cosimplicial algebra $\mathcal{C}^\bullet(R^{\wedge o}_{\overline{K}}, \widehat{E}, L^\wedge)$ is functorial in R.

If $\mathrm{Spec}(R)$ is a disjoint union $\coprod_i \mathrm{Spec}(R_i)$ with each $\mathrm{Spec}(R_i/pR_i)$ connected, we set

$$\mathcal{C}^\bullet(R^{\wedge o}_{\overline{K}}, \widehat{E}, L^\wedge) := \oplus_i \mathcal{C}^\bullet(R^{\wedge o}_{i,\overline{K}}, \widehat{E}_i, L^\wedge), \tag{5.40.4}$$

where $\widehat{E}_i$ denotes the subscheme of E whose image lies in $\mathrm{Spec}(R_i)$. If $U = \mathrm{Spec}(R)$ we also sometimes write $\mathcal{C}^\bullet(U^{\wedge o}_{\overline{K}}, \widehat{E}, L^\wedge)$ for $\mathcal{C}^\bullet(R^{\wedge o}_{\overline{K}}, \widehat{E}, L^\wedge)$.

We denote by $GC(U^{\wedge o}_{\overline{K}}, \widehat{E}, L^\wedge)$ or $GC(R^{\wedge o}_{\overline{K}}, \widehat{E}, L^\wedge)$ the differential graded algebra obtained by applying the functor of Thom–Sullivan cochains to $\mathcal{C}^\bullet(U^{\wedge o}_{\overline{K}}, \widehat{E}, L^\wedge)$. By construction there is a natural map

$$GC(U^o_{\overline{K}}, L, E) \longrightarrow GC(U^{\wedge o}_{\overline{K}}, \widehat{E}, L^\wedge). \tag{5.40.5}$$

5.41. Let X/V be a smooth scheme and $D \subset X$ a divisor with normal crossings relative to V. Denote by X^o the complement of D, and fix a collection of geometric generic points $E \to X_{\overline{K}}$ meeting every connected component. Let L be a differential graded algebra in the category of ind–objects of smooth $\mathbb{Q}_p$–sheaves on $X^o_{\overline{K}}$.

Choose a hypercover $U_\bullet \to X$ with each U_n a disjoint union of $K(\pi,1)$'s, and let $GC^\wedge(E,L)_{U_\bullet}$ (or simply $GC^\wedge(E,L)$ if the reference to $U_\bullet$ is clear) be the differential graded algebra obtained by applying the functor of Thom–Sullivan cochains to the cosimplicial differential graded algebra

$$[n] \mapsto GC(U^{\wedge o}_{n,\overline{K}}, \widehat{E}, L^\wedge). \tag{5.41.1}$$

The maps 5.40.5 induce a morphism $GC(L, E) \to GC^\wedge(L, E)$ in $\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}_p})$. If L is a sheaf of G–equivariant differential graded algebras for some affine group scheme $G/\mathbb{Q}_p$ then we get $GC^\wedge(L, E)$ and $GC(L, E) \to GC^\wedge(L, E)$ in $\mathrm{Ho}(G - \mathrm{dga}_{\mathbb{Q}_p})$.

Remark 5.42. As in 4.37, the above can also be carried out with any fiber functor $\omega : \widetilde{\langle L_{\overline{K}}\rangle}_\otimes \to \mathrm{Mod}_R$ taking values in the category of R–modules for some $\mathbb{Q}_p$–algebra R. If G/R denotes the group scheme of tensor automorphisms of ω then the (G, G)–bimodule $\mathcal{O}_G$ corresponds to an ind–smooth sheaf $\mathbb{V}_\omega$ on $X_{\overline{K},\mathrm{et}}$ with R–module structure and right G–action.

In particular, if $(X, D)/V$ is as in 5.41 and $U_\bullet \to X$ is a hypercover with each U_n a disjoint union of $K(\pi, 1)$'s, then we obtain a morphism of differential graded R–algebras with right G–action

(5.42.1) $$GC(\mathbb{V}_\omega, E)_{U_\bullet} \to GC^\wedge(\mathbb{V}_\omega, E)_{U_\bullet}.$$

Furthermore, if $R \to R'$ is a morphism of $\mathbb{Q}_p$–algebras and $\omega' : \widetilde{\langle L_{\overline{K}}\rangle}_\otimes \to \mathrm{Mod}_{R'}$ is a fiber functor with an isomorphism $\omega' \simeq \omega \otimes_R R'$ then there is a natural commutative diagram

(5.42.2) $$\begin{array}{ccc} GC(\mathbb{V}_\omega, E)_{U_\bullet} & \longrightarrow & GC^\wedge(\mathbb{V}_\omega, E)_{U_\bullet} \\ \downarrow & & \downarrow \\ GC(\mathbb{V}_{\omega'}, E)_{U_\bullet} & \longrightarrow & GC^\wedge(\mathbb{V}_{\omega'}, E)_{U_\bullet}. \end{array}$$

If $R \to R'$ is flat then the vertical arrows become equivalences after tensoring the top row with R'.

6. The comparison Theorem

6.1. Let k be a perfect field of characteristic $p > 0$, V the ring of Witt vectors of k, and K the field of fractions of V. Let X/V be a smooth proper scheme with X_K geometrically connected, $D \subset X$ a divisor with normal crossings relative to V, and $X^o = X - D$. We write M_X be for the log structure on X defined by D.

Following [Fa1], we call an étale $U = \mathrm{Spec}(R) \subset X$ *small* if $\mathrm{Spec}(R/pR)$ is connected and there exists an étale map

(6.1.1) $$\mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(V[T_1, \ldots, T_s, T_{s+1}^{\pm}, \ldots, T_r^{\pm}])$$

for some r and s such that M_U is defined by the divisor $T_1 = \cdots = T_s = 0$. We call U *very small* if U is small and $\mathrm{Spec}(R_{\overline{K}})^o := \mathrm{Spec}(R) \times_X X^o_{\overline{K}}$ is a $K(\pi, 1)$. By [Fa2, Chapter II, 2.1] (see also [Ol3, 5.4]), any étale map $U \to X$ admits a covering by a disjoint union of very small étale X–schemes.

The ring $B_{\mathrm{cris}}(R^\wedge)$ [Fa1, Fo1, Fo2, Fo3, Fo4, Ts1].

6.2. For any small étale $\mathrm{Spec}(R) \to X$ and choice of an algebraic closure $\mathrm{Frac}(R) \hookrightarrow \Omega$, Fontaine's theory gives a ring $B_{\mathrm{cris}}(R^\wedge)$ as follows. Let $R^\wedge$ denote the p–adic completion of R and let $\widehat{\Omega}$ denote the completion of Ω with respect to the topology defined by the p–adic topology on R. There is a natural extension of the inclusion $R \hookrightarrow \Omega$ to an inclusion $R^\wedge \hookrightarrow \widehat{\Omega}$.

Let $L \subset \widehat{\Omega}$ denote the maximal field extension of $\mathrm{Frac}(R^\wedge)$ such that the normalization $\overline{R}^\wedge$ of $R^\wedge$ in L is unramified over $\mathrm{Spec}(R^\wedge) \times_X X_K^o$. Set

$$S := \varprojlim \overline{R}^\wedge / p\overline{R}^\wedge, \tag{6.2.1}$$

where the projective limit is taken with respect to Frobenius. Since S is perfect, the ring of Witt vectors $W(S)$ has a canonical lift of Frobenius. An element $x \in W(S)$ can be represented by a vector $(x_0, x_1, x_2, \dots)$ where each $x_i = (x_{i0}, x_{i1}, \dots)$ is an infinite vector with $x_{ij} \in \overline{R}^\wedge / p\overline{R}^\wedge$.

Let $\overline{R}^\dagger$ denote the p-adic completion of $\overline{R}^\wedge$. There is a natural map

$$\theta : W(S) \longrightarrow \overline{R}^\dagger \tag{6.2.2}$$

defined by sending x as above to

$$\theta(x) = \varinjlim_m (\tilde{x}_{0m}^{p^m} + p\tilde{x}_{1m}^{p^{m-1}} + \cdots + p^m \tilde{x}_{mm}), \tag{6.2.3}$$

where $\tilde{x}_{ij} \in \overline{R}^\wedge$ is any lift of x_{ij}. The assumption that R is small ensures that the map θ is surjective [Ts1, A1.1].

We set $J = \mathrm{Ker}(\theta)$ and define $A_{\mathrm{cris}}(R^\wedge)$ to be the p–adic completion of the divided power envelope $D_J(W(S))$. We thus obtain a diagram

$$\begin{array}{ccc} \mathrm{Spec}(\overline{R}^\dagger) & \longrightarrow & \mathrm{Spec}(A_{\mathrm{cris}}(R^\wedge)) \\ \big\downarrow & & \\ \mathrm{Spec}(R). & & \end{array} \tag{6.2.4}$$

Choose elements $\epsilon_m \in \overline{R}^\wedge$ with $\epsilon_0 = 1$, $\epsilon_{m+1}^p = \epsilon_m$, and $\epsilon_1 \neq 1$. Let $\epsilon \in S$ denote the element obtained from the reductions of the ϵ_i, and let $[\epsilon] \in W(S)$ be the Teichmuller lift of ϵ. Set $\pi_\epsilon := [\epsilon] - 1 \in W(S)$ and

$$t = \log([\epsilon]) = \sum_{m \geq 1} (-1)^{m-1}(m-1)!\pi_\epsilon^{[m]} \in A_{\mathrm{cris}}(R^\wedge). \tag{6.2.5}$$

The ring $B_{\mathrm{cris}}(R^\wedge)$ is defined by

$$B_{\mathrm{cris}}(R^\wedge) := A_{\mathrm{cris}}(R^\wedge)[\frac{1}{t}]. \tag{6.2.6}$$

By [Ts1, A3.2], we have $t^{p-1} \in pA_{\mathrm{cris}}(R^\wedge)$, and therefore p is invertible in $B_{\mathrm{cris}}(R^\wedge)$.

6.3. The ring $A_{\mathrm{cris}}(R^\wedge)$ has a lift of Frobenius $\varphi_{A_{\mathrm{cris}}(R^\wedge)}$ induced by the canonical lift of Frobenius to $W(S)$. This lifting of Frobenius induces a semi–linear automorphism $\varphi_{B_{\mathrm{cris}}(R^\wedge)}$ of $B_{\mathrm{cris}}(R^\wedge)$. In particular, the enlargement

$$\begin{array}{ccc} \mathrm{Spec}(\overline{R}^\wedge / p\overline{R}^\wedge) & \longrightarrow & \mathrm{Spec}(A_{\mathrm{cris}}(R^\wedge)) \\ \big\downarrow & & \\ \mathrm{Spec}(R/pR) & & \end{array} \tag{6.3.1}$$

obtained from 6.2.4 comes equipped with a lift of Frobenius.

Remark 6.4. The fact that 6.3.1 is an enlargement, and not just a widening, follows from the fact that the kernel of the map $A_{\text{cris}}(R^\wedge) \to \overline{R}^\dagger$ is the p-adic completion of a divided power ideal. This implies that any element of

$$\operatorname{Ker}(A_{\text{cris}}(R^\wedge)/pA_{\text{cris}}(R^\wedge) \to \overline{R}^\wedge/p\overline{R}^\wedge) \tag{6.4.1}$$

is nilpotent.

6.5. Define a filtration $I^{[\cdot]}$ on $A_{\text{cris}}(R^\wedge)$ by

$$I^{[r]} := \{x \in A_{\text{cris}}(R^\wedge) | \varphi^n(x) \in \operatorname{Fil}^r A_{\text{cris}}(R^\wedge) \text{ for all } n \geq 0\}, \tag{6.5.1}$$

where $\operatorname{Fil}^r A_{\text{cris}}(R^\wedge)$ denotes the filtration obtained as the p–adic completion of the filtration on $D_J(W(S))$ defined by the PD–ideals $J^{[r]}$. The element $t \in A_{\text{cris}}(R^\wedge)$ lies in $\operatorname{Fil}^1 A_{\text{cris}}(R^\wedge)$ so we obtain a filtration $\operatorname{Fil}_{B_{\text{cris}}(R^\wedge)}$ on $B_{\text{cris}}(R^\wedge)$ by declaring that $1/t$ has degree -1. The automorphism $\varphi_{B_{\text{cris}}(R^\wedge)}$ preserves this filtration.

6.6. The natural action of $\operatorname{Gal}(\overline{R}^\wedge/R^\wedge)$ on S induces an action of $\operatorname{Gal}(\overline{R}^\wedge/R^\wedge)$ on $A_{\text{cris}}(R^\wedge)$ which in turn induces an action $\rho_{B_{\text{cris}}(R^\wedge)}$ of Galois on $B_{\text{cris}}(R^\wedge)$. This action is continuous and compatible with the filtration. Furthermore the induced action on the enlargement 6.3.1 commutes with the lift of Frobenius.

If $s : \operatorname{Spec}(\Omega') \to \operatorname{Spec}(\Omega)$ is a morphism of geometric generic points of $\operatorname{Spec}(R)$, then there is a natural isomorphism

$$\iota_s : s^* B_{\text{cris}}(R^\wedge) \to B_{\text{cris}}(R^\wedge)', \tag{6.6.1}$$

where $B_{\text{cris}}(R^\wedge)'$ denotes the $\operatorname{Gal}(\overline{R}'^\wedge/R^\wedge)$–module obtained by replacing Ω with Ω' in the above construction. It follows that the association $\Omega \mapsto B_{\text{cris}}(R^\wedge)$ defines a Galois module in the sense of 5.2 on $\operatorname{Spec}(R^\wedge) \times_X X_K^o$ equipped with a semi–linear Frobenius automorphism and filtration.

6.7. There is a natural log structure $M_{A_{\text{cris}}(R^\wedge)}$ on $\operatorname{Spec}(A_{\text{cris}}(R^\wedge))$ defined as follows. Choose an étale map as in 6.1.1, and write $t_1, \ldots, t_s \in R$ for the images of the T_i ($i = 1, \ldots, s$). For each i and l, the extension $R[X]/(X^{p^l} - t_i)$ is étale over $R[1/(pt_1 \cdots t_s)]$. It follows that for each i, we can choose a sequence $\tau_{i,n}$ of elements in $\overline{R}^\wedge$ such that $\tau_{i,n}^p = \tau_{i,n-1}$ and $\tau_{i,0} = t_i$. Let $\tau_i \in S$ denote the corresponding element. We then get a map

$$\beta : \mathbb{N}^r \to W(S), \quad e_i \mapsto [\tau_i], \tag{6.7.1}$$

where $[\tau_i]$ denotes the Teichmuller lift of τ_i. This defines a log structure on $W(S)$ and hence in turn also a log structure on $A_{\text{cris}}(R^\wedge)$. Note that the log structure on $\overline{R}^\wedge$ induced by this map β composed with θ is simply the log structure induced by the pulling back M_R via the map $\operatorname{Spec}(\overline{R}^\wedge) \to \operatorname{Spec}(R)$.

We show that the above log structure on $A_{\text{cris}}(R^\wedge)$ is independent of the choices as follows. Consider a second map as in 6.1.1 giving elements $t'_1, \ldots, t'_s \in R$ defining the log structure, and let $\tau'_{i,n}$ be a choice of roots of the t'_i. Then there exists a unique sequence $u_{i,n} \in \overline{R}^{\wedge *}$ such that $u_{i,n}^p = u_{i,n-1}$ and such that $\tau_{i,n} = u_{i,n}\tau'_{i,n}$. Letting u_i denote the corresponding element of S, we see that $[\tau_i] = [u_i] \cdot [\tau'_i]$ in $A_{\text{cris}}(R^\wedge)$, and hence we get a canonical isomorphism between the associated log structures.

It follows from the above discussion that the enlargement 6.3.1 has a natural structure of a logarithmic enlargement

(6.7.2)
$$\begin{array}{ccc} (\mathrm{Spec}(\overline{R}^{\wedge}/p\overline{R}^{\wedge}), M_R|_{\overline{R}^{\wedge}/p\overline{R}^{\wedge}}) & \longrightarrow & (\mathrm{Spec}(A_{\mathrm{cris}}(R^{\wedge})), M_{A_{\mathrm{cris}}(R^{\wedge})}) \\ \downarrow & & \\ (\mathrm{Spec}(R/p), M_{R/p}). & & \end{array}$$

Note also that the action of $\mathrm{Gal}(\overline{R}^{\wedge}/R^{\wedge})$ extends naturally to an action on the log scheme $(\mathrm{Spec}(A_{\mathrm{cris}}(R^{\wedge})), M_{A_{\mathrm{cris}}(R^{\wedge})})$.

6.8. We will apply Faltings' theory of "almost mathematics" (see for example [Ol3, §2]) to modules over $B_{\mathrm{cris}}(V)$. Let $\Lambda \subset \mathbb{Q}$ denote the subring $\mathbb{Z}[1/p]$, and let $\Lambda_+ := \Lambda \cap \mathbb{Q}_{>0}$. Following [Ol3, §11], for every $\alpha \in \Lambda_+$ we define a principal ideal $\mathfrak{m}_{\mathrm{cris},\alpha} \subset B_{\mathrm{cris}}(V)$ as follows. Fix a sequence $(\tau_m)_{m\geq 0}$ of elements of $\overline{V}$ with $\tau_0 = p$ and $\tau_{m+1}^p = \tau_m$ for all $m \geq 0$. We define τ_m to be 0 if $m < 0$. For any $n \in \mathbb{Z}$, define $\lambda_{1/p^n} \in S_V$ to be the element $(a_m)_{m\geq 0}$ with

(6.8.1)
$$a_m = \tau_{n+m},$$

and let $\delta_{1/p^n} \in W(S_V)$ be the Teichmuller lifting of λ_{1/p^n}. For any $\alpha = s/p^n \in \Lambda_+$ we then define

(6.8.2)
$$\delta_\alpha := (\delta_{1/p^n})^s \in W(S_V).$$

Let $\mathfrak{m}_{\mathrm{cris},\alpha} \subset B_{\mathrm{cris}}(V)$ be the ideal generated by δ_α. As explained in [Ol3, §11], the ideals $\mathfrak{m}_{\mathrm{cris},\alpha}$ satisfy the necessary conditions enabling us to apply the almost theory.

In what follows we denote by $\widetilde{B}_{\mathrm{cris}}(V)$ the ring $B_{\mathrm{cris}}(V)[\delta_\alpha^{-1}]_{\alpha\in\Lambda_+}$.

The category $MF_X^{\nabla}(\Phi)$.

6.9. Let $(X, M_X)/V$ be as in 6.1 and let $(X_0, M_{X_0})/k$ be the reduction.

We define the category $MF_X^{\nabla}(\Phi)$ as in [Fa1, Ts2]. If E is an isocrystal on $(X_0, M_{X_0})/V$, let $(\mathcal{E}, \nabla_{\mathcal{E}})$ denote the module with logarithmic connection on (X_K, M_{X_K}) obtained by evaluating E on the enlargement $(X_0, M_{X_0}) \hookrightarrow (X^{\wedge}, M_{X^{\wedge}})$, where $(X^{\wedge}, M_{X^{\wedge}})$ denotes the p–adic completion of $(X, M_X)/V$. The category $MF_X^{\nabla}(\Phi)$ is defined to be the category of triples $(E, \varphi_E, \mathrm{Fil}_{\mathcal{E}})$, where (E, φ_E) is an F–isocrystal on $(X_0, M_{X_0})/V$ and $\mathrm{Fil}_{\mathcal{E}}$ is a decreasing filtration on $\mathcal{E}$ satisfying Griffith's transversality

(6.9.1)
$$\nabla_{\mathcal{E}}(\mathrm{Fil}^i_{\mathcal{E}}) \subset \mathrm{Fil}^{i-1}_{\mathcal{E}} \otimes \Omega^1_{(X_K, M_{X_K})/K}.$$

6.10. If $(E, \mathrm{Fil}_E, \varphi_E) \in MF_X^{\nabla}(\Phi)$ and $\mathrm{Spec}(R) \to X$ is étale and small, we can evaluate E on the enlargement 6.7.2 to get a $A_{\mathrm{cris}}(R^{\wedge}) \otimes \mathbb{Q}$–module $E((\mathrm{Spec}(A_{\mathrm{cris}}(R^{\wedge})), M_{A_{\mathrm{cris}}(R^{\wedge})}))$. Inverting $t \in A_{\mathrm{cris}}(R^{\wedge})$, we get a $B_{\mathrm{cris}}(R^{\wedge})$–module which we denote simply by $E(B_{\mathrm{cris}}(R^{\wedge}))$. The F–isocrystal structure φ_E induces a semi–linear automorphism of the $B_{\mathrm{cris}}(R^{\wedge})$–module $E(B_{\mathrm{cris}}(R^{\wedge}))$.

The $B_{\mathrm{cris}}(R^{\wedge})$–module $E(B_{\mathrm{cris}}(R^{\wedge}))$ also has a natural filtration $\mathrm{Fil}_{E(B_{\mathrm{cris}}(R^{\wedge}))}$ defined as follows. Since $(X, M_X)/V$ is smooth, we can choose a morphism

(6.10.1)
$$r : (\mathrm{Spec}(A_{\mathrm{cris}}(R^{\wedge})), M_{A_{\mathrm{cris}}(R^{\wedge})}) \to (\mathrm{Spec}(R), M_{\mathrm{Spec}(R)})$$

such that the diagram

$$
\begin{tikzcd}
(\mathrm{Spec}(\overline{R}^{\wedge}), M_{R^{\wedge}}) \arrow[r] \arrow[d] & (\mathrm{Spec}(A_{\mathrm{cris}}(R^{\wedge})), M_{A_{\mathrm{cris}}(R^{\wedge})}) \arrow[d, "r"] \\
(\mathrm{Spec}(R), M_{\mathrm{Spec}(R)}) \arrow[r, "\mathrm{id}"] & (\mathrm{Spec}(R), M_{\mathrm{Spec}(R)})
\end{tikzcd} \tag{6.10.2}
$$

commutes. The choice of such an r gives an isomorphism

$$\sigma_r : E(B_{\mathrm{cris}}(R^{\wedge})) \simeq \mathcal{E}(\mathrm{Spec}(R)) \otimes_R B_{\mathrm{cris}}(R^{\wedge}), \tag{6.10.3}$$

and we define $\mathrm{Fil}_{E(B_{\mathrm{cris}}(R^{\wedge}))}$ to be the tensor product filtration of $\mathrm{Fil}_{\mathcal{E}(\mathrm{Spec}(R))}$ and the filtration $\mathrm{Fil}_{B_{\mathrm{cris}}(R^{\wedge})}$ on $B_{\mathrm{cris}}(R^{\wedge})$.

Lemma 6.11. *The filtration* $\mathrm{Fil}_{E(B_{\mathrm{cris}}(R^{\wedge}))}$ *is independent of the choice of* r*.*

Proof. Let $r' : \mathrm{Spec}(A_{\mathrm{cris}}(R^{\wedge})) \to \mathrm{Spec}(R)$ be a second retraction, and let $\tau : E(\mathrm{Spec}(R)) \otimes_{R,r} B_{\mathrm{cris}}(R^{\wedge}) \to E(\mathrm{Spec}(R)) \otimes_{R,r'} B_{\mathrm{cris}}(R^{\wedge})$ be the composite
(6.11.1)

$$E(\mathrm{Spec}(R)) \otimes_{R,r} B_{\mathrm{cris}}(R^{\wedge}) \xrightarrow{\sigma_r^{-1}} E(B_{\mathrm{cris}}(R^{\wedge})) \xrightarrow{\sigma_{r'}} E(\mathrm{Spec}(R)) \otimes_{R,r'} B_{\mathrm{cris}}(R^{\wedge}).$$

Choose an étale morphism as in 6.1.1 and let $\nabla_i : E(\mathrm{Spec}(R)) \to E(\mathrm{Spec}(R))$ denote the induced operator $\nabla_{T_i \frac{\partial}{\partial T_i}}$ (the dual of $\mathrm{dlog}(T_i)$). Then τ is given by the formula

$$\tau(e \otimes 1) = \sum_{\underline{n} \in \mathbb{N}^d} \frac{1}{\underline{n}!} \Big(\prod_{i=1}^{d} (r(T_i) \cdot r'(T_i)^{-1} - 1)^{n_i}\Big) \otimes \Big(\prod_{1 \leq i \leq d} \prod_{0 \leq j < n_i} (\nabla_i - j)\Big)(e). \tag{6.11.2}$$

In particular, if $e \in \mathrm{Fil}^s_{\mathcal{E}}(\mathrm{Spec}(R))$, then

$$\tau(e \otimes 1) \in \sum_{\underline{n} \in \mathbb{N}^d} \mathrm{Fil}^{\sum n_i}_{B_{\mathrm{cris}}(R^{\wedge})} \otimes \mathrm{Fil}^{s - (\sum n_i)}_{\mathcal{E}(\mathrm{Spec}(R))}. \tag{6.11.3}$$

□

6.12. The module $E(B_{\mathrm{cris}}(R^{\wedge}))$ also comes equipped with a continuous action of $\mathrm{Gal}(\overline{R}^{\wedge}/R^{\wedge})$ which commutes with the Frobenius automorphism induced by the F–isocrystal structure as well as the filtration. As in 6.6, this $\mathrm{Gal}(\overline{R}^{\wedge}/R^{\wedge})$–module $E(B_{\mathrm{cris}}(R^{\wedge}))$ is functorial for morphisms $s : \mathrm{Spec}(\Omega') \to \mathrm{Spec}(\Omega)$ of geometric generic points of $\mathrm{Spec}(R)$, and hence $E(B_{\mathrm{cris}}(R^{\wedge}))$ is naturally viewed as a Galois module in the sense of 5.2 with semi–linear automorphism. In what follows it is necessary to avoid choosing a geometric generic point so we will usually view $E(B_{\mathrm{cris}}(R^{\wedge}))$ as a Galois module in this sense. Note in particular that $E(B_{\mathrm{cris}}(R^{\wedge}))$ when viewed in this way is functorial in R. We hope the ambiguous notation does not cause too much confusion.

If $U = \mathrm{Spec}(R) \to X$ is a disjoint union of small and étale X–schemes and $(E, \mathrm{Fil}_E, \varphi_E) \in MF^{\nabla}_X(\Phi)$, we write $E(B_{\mathrm{cris}}(U^{\wedge}))$ (or $E(B_{\mathrm{cris}}(R^{\wedge}))$) for the filtered Galois module with semi–linear automorphisms on $U^{\wedge o}_K := \mathrm{Spec}(R^{\wedge}) \times_X X^o_K$ obtained from the construction 6.10 on each connected component.

Associated sheaves and comparison.

6.13. If L is a smooth $\mathbb{Q}_p$–sheaf on X_K^o and $U \rightarrow X$ is small and étale, the pullback of L to $U_K^{\wedge o}$ is a Galois module on $U_K^{\wedge o}$ which we denote by $L_{U_K^{\wedge o}}$. Define an *association* ι between $(E, \mathrm{Fil}_E, \varphi_E) \in MF_X^{\nabla}(\Phi)$ and a smooth $\mathbb{Q}_p$–sheaf L on X_K^o to be a collection of isomorphisms of Galois modules, one for each small étale $U \rightarrow X$,

$$\iota_U : E(B_{\mathrm{cris}}(U^{\wedge})) \simeq L_{U_K^{\wedge o}} \otimes B_{\mathrm{cris}}(U^{\wedge}) \tag{6.13.1}$$

compatible with the semi–linear Frobenius automorphisms, and the filtrations. Furthermore, we require that the isomorphisms ι_U be compatible with morphisms over X. In what follows it will also be important to consider differential graded algebras with an action of an algebraic group. Let G_{dR}/K and $G_{\mathrm{et}}/\mathbb{Q}_p$ be algebraic groups and assume given an isomorphism $G_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V) \simeq G_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$. If $(E, \mathrm{Fil}_E, \varphi_E)$ is a G_{dR}–equivariant differential graded algebra in $MF_X^{\nabla}(\Phi)$ and L is a G_{et}–equivariant differential graded algebra in the category of smooth $\mathbb{Q}_p$ sheaves on X_K^o then an association ι between $(E, \mathrm{Fil}_E, \varphi)$ and L is also required to be compatible with the algebra structures and $G_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V) \simeq G_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$–actions. We hope that the context makes clear what we mean by "association" in what follows.

Let us also recall that a smooth sheaf L on X_K is called *crystalline* if it is associated to some object in $MF_X^{\nabla}(\Phi)$.

Remark 6.14. In order to make sense of pullback of associations, it is convenient to restrict attention to certain subcategories of the category of disjoint unions of small étale morphisms $U \rightarrow X$. If $\mathcal{U} \subset \mathrm{Et}(X)$ is a full subcategory with each $U \in \mathcal{U}$ a disjoint union of small and étale morphisms and such that every small and étale $V \rightarrow X$ admits a covering by an object of $\mathcal{U}$, then the topos corresponding to $\mathcal{U}$ is equal to X_{et}. Define a $\mathcal{U}$*–association* between $(E, \mathrm{Fil}_E, \varphi_E)$ and L to be the data of compatible isomorphisms 6.13.1 over each $U \in \mathcal{U}$. In the comparison between cohomologies below, it suffices to consider $\mathcal{U}$–associations. We leave it to the reader to make the necessary modifications.

6.15. Fix a geometric generic point $E = \mathrm{Spec}(\Omega) \rightarrow X_{\overline{K}}^o$. Note that the projection $E \rightarrow \mathrm{Spec}(\overline{K})$ determines an inclusion $\overline{K} \hookrightarrow \Omega$.

Let G_{dR}/K and $G_{\mathrm{et}}/\mathbb{Q}_p$ be algebraic groups and assume given an isomorphism $G_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V) \simeq G_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$. To ease notation we write simply $G_{B_{\mathrm{cris}}(V)}$ for $G_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V)$ and $G_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$ identified by the given isomorphism. Let $(E, \mathrm{Fil}, \varphi_E)$ be a G_{dR}–equivariant differential graded algebra in $MF_X^{\nabla}(\Phi)$ associated by ι to a G_{et}–equivariant differential graded algebra L in the category of smooth $\mathbb{Q}_p$–sheaves on X_K^o. We construct a natural equivalence

$$\mathbb{R}\Gamma_{\mathrm{cris}}(E) \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq GC(L, E) \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \tag{6.15.1}$$

in $\mathrm{Ho}(G_{\widetilde{B}_{\mathrm{cris}}(V)} - \mathrm{dga}_{\widetilde{B}_{\mathrm{cris}}(V)})$ compatible with the Frobenius automorphisms and $\mathrm{Gal}(\overline{K}/K)$–actions.

Remark 6.16. In the above the notation $B_{\mathrm{cris}}(V)$ indicates the ring obtained by the construction in 6.2 using the specified embedding $K \hookrightarrow \overline{K}$.

6.17. Let $E \to \mathbb{R}^\bullet$ be as in 4.31. For any étale $U \to X$ which is a disjoint union of very small X–schemes we have a diagram of differential graded algebras

(6.17.1)

$$\begin{array}{ccccc} GC(U^o_{\overline{K}}, E_U, L) & \longrightarrow & GC(U^{\wedge o}_{\overline{K}}, \widehat{E}_U, L^\wedge) & \longrightarrow & GC(U^{\wedge o}_{\overline{K}}, \widehat{E}_U, L^\wedge \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(U^\wedge)) \\ & & & & \downarrow \simeq \\ & & GC(U^{\wedge o}_{\overline{K}}, \widehat{E}_U, \mathbb{R}^\bullet(B_{\mathrm{cris}}(U^\wedge))) & \longleftarrow & GC(U^{\wedge o}_{\overline{K}}, \widehat{E}_U, E(B_{\mathrm{cris}}(U^\wedge))). \end{array}$$

Let $\lambda_U : GC(U^o_{\overline{K}}, E_U, L) \to GC(U^{\wedge o}_{\overline{K}}, \widehat{E}_U, \mathbb{R}^\bullet(B_{\mathrm{cris}}(U^\wedge)))$ denote the composite. Observe that the inclusion $\overline{K} \hookrightarrow \Omega$ induces a natural map $E_U \to \mathrm{Spec}(\overline{K})$. It follows that

$$GC(U^{\wedge o}_{\overline{K}}, \widehat{E}_U, E(B_{\mathrm{cris}}(U^\wedge))) \quad \text{and} \quad GC(U^{\wedge o}_{\overline{K}}, \widehat{E}_U, \mathbb{R}^\bullet(B_{\mathrm{cris}}(U^\wedge))) \tag{6.17.2}$$

are naturally $B_{\mathrm{cris}}(V)$–modules.

The natural map

$$\mathbb{R}^\bullet(((U^\wedge, M_{U^\wedge})/K)_{\mathrm{cris}}) \longrightarrow \mathbb{R}^\bullet(B_{\mathrm{cris}}(U^\wedge)) \tag{6.17.3}$$

induces a map

$$\mathbb{R}^\bullet(((U^\wedge, M_{U^\wedge})/K)_{\mathrm{cris}}) \longrightarrow GC(U^{\wedge o}_{\overline{K}}, \widehat{E}_U, \mathbb{R}^\bullet(B_{\mathrm{cris}}(U^\wedge))) \tag{6.17.4}$$

since $\mathbb{R}^\bullet(((U^\wedge, M_{U^\wedge})/K)_{\mathrm{cris}})$ is a complex of trivial Galois modules. As in 4.33.6, we thus obtain a diagram of $G_{B_{\mathrm{cris}}(V)}$–equivariant differential graded algebras

$$\begin{array}{ccc} GC(U^o_{\overline{K}}, E_U, L) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) & \xrightarrow{\lambda_U \otimes B_{\mathrm{cris}}(V)} & GC(U^{\wedge o}_{\overline{K}}, \widehat{E}_U, \mathbb{R}^\bullet(B_{\mathrm{cris}}(U^\wedge))) \\ & & \uparrow \epsilon \\ & & \mathbb{R}^\bullet(((U^\wedge, M_{U^\wedge})/K)_{\mathrm{cris}}) \otimes_K B_{\mathrm{cris}}(V) \\ & & \downarrow \simeq \\ DR(E)(U^\wedge, M_{U^\wedge}) \otimes_K B_{\mathrm{cris}}(V) & \xrightarrow{\simeq} & DR(\mathbb{R}^\bullet)(U^\wedge, M_{U^\wedge}) \otimes_K B_{\mathrm{cris}}(V). \end{array} \tag{6.17.5}$$

This diagram is functorial in U. In particular, if $U_\bullet \to X$ is a hypercover with each U_n a disjoint union of very small X–schemes, we obtain a diagram of simplicial differential graded algebras

$$\begin{array}{ccc} GC(U^o_{\bullet,\overline{K}}, E_{U_\bullet}, L) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) & \xrightarrow{\lambda_{U_\bullet} \otimes B_{\mathrm{cris}}(V)} & GC(U^{\wedge o}_{\bullet,\overline{K}}, \widehat{E}_{U_\bullet}, \mathbb{R}^\bullet(B_{\mathrm{cris}}(U^\wedge_\bullet))) \\ & & \uparrow \epsilon_\bullet \\ & & \mathbb{R}^\bullet(((U^\wedge_\bullet, M_{U^\wedge_\bullet})/K)_{\mathrm{cris}}) \otimes_K B_{\mathrm{cris}}(V) \\ & & \downarrow \simeq \\ DR(E)(U^\wedge_\bullet, M_{U^\wedge_\bullet}) \otimes_K B_{\mathrm{cris}}(V) & \xrightarrow{\simeq} & DR(\mathbb{R}^\bullet)(U^\wedge_\bullet, M_{U^\wedge_\bullet}) \otimes_K B_{\mathrm{cris}}(V). \end{array} \tag{6.17.6}$$

By the proof of [Fa1, 5.6] (see also [Ol3, 12.5 and 13.21]), the morphisms $\lambda_{U_\bullet} \otimes B_{\mathrm{cris}}(V)$ and $\epsilon_\bullet$ induce equivalences on the differential graded algebras obtained by applying the functor of Thom–Sullivan cochains after inverting the elements $\delta_\alpha \in B_{\mathrm{cris}}(V)$ ($\alpha \in \Lambda_+$). Thus we obtain the desired equivalence 6.15.1 by applying the functor of Thom–Sullivan cochains to the diagram 6.17.6. The naturality of the construction implies that the equivalence 6.15.1 is compatible with the actions of $\mathrm{Gal}(\overline{K}/K)$ as well as the Frobenius automorphisms.

Pullback of associations.

6.18. One can define pullback of associations as follows. Let $f : W \to X$ be a morphism of smooth proper V–schemes, and assume that the inverse image of D in W is a divisor with normal crossings on W. Denote by M_W the associated log structure on W so that (W, M_W) is a log smooth log scheme over V. The morphism f extends in uniquely to a morphism of log schemes $f : (W, M_W) \to (X, M_X)$.

An object $(E, \mathrm{Fil}_E, \varphi_E) \in MF^{\nabla}_X(\Phi)$ can be pulled back to an object $f^*(E, \mathrm{Fil}_E, \varphi_E) \in MF^{\nabla}_W(\Phi)$ with F–isocrystal the usual pull–back of (E, φ) and filtration the one obtained by pullback from Fil_E. Also, if L is a smooth $\mathbb{Q}_p$–sheaf on X^o_K it can be pulled back to a smooth $\mathbb{Q}_p$–sheaf f^*L on W^o_K.

Let $\mathcal{U} \subset \mathrm{Et}(W)$ denote the full subcategory of étale morphisms $U \to W$ which are disjoint unions of small and étale W–schemes such that there exists a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ W & \longrightarrow & X \end{array} \tag{6.18.1}$$

with $V \to X$ a disjoint union of small and étale morphisms. The category $\mathcal{U}$ satisfies the assumptions of 6.14.

Proposition 6.19. *An association ι between $(E, \mathrm{Fil}_E, \varphi_E)$ and L induces a natural $\mathcal{U}$–association $f^*(\iota)$ between $f^*(E, \mathrm{Fil}_E, \varphi_E)$ and f^*L.*

Proof. Let $U = \mathrm{Spec}(P)$ be in $\mathcal{U}$ and choose a diagram

$$\begin{array}{ccc} \mathrm{Spec}(P) & \longrightarrow & \mathrm{Spec}(R) \\ \downarrow & & \downarrow \\ W & \longrightarrow & X \end{array} \tag{6.19.1}$$

with $\mathrm{Spec}(R) \to X$ small and étale. Let $u : R^\wedge \to P^\wedge$ be the induced map on p–adic completions. Choose algebraic closures $\mathrm{Frac}(P) \hookrightarrow \Omega$ and $\mathrm{Frac}(R) \hookrightarrow \Omega'$ and define $\overline{R}^\wedge$ and $\overline{P}^\wedge$ as in 6.2. By the same argument as in [Ts1, 1.4.3], the map u extends to a map $\bar{u} : \overline{R}^\wedge \to \overline{P}^\wedge$, and any two such extensions differ by composing with a unique element of $\mathrm{Gal}(\overline{R}^\wedge/R^\wedge)$. Choose one such extension $\bar{u}$. By associating to $\sigma \in \mathrm{Gal}(\overline{P}^\wedge/P^\wedge)$ the unique element $\lambda \in \mathrm{Gal}(\overline{R}^\wedge/R^\wedge)$ such that $\sigma \circ \bar{u} = \bar{u} \circ \lambda$, we get a continuous homomorphism

$$\lambda : \mathrm{Gal}(\overline{P}^\wedge/P^\wedge) \longrightarrow \mathrm{Gal}(\overline{R}^\wedge/R^\wedge). \tag{6.19.2}$$

If $\rho : \mathrm{Gal}(\overline{R}^\wedge/R^\wedge) \to \mathrm{Aut}(L)$ is a continuous representation corresponding to a smooth $\mathbb{Q}_p$–sheaf, then the pullback sheaf is the sheaf corresponding to the representation $\rho \circ \lambda$. The choice of $\bar{u}$ also induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\overline{P}^{\wedge}) & \longrightarrow & \mathrm{Spec}(A_{\mathrm{cris}}(P^{\wedge})) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{R}^{\wedge}) & \longrightarrow & \mathrm{Spec}(A_{\mathrm{cris}}(R^{\wedge})) \end{array} \tag{6.19.3}$$

which is compatible with the Galois actions, Frobenius automorphisms, filtrations, and log structures. Furthermore, this diagram identifies $f^*(E, \mathrm{Fil}_E, \varphi_E)(B_{\mathrm{cris}}(P^{\wedge}))$ with

$$E(B_{\mathrm{cris}}(R^{\wedge})) \otimes_{B_{\mathrm{cris}}(R^{\wedge})} B_{\mathrm{cris}}(P^{\wedge}) \tag{6.19.4}$$

with filtration induced by that on $E(B_{\mathrm{cris}}(R^{\wedge}))$. We now define $f^*(\iota)$ to be the isomorphism making the diagram

$$\begin{array}{ccc} f^*E(B_{\mathrm{cris}}(P^{\wedge})) & \xrightarrow{f^*(\iota)} & f^*L \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(P^{\wedge}) \\ \simeq\downarrow & & \downarrow\simeq \\ E(B_{\mathrm{cris}}(R^{\wedge})) \otimes_{B_{\mathrm{cris}}(R^{\wedge})} B_{\mathrm{cris}}(P^{\wedge}) & \xrightarrow{\iota \otimes B_{\mathrm{cris}}(P^{\wedge})} & L \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(P^{\wedge}) \end{array} \tag{6.19.5}$$

commute. We leave it to the reader to verify that this isomorphism is independent of the choice of the diagram 6.18.1 and the extension $\bar{u}$ and therefore defines an association. $\square$

6.20. In particular, if $x \in X^o(V)$ is a point then we can pull back associations to $x = \mathrm{Spec}(V)$. Let G_{dR}, G_{et}, $(E, \mathrm{Fil}_E, \varphi_E) \in MF_X^{\nabla}(\Phi)$ and L be as in 6.15, and assume in addition we are given augmentations $e_{\mathrm{dR}} : x^*E \to K$ and $e_{\mathrm{et}} : L_{\bar{x}} \to \mathbb{Q}_p$ such that the induced diagram

$$\begin{array}{ccc} x^*E \otimes_K B_{\mathrm{cris}}(V) & \xrightarrow{x^*(\iota)} & L_{\bar{x}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \\ e_{\mathrm{dR}}\downarrow & & \downarrow e_{\mathrm{et}} \\ B_{\mathrm{cris}}(V) & \xrightarrow{\mathrm{id}} & B_{\mathrm{cris}}(V) \end{array} \tag{6.20.1}$$

commutes.

By 4.34, 5.21, and 5.25 the algebras $\mathbb{R}\Gamma_{\mathrm{cris}}(E)$ and $\mathbb{R}\Gamma_{\mathrm{et}}(L)$ are naturally viewed as objects of $\mathrm{Ho}(G_{\mathrm{dR}} - \mathrm{dga}_{K,/\mathcal{O}_{G_{\mathrm{dR}}}})$ and $\mathrm{Ho}(G_{\mathrm{et}} - \mathrm{dga}_{\mathbb{Q}_p,/\mathcal{O}_{G_{\mathrm{et}}}})$ respectively. Chasing through the above constructions one sees that the equivalence 6.15.1 extends naturally to an equivalence in $\mathrm{Ho}(G_{\widetilde{B}_{\mathrm{cris}}(V)} - \mathrm{dga}_{\widetilde{B}_{\mathrm{cris}}(V),/\mathcal{O}_{G_{\widetilde{B}_{\mathrm{cris}}(V)}}})$. We leave the details of this verification to the reader.

7. Proofs of 1.7–1.13

7.1. Let $(X, M_X)/V$ be as in 6.1 and $x : \mathrm{Spec}(V) \to X^o$ a section. Let $(E, \mathrm{Fil}_E, \varphi_E) \in MF_X^{\nabla}(\Phi)$ be associated to L on X_K^o and assume $E \in V_{\mathrm{nilp}}^{\mathrm{cris}}((Y, M_Y)/K)$, where (Y, M_Y) denotes the reduction of (X, M_X). Denote by $\mathcal{C}_{\mathrm{dR}}$ the smallest Tannakian subcategory of $V_{\mathrm{nilp}}^{\mathrm{cris}}((Y, M_Y)/K)$ closed under extensions and containing E, and by $\mathcal{C}_{\mathrm{et}}$ the smallest Tannakian subcategory of the category of smooth $\mathbb{Q}_p$–sheaves on $X_{\overline{K}}^o$ closed under extensions and containing the restriction of L to $X_{\overline{K}}^o$.

For the remainder of this section we make the following assumption:

Assumption 7.2. *The categories $\langle E\rangle_\otimes \subset V^{\mathrm{cris}}_{\mathrm{nilp}}((Y, M_Y)/K)$ and $\langle L_{\overline{K}}\rangle_\otimes$ are semi–simple. Equivalently, the groups $G_{\mathrm{dR}} := \pi_1(\langle E\rangle_\otimes, \omega_x)$ and $G_{\mathrm{et}} := \pi_1(\langle L\rangle_\otimes, \omega_{\bar{x}})$ are reductive.*

Tannakian considerations.

7.3. Let $\mathcal{D}$ be a semi–simple Tannakian category over a field Υ of characteristic 0 and assume $E \in \mathcal{D}$ is an object such that $\langle E\rangle_\otimes = \mathcal{D}$. The category $\mathcal{D}$ can then be described as follows.

Let $\mathbb{N}[T^\pm] \subset \mathbb{Z}[T^\pm]$ be the subset of elements $\sum a_i T^i$ with $a_i \in \mathbb{N}$ for every i. For any

$$P = \sum_i a_i T^i \in \mathbb{N}[T^\pm], \tag{7.3.1}$$

let $P(E)$ denote $\oplus_i (V^{\otimes i})^{\oplus a_i} \in \mathcal{D}$, where if i is negative $V^{\otimes i}$ denotes the dual of $V^{\otimes(-i)}$. Consider the category $\mathcal{D}'$ defined as follows. The objects of $\mathcal{D}'$ are pairs (P, e) where $P \in \mathbb{N}[T^\pm]$ and $e \in \mathrm{End}_{\mathcal{D}}(P(E))$ is an idempotent. A morphism $(P, e) \to (P', e')$ is defined to be an equivalence class of elements $\lambda \in \mathrm{Hom}_{\mathcal{D}}(P(E), P'(E))$ such that the diagram

$$\begin{array}{ccc} P(E) & \xrightarrow{\lambda} & P'(E) \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ P(E) & \xrightarrow{\lambda} & P'(E) \end{array} \tag{7.3.2}$$

commutes. Here $\lambda \sim \lambda'$ if $\lambda \circ e = \lambda' \circ e$ (or equivalently $e' \circ \lambda = e' \circ \lambda'$). Note that the condition that 7.3.2 commutes is equivalent to saying that λ is in the equalizer of the two maps

$$? \circ e, e' \circ ? : \mathrm{Hom}_{\mathcal{D}}(P(E), P'(E)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(P(E), P'(E)), \tag{7.3.3}$$

and $\lambda \sim \lambda'$ if they map to the same element under $? \circ e$.

There is a natural functor

$$\mathcal{D}' \longrightarrow \mathcal{D} \tag{7.3.4}$$

which sends (P, e) to $\mathrm{Im}(e : P(E) \to P(E))$. It follows from the fact that $\mathcal{D}$ is semi–simple and the definition of $\mathcal{D}'$ that 7.3.4 is an equivalence.

Let $\mathcal{D}_{\mathrm{dR}}$ (resp. $\mathcal{D}_{\mathrm{et}}$) denote the category $\langle E\rangle_\otimes$ (resp. $\langle L\rangle_\otimes$).

Proposition 7.4. *There is a unique equivalence of Tannakian categories*

$$\theta : \mathcal{D}_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V) \to \mathcal{D}_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \tag{7.4.1}$$

such that $\theta(P(E)) = P(L)$ for every $P \in \mathbb{N}[T^\pm]$, and for any other $P' \in \mathbb{N}[T^\pm]$ the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}_{\mathrm{dR}} \otimes B_{\mathrm{cris}}(V)}(P'(E), P(E)) & \xrightarrow{\theta} & \mathrm{Hom}_{\mathcal{D}_{\mathrm{et}} \otimes B_{\mathrm{cris}}(V)}(P'(L), P(L)) \\ {\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle \simeq} \\ H^0_{\mathrm{dR}}(P(E) \otimes P'(E)^*) \otimes_K B_{\mathrm{cris}}(V) & \xrightarrow{\iota} & H^0_{\mathrm{et}}(P(L) \otimes P'(L)^*) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \end{array} \tag{7.4.2}$$

commutes, where ι is the comparison isomorphism. Here uniqueness means that if θ' is another such functor then there exists a unique isomorphism $\lambda : \theta \to \theta'$ such that for every

$P \in \mathbb{N}[T^{\pm}]$ *the diagram*

$$\begin{array}{ccc} \theta(P(E)) & \xrightarrow{\lambda} & \theta'(P(E)) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ P(L) & \xrightarrow{\mathrm{id}} & P(L) \end{array} \tag{7.4.3}$$

commutes.

Proof. By [Sa, II.1.5.3.1], to give a functor θ as in 7.4.1 is equivalent to giving a K–linear functor

$$\mathcal{D}_{\mathrm{dR}} \longrightarrow \mathcal{D}_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V). \tag{7.4.4}$$

Let $\mathcal{D}'_{\mathrm{dR}}$ be as in 7.3. Identifying $\mathcal{D}_{\mathrm{dR}}$ with $\mathcal{D}'_{\mathrm{dR}}$ as in 7.3.4, we see that giving θ is equivalent to giving a functor

$$\tilde{\theta} : \mathcal{D}'_{\mathrm{dR}} \longrightarrow \mathcal{D}_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V). \tag{7.4.5}$$

We define $\tilde{\theta}$ to be the functor which sends (P, e) to the image of

$$\iota(e) : P(L) \otimes B_{\mathrm{cris}}(V) \longrightarrow P(L) \otimes B_{\mathrm{cris}}(V), \tag{7.4.6}$$

where $\iota(e)$ denotes the image of e under the map given by ι

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}_{\mathrm{dR}}}(P(E), P(E)) & & \mathrm{Hom}_{\mathcal{D}_{\mathrm{et}} \otimes B_{\mathrm{cris}}(V)}(P(L) \otimes B_{\mathrm{cris}}(V), P(L) \otimes B_{\mathrm{cris}}(V)) \\ {\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle \simeq} \\ H^0_{\mathrm{dR}}(P(E) \otimes P(E)^*) & \xrightarrow{\iota} & H^0(P(L) \otimes P(L)^*) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V). \end{array} \tag{7.4.7}$$

Similarly there is a functor

$$\tilde{\eta} : \mathcal{D}'_{\mathrm{et}} \longrightarrow \mathcal{D}_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V) \tag{7.4.8}$$

sending $(Q, \ell) \in \mathcal{D}'_{\mathrm{et}}$ to the image of

$$\iota^{-1}(\ell) : Q(E) \otimes B_{\mathrm{cris}}(V) \to Q(E) \otimes B_{\mathrm{cris}}(V). \tag{7.4.9}$$

We leave it to the reader to verify that the resulting functors

$$\theta : \mathcal{D}_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V) \to \mathcal{D}_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V), \quad \eta : \mathcal{D}_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \to \mathcal{D}_{\mathrm{et}} \otimes_K B_{\mathrm{cris}}(V) \tag{7.4.10}$$

are inverses and that θ has the required properties. The uniqueness statement follows from the equivalence $\mathcal{D}_{\mathrm{dR}} \simeq \mathcal{D}'_{\mathrm{dR}}$ and [Sa, II.1.5.3.1]. □

The association between $\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})$ *and* $\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})$.

7.5. Let $\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})$ be the ind–isocrystal defined in 4.35 and $\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})$ the ind–smooth sheaf on $X^o_{\overline{K}}$ defined in 5.28.

Because pullback by Frobenius induces an auto–equivalence of $\langle E \rangle_{\otimes}$, the ind–isocrystal $\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})$ has a natural F–isocrystal structure $\varphi_{\mathcal{O}_{G_{\mathrm{dR}}}}$. This F–isocrystal structure $\varphi_{\mathcal{O}_{G_{\mathrm{dR}}}}$ can be described as follows. By [Sa, II.2.3.2.1], the isocrystal $\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})$ represents the functor on $\langle E \rangle_{\otimes}$ which to any $E' \in \langle E \rangle_{\otimes}$ associates $\omega_x(E')^*$. Since pullback by Frobenius induces an auto–equivalence on $\langle E \rangle_{\otimes}$, there exists for any $E' \in \langle E \rangle_{\otimes}$ a pair (E'', s), where $E'' \in \langle E \rangle_{\otimes}$

and $s : F^*E'' \simeq E'$ is an isomorphism. Moreover, by the full faithfulness of F^* the pair (E'', s) is unique up to unique isomorphism. Hence we have canonical isomorphisms

$$\mathrm{Hom}(E', F^*\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})) \simeq F^*\mathrm{Hom}(E'', \mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})) \simeq F^*\omega_x(E'')^* \simeq \omega_x(E')^*. \tag{7.5.1}$$

Therefore $\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})$ and $F^*\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})$ represent the same functor and hence are canonically isomorphic. This canonical isomorphism is $\varphi_{\mathcal{O}_{G_{\mathrm{dR}}}}$.

7.6. Let r be the rank of E and L, and let M_{dR} (resp. M_{et}) be the dual of $\bigwedge^r E$ (resp. $\bigwedge^r L$). Set

$$\mathcal{A}_{\mathrm{dR}} := \mathrm{Sym}^{\bullet}(E \otimes E(x)^*) \otimes \mathrm{Sym}^{\bullet}(M_{\mathrm{dR}} \otimes M_{\mathrm{dR}}(x)^*), \tag{7.6.1}$$

$$\mathcal{A}_{\mathrm{et}} := \mathrm{Sym}^{\bullet}(L \otimes L_x^*) \otimes \mathrm{Sym}^{\bullet}(M_{\mathrm{et}} \otimes M_{\mathrm{et},x}^*). \tag{7.6.2}$$

The sheaf $\mathcal{A}_{\mathrm{dR}}$ (resp. $\mathcal{A}_{\mathrm{et}}$) has a natural action of G_{dR} (resp. G_{et}) induced by the G_{dR}-action (resp. G_{et}–action) on $E(x)^*$ and $M_{\mathrm{dR}}(x)^*$ (resp. L_x^* and $M_{\mathrm{et},x}^*$). The determinant map $\det : \mathrm{End}(E(x)) \rightarrow \mathrm{End}(M_{\mathrm{dR}}(x)^*)$ (resp. $\det : \mathrm{End}(L_{\bar{x}}) \rightarrow \mathrm{End}(M_{\mathrm{et},\bar{x}}^*)$) sends G_{dR} (resp. G_{et}) to $\mathrm{Aut}(M_{\mathrm{dR}}(x)^*)$ (resp. $\mathrm{Aut}(M_{\mathrm{et},\bar{x}}^*)$). Composing with the natural isomorphism $\mathrm{Aut}(M_{\mathrm{dR}}(x)^*) \rightarrow \mathrm{Aut}(M_{\mathrm{dR}}(x)^*)$ (resp. $\mathrm{Aut}(M_{\mathrm{et},\bar{x}}^*) \rightarrow \mathrm{Aut}(M_{\mathrm{et},\bar{x}}^*)$) sending an automorphism to its inverse (note that these group schemes are abelian) we obtain homomorphisms $t_{\mathrm{dR}} : G_{\mathrm{dR}} \rightarrow \mathrm{Aut}(M_{\mathrm{dR}}(x)^*)$ (resp. $t_{\mathrm{et}} : G_{\mathrm{et}} \rightarrow \mathrm{Aut}(M_{\mathrm{et},\bar{x}}^*)$). As in 7.9, the maps t_{dR} and t_{et} induce maps

$$M_{\mathrm{dR}} \otimes M_{\mathrm{dR}}(x)^* \rightarrow \mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}), \quad M_{\mathrm{et}} \otimes M_{\mathrm{et},x} \rightarrow \mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}}). \tag{7.6.3}$$

which in turn induce surjections

$$\mathcal{A}_{\mathrm{dR}} \rightarrow \mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}), \quad \mathcal{A}_{\mathrm{et}} \rightarrow \mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}}). \tag{7.6.4}$$

Note that since L and M_{et} are crystalline sheaves associated to E and M_{dR} (with their natural filtered F–isocrystal structures), the sheaf $\mathcal{A}_{\mathrm{et}}$ is an ind–crystalline sheaf associated to $\mathcal{A}_{\mathrm{dR}}$. In particular there is a natural filtration on $\mathcal{A}_{\mathrm{dR}}$ and we define $\mathrm{Fil}_{\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})}$ to be the image of this filtration under 7.6.4.

Proposition 7.7. *The sheaf* $\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})$ *is associated to* $(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}), \mathrm{Fil}_{\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})}, \varphi_{\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})})$.

Remark 7.8. We are abusing language in the statement of 7.7 as

$$\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}}) \quad \text{and} \quad (\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}), \mathrm{Fil}_{\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})}, \varphi_{\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})}) \tag{7.8.1}$$

are only ind–objects in the categories of smooth $\mathbb{Q}_p$–sheaves and $MF_X^{\nabla}(\Phi)$ respectively. An association between such ind–sheaves should be interpreted as saying that we can write $\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})$ as an inductive limit $\varinjlim L_i$ of crystalline sheaves such that if $(E_i, \mathrm{Fil}_{E_i}, \varphi_{E_i})$ is the corresponding inductive system in $MF_X^{\nabla}(\Phi)$ then

$$(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}), \mathrm{Fil}_{\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})}, \varphi_{\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})}) \simeq \varinjlim(E_i, \mathrm{Fil}_{E_i}, \varphi_{E_i}). \tag{7.8.2}$$

The proof of 7.7 will be in several steps 7.9–7.14.

7.9. Let $\mathcal{D}$ be a Tannakaian category over some field Υ of characteristic 0. Assume $\mathcal{D} = \langle E \rangle_{\otimes}$ for some $E \in \mathcal{D}$ and that

$$\omega, \omega' : \mathcal{D} \longrightarrow \mathrm{Mod}_R \tag{7.9.1}$$

are two fiber functors, where R is a Υ–algebra. Let $\mathcal{D}_R$ denote the base change $\mathcal{D}\otimes_\Upsilon R$ [Sa, II.1.5.2]. Recall that $\mathcal{D}_R$ is the category of pairs (V,ρ), where V is an ind–object in $\mathcal{D}$ and $\rho : R \to \mathrm{End}_{\mathcal{D}}(L)$ is a Υ–algebra homomorphism. Any fiber functor $\eta : \mathcal{D} \to \mathrm{Mod}_R$ induces a unique functor (which we denote by the same letter) $\eta : \mathcal{D}_R \to \mathrm{Mod}_R$ sending (V,ρ) to $\eta(V)\otimes_{R\otimes_\Upsilon R} R$, where the second factor of $R\otimes_\Upsilon R$ acts on $\eta(V)$ via ρ and the tensor product is taken via the diagonal map $R\otimes_\Upsilon R\to R$ [Sa, II.1.5.3.2].

Let $H = \pi_1(\mathcal{D},\omega)$ and let $\mathcal{O}_H$ be its coordinate ring. The module $\mathcal{O}_H$ has a structure of a (H,H)–module induced by left and right translation. There is a natural inclusion $H \subset GL(\omega(E))$ inducing a surjection

$$\mathrm{Sym}^\bullet(\omega(E)\otimes_R\omega(E)^*)_{\mathrm{det}} \longrightarrow \mathcal{O}_H \tag{7.9.2}$$

compatible with the left and right H–actions. The left action of H corresponds by Tannaka duality to a morphism

$$\mathrm{Sym}^\bullet(E\otimes_\Upsilon\omega(E)^*)_{\mathrm{det}} \longrightarrow \mathbb{L}(\mathcal{O}_H) \tag{7.9.3}$$

of objects of $\mathcal{D}_R$ with right H–action. Applying ω' we obtain a surjection of algebras with right H–action

$$\mathrm{Sym}^\bullet(\omega'(E)\otimes_R\omega(E)^*)_{\mathrm{det}} \longrightarrow \omega'(\mathbb{L}(\mathcal{O}_H)). \tag{7.9.4}$$

The left hand side of 7.9.4 with its right H–action is the coordinate ring of the R–scheme $\underline{\mathrm{Isom}}(\omega'(E),\omega(E))$ with its natural left H–action coming from the action on $\omega(E)$. The algebra $\omega'(\mathbb{L}(\mathcal{O}_H))$ thus is obtained from an H–invariant closed subscheme

$$T\subset \underline{\mathrm{Isom}}(\omega'(E),\omega(E)). \tag{7.9.5}$$

Lemma 7.10. *The subscheme $T\subset \underline{\mathrm{Isom}}(\omega'(E),\omega(E))$ is equal to*

$$\underline{\mathrm{Isom}}^\otimes(\omega',\omega)\subset \underline{\mathrm{Isom}}(\omega'(E),\omega(E)). \tag{7.10.1}$$

Proof. By [Sa, II.2.3.2.1], the functor $\mathrm{Hom}_{\mathcal{D}_R}(\cdot,\mathbb{L}(\mathcal{O}_H))$ with its right H–action is the functor

$$\mathcal{D}_R \longrightarrow (\text{right } H\text{–modules}) \tag{7.10.2}$$

sending V to $\mathrm{Hom}_R(\omega(V),R)$. It follows that for any $V\in\mathcal{D}$ and $W\in\mathcal{D}_R$ there is a natural isomorphism of right H–modules

$$\mathrm{Hom}_{\mathcal{D}_R}(W,V^*\otimes_\Upsilon\mathbb{L}(\mathcal{O}_H))\simeq\omega(W)^*\otimes_R\omega(V)^*\simeq\mathrm{Hom}_{\mathcal{D}_R}(W,\mathbb{L}(\mathcal{O}_H)\otimes_R\omega(V)^*). \tag{7.10.3}$$

By Yoneda's lemma, this isomorphism is obtained from an isomorphism $V^*\otimes_\Upsilon\mathbb{L}(\mathcal{O}_H)\simeq\mathbb{L}(\mathcal{O}_H)\otimes_R\omega(V)^*$ of objects in $\mathcal{D}_R$ with right H–action.

Thus we find that for any $V\in\mathcal{D}$ there is a natural isomorphism

(7.10.4)

$$\mathrm{Hom}_R(\omega'(V),\omega'(\mathbb{L}(\mathcal{O}_H)))\simeq\omega'(V^*\otimes_\Upsilon\mathbb{L}(\mathcal{O}_H))\simeq\omega'(\mathbb{L}(\mathcal{O}_H)\otimes_R\omega(V)^*)\simeq\omega'(\mathbb{L}(\mathcal{O}_H))\otimes_R\omega(V)^*.$$

In other words, for every $V\in\mathcal{D}$ there is a natural isomorphism

$$\omega'(V)^*|_{\mathrm{Spec}(\omega'(\mathbb{L}(\mathcal{O}_H)))}\simeq\omega(V)^*|_{\mathrm{Spec}(\omega'(\mathbb{L}(\mathcal{O}_H)))}. \tag{7.10.5}$$

This defines a map

$$\mathrm{Spec}(\omega'(\mathbb{L}(\mathcal{O}_H)))\longrightarrow\underline{\mathrm{Isom}}^\otimes(\omega',\omega) \tag{7.10.6}$$

over R. By construction this map is compatible with the H–actions and the inclusions into $\underline{\mathrm{Isom}}(\omega'(E),\omega(E))$, and since both are H–torsors it is an isomorphism. $\square$

7.11. Let $U = \mathrm{Spec}(R) \to X$ be a small étale morphism and fix a geometric generic point $\mathrm{Spec}(\Omega) \to \mathrm{Spec}(R_K)$. Define

$$\omega^{\mathrm{dR}}_{B_{\mathrm{cris}}(U^\wedge)} : \mathcal{D}_{\mathrm{dR}} \longrightarrow \mathrm{Mod}_{B_{\mathrm{cris}}(U^\wedge)}, \quad F \mapsto F(B_{\mathrm{cris}}(U^\wedge)), \tag{7.11.1}$$

and

$$\omega^{\mathrm{et}}_{B_{\mathrm{cris}}(U^\wedge)} : \mathcal{D}_{\mathrm{et}} \longrightarrow \mathrm{Mod}_{B_{\mathrm{cris}}(U^\wedge)} \tag{7.11.2}$$

to be the functor sending $A \in \mathcal{D}_{\mathrm{et}}$ to the $\mathbb{Q}_p$–vector space which is the stalk of A at $\mathrm{Spec}(\Omega) \to \mathrm{Spec}(R_K)$ tensored with $B_{\mathrm{cris}}(U^\wedge)$ (here we write $B_{\mathrm{cris}}(U^\wedge)$ for the ring obtained by applying the construction of 6.2 using the chosen geometric generic point of U). Here $\mathcal{D}_{\mathrm{dR}}$ and $\mathcal{D}_{\mathrm{et}}$ are as in 7.4.

Lemma 7.12. *The two tensor functors*

$$\omega_x \otimes_K B_{\mathrm{cris}}(V), (\omega_{\bar{x}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)) \circ \theta : \mathcal{D}_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V) \longrightarrow \mathrm{Mod}_{B_{\mathrm{cris}}(V)} \tag{7.12.1}$$

are naturally isomorphic. Similarly the two functors

$$\omega^{\mathrm{dR}}_{B_{\mathrm{cris}}(U^\wedge)}, \omega^{\mathrm{et}}_{B_{\mathrm{cris}}(U^\wedge)} \circ \theta : \mathcal{D}_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V) \longrightarrow \mathrm{Mod}_{B_{\mathrm{cris}}(U^\wedge)} \tag{7.12.2}$$

are naturally isomorphic.

Proof. Observe first that the functor $\omega_x \otimes_K B_{\mathrm{cris}}(V)$ is isomorphic to the functor which to any $F \in \mathcal{D}_{\mathrm{dR}}$ associates $x^*F(B_{\mathrm{cris}}(V))$. Let $\mathcal{D}'_{\mathrm{dR}}$ be as in 7.3. For any $(P, e) \in \mathcal{D}'_{\mathrm{dR}}$, the image under $\omega_x \otimes_K B_{\mathrm{cris}}(V)$ of (P, e) is equal to the image of the map

$$e : x^*P(E)(B_{\mathrm{cris}}(V)) \longrightarrow x^*P(E)(B_{\mathrm{cris}}(V)) \tag{7.12.3}$$

induced by e. The value of $(\omega_{\bar{x}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)) \circ \theta$ on (P, e) is equal to the image of the map

$$\theta(e) : P(L)_{\bar{x}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \longrightarrow P(L)_{\bar{x}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \tag{7.12.4}$$

induced by e under the isomorphism $H^0_{\mathrm{et}}(P(L) \otimes P(L)^*) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \simeq H^0_{\mathrm{dR}}(P(E) \otimes P(E)^*) \otimes_K B_{\mathrm{cris}}(V)$ obtained from the association ι. From the construction of pullback of associations 6.19, it follows that ι induces an isomorphism $\lambda : x^*P(E)(B_{\mathrm{cris}}(V)) \simeq P(L)_{\bar{x}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$ such that the diagram

$$\begin{tikzcd} x^*P(E)(B_{\mathrm{cris}}(V)) \arrow[r, "\lambda"] \arrow[d, "e"'] & P(L)_{\bar{x}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \arrow[d, "\theta(e)"] \\ x^*P(E)(B_{\mathrm{cris}}(V)) \arrow[r, "\lambda"] & P(L)_{\bar{x}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \end{tikzcd} \tag{7.12.5}$$

commutes. This gives the isomorphism between the functors in 7.12.1. A similar argument left to the reader defines an isomorphism between the functors in 7.12.2. $\square$

7.13. Let ι denote the association between $\mathcal{A}_{\mathrm{et}}$ and $\mathcal{A}_{\mathrm{dR}}$. We claim that there is a unique association $\bar{\iota}$ between $(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}), \mathrm{Fil}_{\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})}, \varphi_{\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})})$ and $\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})$ such that for every very small $U \to X$ the induced diagram of Galois modules

$$\begin{tikzcd} \mathcal{A}_{\mathrm{dR}}(B_{\mathrm{cris}}(U^\wedge)) \arrow[r] \arrow[d, "\iota"'] & \mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})(B_{\mathrm{cris}}(U^\wedge)) \arrow[d, "\bar{\iota}"] \\ \mathcal{A}_{\mathrm{et}} \otimes B_{\mathrm{cris}}(U^\wedge) \arrow[r] & \mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}}) \otimes B_{\mathrm{cris}}(U^\wedge) \end{tikzcd} \tag{7.13.1}$$

commutes. Observe that any such $\bar{\iota}$ is automatically compatible with Frobenius, the Galois action, and the filtrations.

To prove the existence of $\bar{\iota}$, note that by 7.10 to give the map $\bar{\iota}$ is equivalent to giving an isomorphism $\bar{\iota}^*$ such that the diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}^{\otimes}(\omega^{\mathrm{dR}}_{B_{\mathrm{cris}}(U^\wedge)}, \omega_x \otimes_K B_{\mathrm{cris}}(U^\wedge)) \arrow[r] \arrow[d, "\bar{\iota}^*"'] & \underline{\mathrm{Isom}}(\omega^{\mathrm{dR}}_{B_{\mathrm{cris}}(U^\wedge)}(E), \omega_x(E) \otimes_K B_{\mathrm{cris}}(U^\wedge)) \arrow[d, "\iota"] \\ \underline{\mathrm{Isom}}^{\otimes}(\omega^{\mathrm{et}}_{B_{\mathrm{cris}}(U^\wedge)}, \omega_{\bar{x}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(U^\wedge)) \arrow[r] & \underline{\mathrm{Isom}}(\omega^{\mathrm{et}}_{B_{\mathrm{cris}}(U^\wedge)}(L), \omega_{\bar{x}}(L) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(U^\wedge)) \end{tikzcd} \tag{7.13.2}$$

commutes. By construction, the isomorphism ι^* is that induced by the isomorphisms in 7.12 together with the identifications

$$\omega_x \otimes_K B_{\mathrm{cris}}(U^\wedge) \simeq \omega_x \otimes_K B_{\mathrm{cris}}(V) \otimes_{B_{\mathrm{cris}}(V)} B_{\mathrm{cris}}(U^\wedge), \tag{7.13.3}$$

$$\omega_{\bar{x}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(U^\wedge) \simeq \omega_{\bar{x}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \otimes_{B_{\mathrm{cris}}(V)} B_{\mathrm{cris}}(U^\wedge). \tag{7.13.4}$$

The existence of $\bar{\iota}^*$ therefore follows from 7.12.

Corollary 7.14. *The sheaf* $\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})$ *is an ind–crystalline sheaf on* X^o_K.

Proof. With notation as in 7.6, let $F_s \subset \mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})$ be the sub–isocrystal which is the image of $\oplus_{i+j\leq s}\mathrm{Sym}^i(E \otimes E(x)^*) \otimes \mathrm{Sym}^j(M_{\mathrm{dR}} \otimes M_{\mathrm{dR}}(x)^*)$, and let $L_s \subset \mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})$ be the image of $\oplus_{i+j\leq s}\mathrm{Sym}^i(L \otimes L^*_{\bar{x}}) \otimes \mathrm{Sym}^j(M_{\mathrm{et}} \otimes M_{\mathrm{et},x})$. Then by construction the association between $\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})$ and $\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})$ induces an association between L_s and F_s. Since $\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})$ is equal to $\varinjlim L_s$ the corollary follows. ☐

This completes the proof of 7.7. ☐

Corollary 7.15. *There is a natural isomorphism* $G_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V) \simeq G_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$ *and the association of* 7.7 *is compatible with the* G_{dR} *and* G_{et} *actions.*

Proof. This follows from 7.12 and the construction. ☐

Corollary 7.16. *There is a natural isomorphism*

$$\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})) \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq GC(L) \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \tag{7.16.1}$$

in $\mathrm{Ho}(G_{\widetilde{B}_{\mathrm{cris}}(V)} - \mathrm{dga}_{\widetilde{B}_{\mathrm{cris}}(V),/\mathcal{O}_{G_{\widetilde{B}_{\mathrm{cris}}(V)}}})$*, where* $G_{\widetilde{B}_{\mathrm{cris}}(V)}$ *denotes* $G_{\mathrm{dR}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq G_{\mathrm{et}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V)$.

Proof. This follows from 6.13–6.20. ☐

Theorem 1.7 follows from this corollary, the construction of $X_{\mathcal{C}_{\mathrm{dR}}}$ and $X_{\mathcal{C}_{\mathrm{et}}}$, and 4.11. ☐

Remark 7.17. The assumption that the groups G_{dR} and G_{et} are reductive could be eliminated if we knew that any smooth subsheaf of a crystalline sheaf is again crystalline. For then using the filtration defined by the unipotent radical as in 5.30, one can reduce to the case when G_{dR} and G_{et} are reductive. Recent work of Tsuji in this direction (private communication) may enable one to remove this assumption.

Proof of 1.8.

The isomorphism $X_{\mathcal{C}_{\mathrm{dR}}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq X_{\mathcal{C}_{\mathrm{et}}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V)$ induces an isomorphism
(7.17.1)
$$\pi_1(\mathcal{C}_{\mathrm{dR}}, x) \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq \pi_1(X_{\mathcal{C}_{\mathrm{dR}}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V)) \simeq \pi_1(X_{\mathcal{C}_{\mathrm{et}}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V)) \simeq \pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x}) \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V).$$
Equivalently we obtain an isomorphism of Hopf algebras

$$\mathcal{O}_{\pi_1(\mathcal{C}_{\mathrm{dR}}, x)} \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq \mathcal{O}_{\pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x})} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \tag{7.17.2}$$

compatible with Frobenius and the Galois actions. By D.3 this isomorphism is obtained from an isomorphism of Hopf algebras

$$\mathcal{O}_{\pi_1(\mathcal{C}_{\mathrm{dR}}, x)} \otimes_K B_{\mathrm{cris}}(V) \simeq \mathcal{O}_{\pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x})} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \tag{7.17.3}$$

compatible with Frobenius and Galois. This implies 1.8.

Proof of 1.9.

Since $\pi_i(X_{\mathcal{C}_{\mathrm{dR}}})$ and $\pi_i(X_{\mathcal{C}_{\mathrm{et}}})$ are pro–algebraic group schemes and $\pi_i(X_{\mathcal{C}_{\mathrm{dR}}}) \otimes_K B_{\mathrm{cris}}(V) \simeq \pi_i(X_{\mathcal{C}_{\mathrm{et}}}) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$, Theorem 1.9 follows from the following lemma:

Lemma 7.18. *Let $\mathcal{G}_{\mathrm{et}}/\mathbb{Q}_p$ be a pro–algebraic group scheme with action of* $\mathrm{Gal}(\overline{K}/K)$ *and assume $\mathcal{G}_{\mathrm{dR}}/K$ is another pro–algebraic group scheme with an isomorphism*

$$\iota : \mathcal{G}_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \simeq \mathcal{G}_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V) \tag{7.18.1}$$

compatible with the actions of $\mathrm{Gal}(\overline{K}/K)$*. Then* $\mathrm{Lie}(\mathcal{G}_{\mathrm{et}})$ *can be written as a projective limit* $\varprojlim L_i$ *of finite dimensional $\mathbb{Q}_p$–Lie algebras with* $\mathrm{Gal}(\overline{K}/K)$*–action such that each L_i is a crystalline representation.*

Proof. Let S_{et} (resp. S_{dR}) denote $I_{\mathrm{et}}/I_{\mathrm{et}}^2$ (resp. $I_{\mathrm{dR}}/I_{\mathrm{dR}}^2$), where I_{et} (resp. I_{dR}) denotes the ideal of the identity in $\mathcal{O}_{\mathcal{G}_{\mathrm{et}}}$ (resp. $\mathcal{O}_{\mathcal{G}_{\mathrm{dR}}}$). Then $\mathrm{Lie}(\mathcal{G}_{\mathrm{et}})$ (resp. $\mathrm{Lie}(\mathcal{G}_{\mathrm{dR}})$) is equal to the dual of S_{et} (resp. S_{dR}).

The isomorphism ι induces an isomorphism

$$\iota^* : S_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \simeq S_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V) \tag{7.18.2}$$

compatible with the $\mathrm{Gal}(\overline{K}/K)$–actions. Write $S_{\mathrm{dR}} = \varinjlim M_j$ where each M_j is a finite–dimensional K–vector space, and set

$$V_j := S_{\mathrm{et}} \cap (M_j \otimes_K B_{\mathrm{cris}}(V)) \subset S_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V). \tag{7.18.3}$$

Then $V_j \subset S_{\mathrm{et}}$ is Galois stable and finite–dimensional since $V_j \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$ injects into $M_j \otimes_K B_{\mathrm{cris}}(V)$ and hence has finite rank (since $B_{\mathrm{cris}}(V)$ is an integral domain for example by [Ts1, A3.3]). Furthermore $S_{\mathrm{et}} = \varinjlim V_j$.

The Lie algebra structure on $\mathrm{Lie}(\mathcal{G}_{\mathrm{et}})$ is given by the dual of a map

$$\rho : S_{\mathrm{et}} \longrightarrow S_{\mathrm{et}} \otimes S_{\mathrm{et}}. \tag{7.18.4}$$

Furthermore, to give a quotient $\mathrm{Lie}(\mathcal{G}_{\mathrm{et}}) \to L$ of Lie algebras is equivalent to giving a subspace $L^* \subset S_{\mathrm{et}}$ such that $\rho(L^*)$ is contained in $L^* \otimes L^*$. Since $\mathcal{G}_{\mathrm{et}}$ is pro–algebraic, we can write $S_{\mathrm{et}} = \varinjlim S^i$ where each $S^i \subset S_{\mathrm{et}}$ is a finite–dimensional subspace and $\rho(S^i) \subset S^i \otimes S^i$. Let $\overline{S}^i$ be the intersection of all sub–Galois representations of S_{et} containing S^i. Since $S^i \subset \rho^{-1}(\overline{S}^i \otimes \overline{S}^i)$, we have $\rho(\overline{S}^i) \subset \overline{S}^i \otimes \overline{S}^i$. Also, since $S_{\mathrm{et}} = \varinjlim V_j$ and S^i is finite dimensional we must have S^i, and hence also $\overline{S}^i$, contained in some V_j. This implies that $\overline{S}^i$ is finite dimensional. Dualizing we obtain the first part of the lemma. The second statement follows from the following which shows that each of the $\overline{S}^i$ is a crystalline representation. □

Sub-Lemma 7.19. *Let $W' \subset W$ be an inclusion of $\mathrm{Gal}(\overline{K}/K)$–representations which are direct limits of finite dimensional Galois representations. Then the diagram*

(7.19.1)
$$\begin{array}{ccc} \mathbf{D}(W')\otimes_K B_{\mathrm{cris}}(V) & \longrightarrow & \mathbf{D}(W)\otimes_K B_{\mathrm{cris}}(V) \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ W'\otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) & \longrightarrow & W\otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \end{array}$$

is cartesian. In particular, if α is an isomorphism and W' is finite dimensional, then W' is crystalline.

Proof. Let $W'' = W/W'$ and consider the diagram

(7.19.2)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{D}(W')\otimes_K B_{\mathrm{cris}}(V) & \longrightarrow & \mathbf{D}(W)\otimes_K B_{\mathrm{cris}}(V) & \xrightarrow{\pi} & \mathbf{D}(W'')\otimes_K B_{\mathrm{cris}}(V) & & \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \longrightarrow & W'\otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) & \longrightarrow & W\otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) & \longrightarrow & W''\otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) & \longrightarrow & 0. \end{array}$$

By [Fo1, 5.1.2 (ii)] all the vertical maps are injective. From this and a diagram chase the result follows. □

Proof of 1.12.

Let M (resp. S) be a representation of $\pi_1(X_{\mathcal{C}_{\mathrm{dR}}})\otimes \widetilde{B}_{\mathrm{cris}}(V)$ (resp. $\pi_1(X_{\mathcal{C}_{\mathrm{et}}})\otimes \widetilde{B}_{\mathrm{cris}}(V)$) over $\widetilde{B}_{\mathrm{cris}}(V)$.

Lemma 7.20. *There is a natural isomorphism*

(7.20.1)
$$H^*(X_{\mathcal{C}_{\mathrm{dR}}}\otimes_K \widetilde{B}_{\mathrm{cris}}(V), M) \simeq H^*(X_{\mathcal{C}_{\mathrm{dR}}}, M^f) \quad (\textit{resp. } H^*(X_{\mathcal{C}_{\mathrm{et}}}\otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V), S) \simeq H^*(X_{\mathcal{C}_{\mathrm{et}}}, S^f)),$$

where M^f (resp. S^f) denotes the representation of $\pi_1(X_{\mathcal{C}_{\mathrm{dR}}})$ (resp. $\pi_1(X_{\mathcal{C}_{\mathrm{et}}})$) obtained by viewing M as a K–vector space (resp. $\mathbb{Q}_p$–vector space).

Proof. We give the proof for $\mathcal{C}_{\mathrm{dR}}$, leaving the proof for $\mathcal{C}_{\mathrm{et}}$ to the reader (using the same argument).

Write π for $\pi_1(X_{\mathcal{C}_{\mathrm{dR}}})$ and $\pi_{\widetilde{B}_{\mathrm{cris}}(V)}$ for the base change to $\widetilde{B}_{\mathrm{cris}}(V)$. The projection $X_{\mathcal{C}_{\mathrm{dR}}} \rightarrow \tau_{\leq 1}X_{\mathcal{C}_{\mathrm{dR}}} \simeq B\pi$ gives $X_{\mathcal{C}_{\mathrm{dR}}}$ the structure of an object of $\mathrm{Ho}(\mathrm{SPr}_*(K)|_{B\pi})$. Let Aff_K^π denote the site whose objects are affine K–schemes and for which a morphism $T' \rightarrow T$ in Aff_K^π is a pair $(f, g \in \pi(T'))$, where $f : T' \rightarrow T$ is a K–morphism and $g \in \pi(T')$ is an element. If $(s, h \in \pi(T''))$ is a morphism $T'' \rightarrow T'$ for some third object $T' \in \mathrm{Aff}_K$, then the composite $(s,h)\circ(f,g)$ is the morphism $(s\circ f, h\cdot s^*(g))$. We view Aff_K^π as a site by declaring a morphism (f,g) to be a covering if f is faithfully flat and quasi–compact. The category of presheaves on Aff_K^π is naturally equivalent to the category of presheaves on Aff_K with action of the sheaf π, and by [KPT, 1.2.1] there is a natural equivalence

(7.20.2)
$$\mathrm{Ho}(\mathrm{SPr}_*(K)|_{B\pi}) \simeq \mathrm{Ho}(\mathrm{SPr}_*(\mathrm{Aff}_K^\pi)).$$

Let $F_{\mathrm{dR}} \in \mathrm{Ho}(\mathrm{SPr}_*(\mathrm{Aff}^{\pi}_K))$ be the object corresponding to $X_{\mathcal{C}_{\mathrm{dR}}}$.

Replacing K by $\widetilde{B}_{\mathrm{cris}}(V)$ in the above we obtain a site $\mathrm{Aff}^{\pi_{\widetilde{B}_{\mathrm{cris}}(V)}}_{\widetilde{B}_{\mathrm{cris}}(V)}$ and an equivalence

$$\mathrm{Ho}(\mathrm{SPr}_*(\widetilde{B}_{\mathrm{cris}}(V))|_{B\pi_{\widetilde{B}_{\mathrm{cris}}(V)}}) \simeq \mathrm{Ho}(\mathrm{SPr}_*(\mathrm{Aff}^{\pi_{\widetilde{B}_{\mathrm{cris}}(V)}}_{\widetilde{B}_{\mathrm{cris}}(V)})). \tag{7.20.3}$$

There is a morphism of sites $\mathrm{Aff}^{\pi}_K \to \mathrm{Aff}^{\pi_{\widetilde{B}_{\mathrm{cris}}(V)}}_{\widetilde{B}_{\mathrm{cris}}(V)}$ sending $T \to \mathrm{Spec}(K)$ to $T \otimes_K \widetilde{B}_{\mathrm{cris}}(V)$. We therefore have adjoint functors

$$f^* : \mathrm{SPr}_*(\mathrm{Aff}^{\pi}_K) \to \mathrm{SPr}_*(\mathrm{Aff}^{\pi_{\widetilde{B}_{\mathrm{cris}}(V)}}_{\widetilde{B}_{\mathrm{cris}}V}), \quad f_* : \mathrm{SPr}_*(\mathrm{Aff}^{\pi_{\widetilde{B}_{\mathrm{cris}}(V)}}_{\widetilde{B}_{\mathrm{cris}}V}) \to \mathrm{SPr}_*(\mathrm{Aff}^{\pi}_K). \tag{7.20.4}$$

For any $F \in \mathrm{SPr}_*(\mathrm{Aff}^{\pi}_K)$ the pullback f^*F is the simplicial presheaf

$$(W \to \mathrm{Spec}(\widetilde{B}_{\mathrm{cris}}(V))) \mapsto F(W \to \mathrm{Spec}(\widetilde{B}_{\mathrm{cris}}(V)) \to \mathrm{Spec}(K)). \tag{7.20.5}$$

In particular, f^* preserves cofibrations and equivalences so the pair (f^*, f_*) is a Quillen adjunction and f^* derives trivially.

It follows from the construction of the equivalence 7.20.3 that f^*F_{dR} corresponds under the equivalence 7.20.3 to $X_{\mathcal{C}_{\mathrm{dR}}} \otimes \widetilde{B}_{\mathrm{cris}}(V)$. The representation M of π corresponds to a sheaf $\mathcal{M}$ on $\mathrm{Aff}^{\pi_{\widetilde{B}_{\mathrm{cris}}(V)}}_{\widetilde{B}_{\mathrm{cris}}(V)}$ and the induced representation M^f of π corresponds to the sheaf $f_*\mathcal{M}$. Fix an integer m and let $K(\mathcal{M}, m) \in \mathrm{Ho}(\mathrm{SPr}_*(\mathrm{Aff}^{\pi_{\widetilde{B}_{\mathrm{cris}}(V)}}_{\widetilde{B}_{\mathrm{cris}}(V)}))$ (resp. $K(f_*\mathcal{M}, m) \in \mathrm{Ho}(\mathrm{SPr}_*(\mathrm{Aff}^{\pi}_K)))$ be the corresponding classifying stack [To1, 1.3]. By definition of cohomology [To1, 1.3] there are isomorphisms

$$H^m(X_{\mathcal{C}_{\mathrm{dR}}} \otimes \widetilde{B}_{\mathrm{cris}}(V), M) \simeq [f^*F_{\mathrm{dR}}, K(\mathcal{M}, m)]_{\mathrm{Ho}(\mathrm{SPr}_*(\mathrm{Aff}^{\pi_{\widetilde{B}_{\mathrm{cris}}(V)}}_{\widetilde{B}_{\mathrm{cris}}(V)}))}, \tag{7.20.6}$$

$$H^m(X_{\mathcal{C}_{\mathrm{dR}}}, M^f) \simeq [F_{\mathrm{dR}}, K(f_*\mathcal{M}, m)]_{\mathrm{Ho}(\mathrm{SPr}_*(\mathrm{Aff}^{\pi}_K))}. \tag{7.20.7}$$

Since

$$[f^*F_{\mathrm{dR}}, K(\mathcal{M}, m)]_{\mathrm{Ho}(\mathrm{SPr}_*(\mathrm{Aff}^{\pi_{\widetilde{B}_{\mathrm{cris}}(V)}}_{\widetilde{B}_{\mathrm{cris}}(V)}))} \simeq [F_{\mathrm{dR}}, \mathbb{R}f_*K(\mathcal{M}, m)]_{\mathrm{Ho}(\mathrm{SPr}_*(\mathrm{Aff}^{\pi}_K))} \tag{7.20.8}$$

to prove the lemma it suffices to exhibit a natural isomorphism $\mathbb{R}f_*K(\mathcal{M}, m) \simeq K(f_*\mathcal{M}, m)$.

Let $\mathcal{M} \to I^\bullet$ be an injective resolution in the category of abelian sheaves on $\mathrm{Aff}^{\pi_{\widetilde{B}_{\mathrm{cris}}(V)}}_{\widetilde{B}_{\mathrm{cris}}(V)}$. For any $T \in \mathrm{Aff}^{\pi}_K$, the complex $f_*I(T)$ computes the cohomology of $\mathcal{M}$ restricted to the site $\mathrm{Aff}^{\pi_{\widetilde{B}_{\mathrm{cris}}(V)}}_{\widetilde{B}_{\mathrm{cris}}(V)}|_{T \otimes \widetilde{B}_{\mathrm{cris}}(V)}$. Since this site is equivalent to the site of affine schemes over $T \otimes \widetilde{B}_{\mathrm{cris}}(V)$ with the fpqc topology and the restriction of $\mathcal{M}$ to this site is quasi–coherent, all higher cohomology groups are zero. Thus the complex $f_*I^\bullet$ is exact.

Let $\mathcal{I}$ be the complex $\tau_{\leq m}I^\bullet[m]$ and note that there is a natural quasi–isomorphism $\mathcal{M}[m] \to \mathcal{I}$. Applying the denormalization functor [G-J, III.2.3], we obtain an equivalence of simplicial presheaves $D(\mathcal{M}[m]) \to D(\mathcal{I})$. Since $\pi_i(\mathcal{M}[m]) \simeq H_i(\mathcal{M}[m])$ for every i [G-J, III.2.7], the simplicial presheaf $D(\mathcal{M})$, and hence also $D(\mathcal{I})$, is a representative for $K(\mathcal{M}, m)$.

The simplicial presheaf $D(\mathcal{I})$ is also fibrant in $\mathrm{SPr}_*(\mathrm{Aff}^{\pi_{\widetilde{B}_{\mathrm{cris}}(V)}}_{\widetilde{B}_{\mathrm{cris}}(V)})$. To see this note that since $D(\mathcal{I})$ is a simplicial presheaf of abelian groups, for any object $W \in \mathrm{Aff}^{\pi_{\widetilde{B}_{\mathrm{cris}}(V)}}_{\widetilde{B}_{\mathrm{cris}}(V)}$ the simplicial set $D(\mathcal{I})(W)$ is fibrant in the category of simplicial sets [G-J, I.3.4]. By [To1, 1.1.2] to prove that $D(\mathcal{I})$ is fibrant it suffices to show that for every hypercover $U_\bullet \to W$ of an object W the natural map

$$D(\mathcal{I})(W) \longrightarrow \mathrm{holim}_\Delta D(\mathcal{I})(U_n) \tag{7.20.9}$$

is an equivalence. This is done in [DHI, 7.9].

Since $D(\mathcal{I})$ is fibrant $\mathbb{R}f_*K(\mathcal{M},m)$ is isomorphic to $f_*D(\mathcal{I}) \simeq D(f_*\mathcal{I})$. On the other hand, since $f_*\mathcal{M} \to f_*I^\bullet$ is a quasi–isomorphism, the natural map $f_*\mathcal{M}[m] \to f_*\mathcal{I}$ is also a quasi–isomorphism, and hence $D(f_*\mathcal{I}) \simeq D(f_*\mathcal{M}[m])$. It follows that $\mathbb{R}f_*K(\mathcal{M},m) \simeq K(f_*\mathcal{M},m)$ as desired. □

To prove 1.12, note that by 4.28 and 5.32.2, we have

$$H^*_{\mathrm{dR}}(M) \simeq H^*(X_{\mathcal{C}_{\mathrm{dR}}}, M^f), \quad H^*_{\mathrm{et}}(S) \simeq H^*(X_{\mathcal{C}_{\mathrm{et}}}, S^f). \tag{7.20.10}$$

Hence to prove 1.12 it suffices to exhibit a natural isomorphism

$$H^*(X_{\mathcal{C}_{\mathrm{dR}}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V), M) \simeq H^*(X_{\mathcal{C}_{\mathrm{et}}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V), S). \tag{7.20.11}$$

Such an isomorphism is provided by 1.7. □

Formality and proof of 1.13.

7.21. Assume k is a finite field, $D = \emptyset$, and fix an embedding $\iota : K \hookrightarrow \mathbb{C}$. Let $(E, \mathrm{Fil}_E, \varphi_E)$ and L be as in 7.1, and assume in addition that (E, φ_E) is ι–pure in the sense of [Ke]. Denote by $H^*_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}))$ (resp. $H^*_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}}))$ the crystalline cohomology (resp. étale cohomology) of $\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})$ (resp. $\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})$). Cup–product gives $H^*_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}))$ (resp. $H^*_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})))$ the structure of a G_{dR}–equivariant (resp. G_{et}–equivariant) differential graded algebra. In [Ol1, proof of 4.25], it is shown that there is an isomorphism

$$\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})) \simeq H^*_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})) \tag{7.21.1}$$

in $\mathrm{Ho}(G_{\mathrm{dR}} - \mathrm{dga}_K)$. On the other hand, Faltings' cohomological comparison isomorphism gives an isomorphism

$$H^*_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})) \otimes_K B_{\mathrm{cris}}(V) \simeq H^*_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \tag{7.21.2}$$

in $\mathrm{Ho}(G_{B_{\mathrm{cris}}(V)} - \mathrm{dga}_{B_{\mathrm{cris}}(V)})$, where we write $G_{B_{\mathrm{cris}}(V)}$ for $G_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V) \simeq G_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$.

Theorem 7.22. *There exists an isomorphism in* $\mathrm{Ho}(G_{\widetilde{B}_{\mathrm{cris}}(V)} - \mathrm{dga}_{\widetilde{B}_{\mathrm{cris}}(V)})$ *compatible with the* $\mathrm{Gal}(\overline{K}/K)$*–actions*

$$GC(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})) \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \simeq H^*_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})) \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V). \tag{7.22.1}$$

Proof. There are isomorphisms in $\mathrm{Ho}(G_{\widetilde{B}_{\mathrm{cris}}(V)} - \mathrm{dga}_{\widetilde{B}_{\mathrm{cris}}(V)})$ compatible with the $\mathrm{Gal}(\overline{K}/K)$–actions

(7.22.2)

$$GC(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})) \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \simeq \mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})) \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq H^*_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})) \otimes_K \widetilde{B}_{\mathrm{cris}}(V)$$

and by 7.21.2

$$H^*_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})) \otimes_K B_{\mathrm{cris}}(V) \simeq H^*_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V). \tag{7.22.3}$$

□

To obtain 1.13 from this, note that the isomorphism

$$\pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x}) \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \simeq \pi_1([\mathbb{R}\mathrm{Spec}_{G_{\mathrm{et}}}(H^*(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})))/G_{\mathrm{et}}]) \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \tag{7.22.4}$$

defined by 7.22 fits into a commutative diagram of isomorphisms

$$\begin{tikzcd} \pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x}) \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \arrow[r] \arrow[d] & \pi_1([\mathbb{R}\mathrm{Spec}_{G_{\mathrm{et}}}(H^*(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})))/G_{\mathrm{et}}]) \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \arrow[d] \\ \pi_1(X_{\mathcal{C}_{\mathrm{dR}}}) \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \arrow[r] & \pi_1([\mathbb{R}\mathrm{Spec}_{G_{\mathrm{dR}}}(H^*_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})))/G_{\mathrm{dR}}]) \otimes_K \widetilde{B}_{\mathrm{cris}}(V). \end{tikzcd} \tag{7.22.5}$$

By D.3 this diagram is induced by a commutative diagram of isomorphisms over $B_{\mathrm{cris}}(V)$

$$\begin{tikzcd} \pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x}) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \arrow[r] \arrow[d] & \pi_1([\mathbb{R}\mathrm{Spec}_{G_{\mathrm{et}}}(H^*(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})))/G_{\mathrm{et}}]) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \arrow[d] \\ \pi_1(X_{\mathcal{C}_{\mathrm{dR}}}) \otimes_K B_{\mathrm{cris}}(V) \arrow[r] & \pi_1([\mathbb{R}\mathrm{Spec}_{G_{\mathrm{dR}}}(H^*_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})))/G_{\mathrm{dR}}]) \otimes_K B_{\mathrm{cris}}(V). \end{tikzcd} \tag{7.22.6}$$

We therefore get an isomorphism of Galois representations

$$\mathrm{Lie}(\pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x})) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \simeq \mathrm{Lie}(\pi_1([\mathbb{R}\mathrm{Spec}_{G_{\mathrm{et}}}(H^*(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})))/G_{\mathrm{et}}])) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V). \tag{7.22.7}$$

In particular, $\mathrm{Lie}(\pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x})) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$ is determined by $H^*_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}}))$. We also have a Galois invariant isomorphism

$$\mathrm{Lie}(\pi_1([\mathbb{R}\mathrm{Spec}_{G_{\mathrm{et}}}(H^*(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})))/G_{\mathrm{et}}])) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \simeq \mathrm{Lie}(\pi_1(X_{\mathcal{C}_{\mathrm{dR}}})) \otimes_K B_{\mathrm{cris}}(V), \tag{7.22.8}$$

and by [Ol1, 4.2]

$$\mathrm{Lie}(\pi_1(X_{\mathcal{C}_{\mathrm{dR}}})) \simeq \mathbb{L}H_1(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}))/(\text{quadratic relations}), \tag{7.22.9}$$

where $H_1(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}))$ denotes the dual of $H^1_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}))$. Base changing to $B_{\mathrm{cris}}(V)$ and using the isomorphism $H^1_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})) \otimes_K B_{\mathrm{cris}}(V) \simeq H^1_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$ we find that

$$\mathrm{Lie}(\pi_1(\mathcal{C}_{\mathrm{et}}, \bar{x})) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \simeq \mathbb{L}H_1(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)/(\text{quadratic relations}). \tag{7.22.10}$$

8. A base point free version

In this section we describe a base point free version of 1.7 and some consequences.

Review of twisted theory [Ol1, 3.7–3.28]

8.1. Recall that a *gerbe* over a site C is a stack $\mathcal{G}$ over C such that the following two conditions hold:

(i) For every object $U \in C$ there exists a covering $\{U_i \to U\}_{i\in I}$ of U such that $\mathcal{G}(U_i)$ is nonempty for all $i \in I$.
(ii) For any $U \in C$ and two objects $\alpha, \beta \in \mathcal{G}(U)$, there exists a covering $\{U_i \to U\}_{i\in I}$ such that the restrictions of α and β to each U_i are isomorphic.

8.2. If C is a site and $\mathcal{G}$ is a gerbe on C, the $\mathcal{G}$*–twisted site*, denoted $C_{\mathcal{G}}$, is the site whose objects are pairs (U, ω), where $U \in C$ is an object and $\omega \in \mathcal{G}(U)$ is an object. A morphism $(U', \omega') \to (U, \omega)$ is a pair (f, f^b), where $f : U' \to U$ is a morphism in U and $f^b : f^*(\omega) \to \omega'$ is an isomorphism in $\mathcal{G}(U')$. A collection of maps $\{(U'_i, \omega'_i) \to (U, \omega)\}$ is a covering family if the underlying family $\{U'_i \to U\}$ of maps in C is a covering family.

Note that for any object $(U,\omega) \in C_{\mathcal{G}}$ the site $C_{\mathcal{G}}|_{(U,\omega)}$ is naturally equivalent to $C|_U$.

Assume that for any object $U \in C$ and $\omega \in \mathcal{G}(U)$, the sheaf $\underline{\mathrm{Aut}}(\omega)$ on $C|_U$ which to any $f : U' \to U$ associates $\mathrm{Aut}(f^*\omega)$ is cofibrant viewed as a constant object in $\mathrm{SPr}(C|_U)$. This holds for example if $\underline{\mathrm{Aut}}(\omega)$ is representable.

Let $B\mathcal{G} \in \mathrm{SPr}(C)$ be the simplicial presheaf which to any U associates the nerve of the category $\mathcal{G}(U)$.

The following generalization of [KPT, 1.2.1] is shown in [Ol1, 3.12]:

Lemma 8.3. *There is a natural equivalence of homotopy categories*

$$\mathrm{Ho}(\mathrm{SPr}(C|_{\mathcal{G}})) \simeq \mathrm{Ho}(\mathrm{SPr}(C)|_{B\mathcal{G}}). \tag{8.3.1}$$

8.4. Let R be a $\mathbb{Q}$–algebra, and consider the site Aff_R of affine R-schemes with the fpqc topology. Following [Sa, III.2.2.2], we say that a gerbe $\mathcal{G}$ over Aff_R is *Tannakian* if fpqc locally on $\mathrm{Spec}(R)$ the gerbe $\mathcal{G}$ is isomorphic to BG for an affine and flat group scheme G. The condition that a gerbe $\mathcal{G}$ over Aff_R is Tannakian is equivalent to the following two conditions:

(i) The diagonal

$$\Delta : \mathcal{G} \to \mathcal{G} \times \mathcal{G} \tag{8.4.1}$$

is representable and affine.

(ii) There exists an fpqc covering $\mathrm{Spec}(R') \to \mathcal{G}$ for some flat R-algebra R'.

8.5. For a Tannakian gerbe over Aff_R, denote by $\mathrm{Aff}_{R,\mathcal{G}}$ be the resulting $\mathcal{G}$–twisted site. Define a sheaf $\mathcal{F}$ on $\mathrm{Aff}_{R,\mathcal{G}}$ to be quasi–coherent if for any object $(S,\omega) \in \mathrm{Aff}_{R,\mathcal{G}}$ the restriction of $\mathcal{F}$ to $\mathrm{Aff}_{R,\mathcal{G}}|_{(S,\omega)} \simeq \mathrm{Aff}_S$ is a quasi–coherent sheaf. Denote by $\mathcal{G} - \mathrm{dga}_R$ (resp. $\mathcal{G} - \mathrm{Alg}_R^{\Delta}$) the category of differential graded algebras (resp. cosimplicial algebras) in the category of quasi–coherent sheaves on $\mathrm{Aff}_{R,\mathcal{G}}$. There are natural model category structures on $\mathcal{G} - \mathrm{dga}_R$ and $\mathcal{G} - \mathrm{Alg}_R^{\Delta}$ just as in 2.21, and the Dold–Kan correspondence induces an equivalence of categories

$$\mathrm{Ho}(\mathcal{G} - \mathrm{dga}_R) \simeq \mathrm{Ho}(\mathcal{G} - \mathrm{Alg}_R^{\Delta}). \tag{8.5.1}$$

There is also a functor

$$\mathrm{Spec}_{\mathcal{G}} : (\mathcal{G} - \mathrm{Alg}_R^{\Delta})^{\mathrm{op}} \longrightarrow \mathrm{SPr}(\mathrm{Aff}_{R,\mathcal{G}}) \tag{8.5.2}$$

sending A to the simplicial presheaf

$$(\mathrm{Spec}(S),\omega) \mapsto ([n] \mapsto \mathrm{Hom}_S(A(\mathrm{Spec}(S),\omega)_n, S)). \tag{8.5.3}$$

Here the transition maps are defined as follows. If

$$(\mathrm{Spec}(S'),\omega') \to (\mathrm{Spec}(S),\omega) \tag{8.5.4}$$

is a morphism in $\mathrm{Aff}_{R,\mathcal{G}}$, then since A is quasi-coherent the natural map

$$S' \otimes_S A(\mathrm{Spec}(S),\omega) \to A(\mathrm{Spec}(S'),\omega') \tag{8.5.5}$$

is an isomorphism. For any $[n] \in \Delta$ we there obtain a map

(8.5.6)

$$\mathrm{Hom}_S(A(\mathrm{Spec}(S),\omega)_n, S) \to \mathrm{Hom}_{S'}(S' \otimes_S A(\mathrm{Spec}(S),\omega)_n, S') \simeq \mathrm{Hom}_{S'}(A(\mathrm{Spec}(S'),\omega'), S').$$

As in [KPT, p. 16] the functor 8.5.2 is right Quillen and we denote by

$$\mathbb{R}\mathrm{Spec}_{\mathcal{G}} : \mathrm{Ho}(\mathcal{G} - \mathrm{Alg}_R)^{\mathrm{op}} \longrightarrow \mathrm{Ho}(\mathrm{SPr}(\mathrm{Aff}_{R,\mathcal{G}})) \tag{8.5.7}$$

the resulting derived functor. We denote by

$$[\mathbb{R}\mathrm{Spec}_{\mathcal{G}}(-)/\mathcal{G}] : \mathrm{Ho}(\mathcal{G} - \mathrm{Alg}_R)^{\mathrm{op}} \longrightarrow \mathrm{Ho}(\mathrm{SPr}(R)|_{B\mathcal{G}}) \tag{8.5.8}$$

the composite of this functor with the equivalence 8.3.1.

Remark 8.6. The site $\mathrm{Aff}_{R,\mathcal{G}}$ is equivalent to the big fpqc site of $\mathcal{G}$, and the above notion of quasi–coherent sheaf agrees with that of [LM-B, 13.2.2].

Gerbes and 1*–truncated stacks*

8.7. Fix a base ring R. A simplicial presheaf $F \in \mathrm{SPr}(R)$ is 1*–truncated* if the map $F \to \tau_{\leq 1}F$ is an equivalence, where $\tau_{\leq 1}F$ is the simplicial presheaf sending $U \in \mathcal{C}$ to $\tau_{\leq 1}F(U)$ (see [G-J, VI.3.4]) for the definition of $\tau_{\leq n}$). Equivalently, F is 1–truncated if for any object $U \in \mathrm{Aff}_R$ and point $* \to F(U)$ the sheaves $\pi_i(F|_U, *)$ on Aff_U are zero for $i > 1$. In particular, if $F \to F'$ is an equivalence then F is 1–truncated if and only if F' is 1–truncated. Thus it makes sense to say that a stack $F \in \mathrm{Ho}(\mathrm{SPr}(R))$ is 1–truncated. Let $\mathrm{Ho}(\mathrm{SPr}^{\leq 1}(R)) \subset \mathrm{Ho}(\mathrm{SPr}(R))$ denote the full subcategory of connected 1–truncated stacks.

If $\mathcal{G}$ is a gerbe over R we obtain a 1–truncated stack $B\mathcal{G}$ by associating to any U/R the nerve of the groupoid $\mathcal{G}(U)$. Let $\underline{\mathrm{Gerbe}}_R$ denote the category whose objects are gerbes over R and whose morphisms are equivalences classes of morphisms, where $f, g : \mathcal{G} \to \mathcal{G}'$ are equivalent if there exists an isomorphism of functors $\sigma : f \to g$.

The following proposition is well–known though we were unable to find a proof in the literature.

Proposition 8.8. *The functor $\mathcal{G} \mapsto B\mathcal{G}$ defines an equivalence $\underline{\mathrm{Gerbe}}_R \simeq \mathrm{Ho}(\mathrm{SPr}^{\leq 1}(R))$.*

Proof. For a simplicial presheaf $F \in \mathrm{SPr}(R)$ such that for every $U \in \mathrm{Aff}_R$ the simplicial set $F(U)$ is fibrant (this holds for example if F is fibrant in $\mathrm{SPr}(R)$ by the definition [To1, 1.1.1]) define $\pi^{ps}(F)$ to be prestack which to any $U \in \mathrm{Aff}_R$ associates the fundamental groupoid [G-J, I.8] of $F(U)$, and let $\pi(F)$ be the associated stack. If $F \to F'$ is an equivalence of fibrant objects in $\mathrm{SPr}(R)$ then the induced morphism of stacks $\pi(F) \to \pi(F')$ is an equivalence, and hence π induces a functor

$$\pi : \mathrm{Ho}(\mathrm{SPr}^{\leq 1}(R)) \to \underline{\mathrm{Gerbe}}_R. \tag{8.8.1}$$

It follows from the construction that for F an object of $\mathrm{Ho}(\mathrm{SPr}^{\leq 1}(R))$ there is a natural equivalence $F \to B\pi(F)$. Also if $\mathcal{G}$ is a gerbe then by [G-J, I.3.5] for any $U \in \mathrm{Aff}_R$ the simplicial set $B\mathcal{G}(U)$ is fibrant, and it follows from the definitions that there is a natural equivalence $\mathcal{G} \simeq \pi B\mathcal{G}$. In particular the functor $\mathcal{G} \mapsto B\mathcal{G}$ induces an equivalence of categories. □

Equivariant algebras

8.9. Let $\mathcal{G}/R$ be a Tannakian gerbe. Denote by $\widetilde{\mathrm{Aff}}_{R,\mathcal{G}}$ the small fpqc site of $\mathcal{G}$, as defined in C.1.

Let $\mathcal{G}-\widetilde{\mathrm{dga}}_R$ (resp. $\mathcal{G}-\widetilde{\mathrm{Alg}}^{\Delta}_R$) denote the category of differential graded algebras (resp. cosimplicial algebras) in the category of sheaves of $\mathcal{O}$–modules on $\widetilde{\mathrm{Aff}}_{R,\mathcal{G}}$ whose cohomology sheaves are quasi–coherent, where $\mathcal{O}$ denotes the structure sheaf (see C.4 for the notion of a quasi-coherent sheaf in this context). The model category structure provided by 2.21 on the category of cosimplicial algebras (resp. differential graded algebras) in the category of sheaves of $\mathcal{O}$–modules on $\widetilde{\mathrm{Aff}}_{R,\mathcal{G}}$ induces a model category structure on $\mathcal{G}-\widetilde{\mathrm{dga}}_R$ (resp. $\mathcal{G}-\widetilde{\mathrm{Alg}}^{\Delta}_R$) in which a morphism $f : A \to B$ is a fibration if the morphism on underlying complexes (resp. normalized complexes) is a level–wise surjection with injective kernel. A morphism f is an equivalence if the underlying morphism of complexes of sheaves (resp. normalized complexes) is a quasi–isomorphism.

There are natural inclusions $j_{\mathrm{dga}} : \mathcal{G}-\mathrm{dga}_R \hookrightarrow \mathcal{G}-\widetilde{\mathrm{dga}}_R$ (resp. $j_{\mathrm{Alg}} : \mathcal{G}-\mathrm{Alg}^{\Delta}_R \hookrightarrow \mathcal{G}-\widetilde{\mathrm{Alg}}^{\Delta}_R$).

Lemma 8.10. *The inclusions j_{dga} and j_{Alg} have right adjoints u_{dga} and u_{Alg}.*

Proof. Recall from C.6 that the inclusion

(8.10.1) $$j : (\text{quasi–coherent sheaves on } \mathrm{Aff}_{R,\mathcal{G}}) \subset (\text{sheaves of } \mathcal{O}\text{–modules on } \widetilde{\mathrm{Aff}}_{R,\mathcal{G}})$$

has a right adjoint

(8.10.2) $$u : (\text{sheaves of } \mathcal{O}\text{–modules on } \widetilde{\mathrm{Aff}}_{R,\mathcal{G}}) \to (\text{quasi–coherent sheaves on } \mathrm{Aff}_{R,\mathcal{G}}).$$

The functor u is constructed as follows.

Since $\mathcal{G}$ is a Tannakian there exists a flat surjection $\mathrm{Spec}(R') \to \mathcal{G}$ corresponding to an object $(R', \omega') \in \mathrm{Aff}_{R,\mathcal{G}}$. Let R'' be the coordinate ring of $\mathrm{Spec}(R') \times_{\mathcal{G}} \mathrm{Spec}(R')$ and let $\mathrm{pr}_i^* : R' \to R''$ ($i = 1, 2$) be the maps induced by the two projections. Denote by $\omega' : \mathrm{Spec}(R') \to \mathcal{G}$ and $\omega'' : \mathrm{Spec}(R'') \to \mathcal{G}$ the projections. For any sheaf of $\mathcal{O}$–modules $\mathcal{F}$ on $\widetilde{\mathrm{Aff}}_{R,\mathcal{G}}$ the quasi-coherent sheaf $u\mathcal{F}$ is defined to be the equalizer of the two maps

(8.10.3) $$\omega'_*\mathcal{F}(R', \omega')^{\sim} \rightrightarrows \omega''_*\mathcal{F}(R'', \omega'')^{\sim}.$$

Observe that for two sheaves of $\mathcal{O}$–modules $\mathcal{F}$ and $\mathcal{F}'$ there is a natural map $u\mathcal{F} \otimes u\mathcal{F}' \to u(\mathcal{F} \otimes \mathcal{F}')$. It follows that the functor u extends in a natural way to the category of cosimplicial $\mathcal{O}$–modules (resp. differential graded $\mathcal{O}$–modules), and if A is a sheaf of cosimplicial algebras (resp. differential graded algebras) then uA also has a natural structure of a cosimplicial algebra (resp. differential graded algebra). It then follows from the fact that u is right adjoint to j that the resulting functors u_{dga} and u_{Alg} are right adjoint to j_{dga} and j_{Alg} respectively. □

The functor j_{dga} (resp. j_{Alg}) clearly preserves cofibrations and trivial cofibrations so the pair $(j_{\mathrm{dga}}, u_{\mathrm{dga}})$ (resp. $(j_{\mathrm{Alg}}, u_{\mathrm{Alg}})$) is a Quillen adjunction.

Proposition 8.11. *The adjunctions $(j_{\mathrm{dga}}, u_{\mathrm{dga}})$ (resp. $(j_{\mathrm{Alg}}, u_{\mathrm{Alg}})$) are Quillen equivalences.*

Proof. It is clear that a morphism $f : A \to B$ in $\mathcal{G}-\mathrm{dga}_R$ (resp. $\mathcal{G}-\mathrm{Alg}^{\Delta}_R$) is an equivalence if and only if $j_{\mathrm{dga}}(f)$ (resp. $j_{\mathrm{Alg}}(f)$) is an equivalence, so by [Ho, 1.3.16] it suffices to show that if $Y \in \mathcal{G}-\widetilde{\mathrm{dga}}_R$ (resp. $\mathcal{G}-\widetilde{\mathrm{Alg}}^{\Delta}_R$) is fibrant then the natural map $j_{\mathrm{dga}}u_{\mathrm{dga}}Y \to Y$ (resp. $j_{\mathrm{Alg}}u_{\mathrm{Alg}}Y \to Y$) is an equivalence. For this it suffices to show that if $I^\bullet$ is a complex of injective $\mathcal{O}$–modules with quasi–coherent cohomology sheaves then the natural map $juI^\bullet \to I^\bullet$ is an equivalence, where j and u are as in the proof of 8.10. This follows from C.9. □

Corollary 8.12. *The functors j_{dga} and j_{Alg} induce equivalences*

$$\mathrm{Ho}(\mathcal{G}-\mathrm{dga}_R)\simeq\mathrm{Ho}(\mathcal{G}-\widetilde{\mathrm{dga}_R}),\quad \mathrm{Ho}(\mathcal{G}-\mathrm{Alg}_R^{\Delta})\simeq\mathrm{Ho}(\mathcal{G}-\widetilde{\mathrm{Alg}_R^{\Delta}}). \tag{8.12.1}$$

In particular, the functors 8.5.7 *and* 8.5.8 *induce functors*

$$\mathbb{R}\mathrm{Spec}_{\mathcal{G}}:\mathrm{Ho}(\mathcal{G}-\widetilde{\mathrm{Alg}_R^{\Delta}})^{\mathrm{op}}\to\mathrm{Ho}(\mathrm{SPr}(\mathrm{Aff}_{R,\mathcal{G}})), \tag{8.12.2}$$

$$[\mathbb{R}\mathrm{Spec}_{\mathcal{G}}(-)/\mathcal{G}]:\mathrm{Ho}(\mathcal{G}-\widetilde{\mathrm{Alg}_R^{\Delta}})^{\mathrm{op}}\to\mathrm{Ho}(\mathrm{SPr}(R)|_{B\mathcal{G}}). \tag{8.12.3}$$

8.13. Let $(X,M_X)/V$, $(E,\mathrm{Fil}_E,\varphi_E)$ and L be as in 7.1 and assume that 7.2 holds. We do not, however, choose a point x. Let $\mathcal{D}_{\mathrm{dR}}$ (resp. $\mathcal{D}_{\mathrm{et}}$) denote $\langle E\rangle_\otimes$ (resp. $\langle L_{\overline{K}}\rangle_\otimes$) and let $\mathcal{G}_{\mathrm{dR}}/K$ (resp. $\mathcal{G}_{\mathrm{et}}/\mathbb{Q}_p$) be the gerbe of fiber functors for $\mathcal{D}_{\mathrm{dR}}$ (resp. $\mathcal{D}_{\mathrm{et}}$). By 7.4, there is a natural equivalence $\theta:\mathcal{D}_{\mathrm{dR}}\otimes_K B_{\mathrm{cris}}(V)\simeq\mathcal{D}_{\mathrm{et}}\otimes_{\mathbb{Q}_p}B_{\mathrm{cris}}(V)$ which induces an equivalence of gerbes $\mathcal{G}_{\mathrm{dR}}\otimes_K B_{\mathrm{cris}}(V)\simeq\mathcal{G}_{\mathrm{et}}\otimes_{\mathbb{Q}_p}B_{\mathrm{cris}}(V)$ compatible with Frobenius structures and Galois actions (note that the proof of 7.4 does not require the existence of the point x).

We now construct stacks $Y_{\mathrm{dR}}\in\mathrm{Ho}(\mathrm{SPr}(K)|_{B\mathcal{G}_{\mathrm{dR}}})$ and $Y_{\mathrm{et}}\in\mathrm{Ho}(\mathrm{SPr}(\mathbb{Q}_p)|_{B\mathcal{G}_{\mathrm{et}}})$ and an equivalence $\tilde\theta:Y_{\mathrm{dR}}\otimes_K B_{\mathrm{cris}}(V)\simeq Y_{\mathrm{et}}\otimes_{\mathbb{Q}_p}B_{\mathrm{cris}}(V)$ in $\mathrm{Ho}(\mathrm{SPr}(B_{\mathrm{cris}}(V))|_{B\mathcal{G}_{B_{\mathrm{cris}}(V)}})$, where we have written $\mathcal{G}_{B_{\mathrm{cris}}(V)}$ for $\mathcal{G}_{\mathrm{dR}}\otimes_K B_{\mathrm{cris}}(V)\simeq\mathcal{G}_{\mathrm{et}}\otimes_{\mathbb{Q}_p}B_{\mathrm{cris}}(V)$.

The stacks Y_{dR}, Y_{et}, and the isomorphism $\tilde\theta$ have the following property. If we choose a point x as in 7.1, then the stacks Y_{dR} and Y_{et} are canonically isomorphic to the stacks obtained from $X_{\mathcal{C}_{\mathrm{dR}}}$ and $X_{\mathcal{C}_{\mathrm{et}}}$ by forgetting the base points. Furthermore, $\tilde\theta$ is equal to the isomorphism obtained from 1.7.

8.14. If $\widetilde{\mathcal{G}}_{\mathrm{dR}}$ (resp. $\widetilde{\mathcal{G}}_{\mathrm{et}}$) denotes the stack of fiber functors for $\mathcal{C}_{\mathrm{dR}}$ (resp. $\mathcal{C}_{\mathrm{et}}$) then there is a natural isomorphism in $\underline{\mathrm{Gerbe}}_K$ (resp. $\underline{\mathrm{Gerbe}}_{\mathbb{Q}_p}$) between $\widetilde{\mathcal{G}}_{\mathrm{dR}}$ (resp. $\widetilde{\mathcal{G}}_{\mathrm{et}}$) and the gerbe associated by 8.8 to $\tau_{\leq 1}Y_{\mathrm{dR}}$ (resp. $\tau_{\leq 1}Y_{\mathrm{et}}$).

For this recall (see [Sa, III.2.3.2.2]) that for any K–algebra R (resp. $\mathbb{Q}_p$–algebra R) the Tannakian category $\mathcal{C}_{\mathrm{dR}}\otimes_K R$ (resp. $\mathcal{C}_{\mathrm{et}}\otimes_{\mathbb{Q}_p}R$) can be recovered as the tensor category of morphisms of stacks over R

$$\widetilde{\mathcal{G}}_{\mathrm{dR}}\otimes_K R\to\underline{\mathrm{Vec}}_R\quad(\text{resp. }\widetilde{\mathcal{G}}_{\mathrm{et}}\otimes_{\mathbb{Q}_p}R\to\underline{\mathrm{Vec}}_R), \tag{8.14.1}$$

where $\underline{\mathrm{Vec}}_R$ denotes the stack over R which to any $R\to R'$ associates the groupoid of coherent sheaves on $\mathrm{Spec}(R')$.

Let $\underline{\mathrm{Tan}}_R$ denote the category whose objects are Tannakian categories over R, and whose morphisms $\mathcal{A}\to\mathcal{B}$ are equivalence classes of tensor $\mathcal{A}\to\mathcal{B}$, where $\omega\sim\omega'$ if there exists an isomorphism of functors $\omega\simeq\omega'$. Then there is a fully faithful functor $\underline{\mathrm{Tan}}_R^{\mathrm{op}}\hookrightarrow\underline{\mathrm{Gerbe}}_R$ sending a Tannakian category $\mathcal{C}$ to its gerbe of fiber functors.

In particular, $\tilde\theta$ and the identifications $\tau_{\leq 1}Y_{\mathrm{dR}}\simeq B\widetilde{\mathcal{G}}_{\mathrm{dR}}$ and $\tau_{\leq 1}Y_{\mathrm{et}}\simeq B\widetilde{\mathcal{G}}_{\mathrm{et}}$ induce an equivalence $\mathcal{C}_{\mathrm{dR}}\otimes_K\widetilde{B}_{\mathrm{cris}}(V)\simeq\mathcal{C}_{\mathrm{et}}\otimes_{\mathbb{Q}_p}\widetilde{B}_{\mathrm{cris}}(V)$ in $\underline{\mathrm{Tan}}_{B_{\mathrm{cris}}(V)}$. This equivalence agrees with the one in 1.8 obtained after choosing a base point.

Construction of Y_{dR}.

8.15. Fix a hypercover $U_\bullet \to X$ with each U_n a disjoint union of very small étale X–schemes. This hypercover defines a diagram of formal log schemes

$$\begin{tikzcd}(U_\bullet, M_{U_\bullet}) \otimes \mathbb{Z}/(p) \arrow[r] \arrow[d] & (U_\bullet^\wedge, M_{U_\bullet^\wedge}) \\ (Y, M_Y) & \end{tikzcd} \tag{8.15.1}$$

as in 4.36.1.

Consider the presheaf $\mathcal{A}^{\mathrm{ps}}$ of differential graded algebras on $\mathrm{Aff}_{K,\mathcal{G}_{\mathrm{dR}}}$ which to any (R, ω) associates $DR(\mathbb{L}_\omega)_{\overline{U}_\bullet}$ (see 4.37 for the notation), and let $\mathcal{A}$ denotes the associated sheaf of differential graded algebras on $\mathrm{Aff}_{K,\mathcal{G}_{\mathrm{dR}}}$.

Lemma 8.16. *The algebra $\mathcal{A}$ is in $\mathcal{G}_{\mathrm{dR}} - \widetilde{\mathrm{dga}}_K$.*

Proof. We have to show that the sheaf associated to the presheaf $(R, \omega) \mapsto H^i(\mathcal{A}^{\mathrm{ps}}(R, \omega))$ is quasi–coherent. As mentioned in 4.37 for any flat morphism $R \to R'$ the map

$$H^i(\mathcal{A}^{\mathrm{ps}}(R, \omega)) \otimes_R R' \to H^i(\mathcal{A}^{\mathrm{ps}}(R', \omega|_{R'})) \tag{8.16.1}$$

is an isomorphism. We need to show that this also holds for arbitrary morphisms $h : R \to R'$ fitting into a commutative diagram

$$\begin{tikzcd}\mathrm{Spec}(R') \arrow[rr, "h"] \arrow[dr, "\omega'"'] & & \mathrm{Spec}(R) \arrow[dl, "\omega"] \\ & \mathcal{G}_{\mathrm{dR}}, & \end{tikzcd} \tag{8.16.2}$$

where ω and ω' are flat.

To verify that the morphism 8.16.1 is an isomorphism, it suffices to verify that it becomes an isomorphism after a faithfully flat base change $R \to \widetilde{R}$. For such a faithfully flat map there is a commutative square

$$\begin{tikzcd}H^i(\mathcal{A}^{\mathrm{ps}}(R, \omega)) \otimes_R (R' \otimes_R \widetilde{R}) \arrow[r] \arrow[d] & H^i(\mathcal{A}^{\mathrm{ps}}(R', \omega|_{R'})) \otimes_R \widetilde{R} \arrow[d] \\ H^i(\mathcal{A}^{\mathrm{ps}}(\widetilde{R}, \omega|_{\widetilde{R}})) \otimes_R R' \arrow[r] & H^i(\mathcal{A}^{\mathrm{ps}}(R' \otimes_R \widetilde{R}, \omega|_{R' \otimes_R \widetilde{R}})), \end{tikzcd} \tag{8.16.3}$$

where the vertical maps are isomorphisms. Hence to prove that 8.16.1 is an isomorphism we may replace R by $\widetilde{R}$ and $\widetilde{R} \otimes_R R'$. Since any two fiber functors for $\mathcal{G}_{\mathrm{dR}}$ are fpqc–locally isomorphic we may therefore assume that there exists a fiber functor $\omega_0 : \mathcal{G}_{\mathrm{dR}} \to \mathrm{Vec}_{K'}$ for some field K' and a morphism $K' \to R$ such that $\omega = \omega_0|_R$. In this case we have a diagram

$$H^i(\mathcal{A}^{\mathrm{ps}}(K', \omega_0)) \otimes_{K'} R' \xrightarrow{\alpha} H^i(\mathcal{A}^{\mathrm{ps}}(R, \omega)) \otimes_R R' \xrightarrow{\beta} H^i(\mathcal{A}^{\mathrm{ps}}(R', \omega|_{R'})) \tag{8.16.4}$$

and α and $\beta \circ \alpha$ are isomorphisms. It follows that β is also an isomorphism. $\square$

8.17. The stack Y_{dR} is defined to be the image of $\mathcal{A}$ under

(8.17.1)

$$\mathrm{Ho}(\mathcal{G}_{\mathrm{dR}} - \widetilde{\mathrm{dga}}_K) \xrightarrow{\text{2.21.2 and 8.12.1}} \mathrm{Ho}(\mathcal{G}_{\mathrm{dR}} - \mathrm{Alg}_K^\Delta) \xrightarrow{[\mathbb{R}\mathrm{Spec}_{\mathcal{G}_{\mathrm{dR}}}(-)/\mathcal{G}_{\mathrm{dR}}]} \mathrm{Ho}(\mathrm{SPr}(K)|_{B\mathcal{G}_{\mathrm{dR}}}).$$

8.18. To construct the isomorphism between $\tau_{\leq 1}Y_{\mathrm{dR}}$ and the gerbe of fiber functors for $\mathcal{C}_{\mathrm{dR}}$, consider the site $\widetilde{\mathrm{Aff}}_{K,\widetilde{\mathcal{G}}_{\mathrm{dR}}}$. Let $(R,\omega) \in \widetilde{\mathrm{Aff}}_{K,\widetilde{\mathcal{G}}_{\mathrm{dR}}}$ be an object and let $\widetilde{G}_\omega$ denote the group scheme $\underline{\mathrm{Aut}}^{\otimes}(\omega)$. The left and right translation actions of $\widetilde{G}_\omega$ on $\mathcal{O}_{\widetilde{G}_\omega}$ induce an ind–isocrystal $\widetilde{\mathbb{L}}_\omega$ with R–module structure and right $\widetilde{G}_\omega$–action on X/K and can form $DR(\widetilde{\mathbb{L}}_\omega)_{\overline{U}_\bullet}$ as in 4.37. Let $\widetilde{\mathcal{A}}^{\mathrm{ps}}$ be the presheaf of differential graded algebras on $\widetilde{\mathrm{Aff}}_{K,\widetilde{\mathcal{G}}_{\mathrm{dR}}}$ which to any (R,ω) associates $DR(\widetilde{L}_\omega)_{\overline{U}_\bullet}$ and let $\widetilde{\mathcal{A}}$ be the associated sheaf of differential graded algebras. By the same argument as in the proof of 8.16 the algebra $\widetilde{\mathcal{A}}$ is in $\widetilde{\mathcal{G}}_{\mathrm{dR}} - \widetilde{\mathrm{dga}}_K$.

8.19. There is a morphism of topoi

$$r : (\widetilde{\mathrm{Aff}}_{K,\widetilde{\mathcal{G}}_{\mathrm{dR}}})^{\smile} \to (\widetilde{\mathrm{Aff}}_{K,\mathcal{G}_{\mathrm{dR}}})^{\smile} \tag{8.19.1}$$

defined as follows. The functor r^* sends a sheaf $F \in (\widetilde{\mathrm{Aff}}_{K,\mathcal{G}_{\mathrm{dR}}})$ to the sheaf sending $(R,\omega) \in \widetilde{\mathrm{Aff}}_{K,\widetilde{\mathcal{G}}_{\mathrm{dR}}}$ to $F(R,\omega|_{\mathcal{D}_{\mathrm{dR}}})$. Note that the projection $\widetilde{\mathcal{G}}_{\mathrm{dR}} \to \mathcal{G}_{\mathrm{dR}}$ is flat so if $(R,\omega) \in \widetilde{\mathrm{Aff}}_{K,\widetilde{\mathcal{G}}_{\mathrm{dR}}}$ then $(R,\omega|_{\mathcal{D}_{\mathrm{dR}}}) \in \widetilde{\mathrm{Aff}}_{K,\widetilde{\mathcal{G}}_{\mathrm{dR}}}$.

For a flat morphism $\omega : \mathrm{Spec}(R) \to \mathcal{G}_{\mathrm{dR}}$, let $\widetilde{\mathcal{G}}_\omega$ denote the fiber product $\mathrm{Spec}(R) \times_{\mathcal{G}_{\mathrm{dR}}} \widetilde{\mathcal{G}}_{\mathrm{dR}}$. Then the projection morphism $\widetilde{\mathcal{G}}_\omega \to \widetilde{\mathcal{G}}_{\mathrm{dR}}$ is flat, and the functor r_* sends a sheaf G on $\widetilde{\mathrm{Aff}}_{K,\widetilde{\mathcal{G}}_{\mathrm{dR}}}$ to the sheaf

$$(R,\omega) \mapsto \Gamma(\widetilde{\mathrm{Aff}}_{R,\widetilde{\mathcal{G}}_\omega}, G). \tag{8.19.2}$$

For any $A \in \mathcal{G}_{\mathrm{dR}} - \mathrm{dga}_K$ (resp. $A \in \mathcal{G}_{\mathrm{dR}} - \widetilde{\mathrm{dga}}_K$) the pullback r^*A is an object of $\widetilde{\mathcal{G}}_{\mathrm{dR}} - \mathrm{dga}_K$ (resp. $\widetilde{\mathcal{G}}_{\mathrm{dR}} - \widetilde{\mathrm{dga}}_K$). The functor r^* clearly also preserves arbitrary equivalences and hence induces a functor

$$r^* : \mathrm{Ho}(\mathcal{G}_{\mathrm{dR}} - \mathrm{dga}_K) \to \mathrm{Ho}(\widetilde{\mathcal{G}}_{\mathrm{dR}} - \mathrm{dga}_K). \tag{8.19.3}$$

Similarly there is a natural functor

$$r^* : \mathrm{Ho}(\mathcal{G}_{\mathrm{dR}} - \mathrm{Alg}_K^{\Delta}) \to \mathrm{Ho}(\widetilde{\mathcal{G}}_{\mathrm{dR}} - \mathrm{Alg}_K^{\Delta}). \tag{8.19.4}$$

As in 2.22 the functors 8.19.3 and 8.19.4 are part of a Quillen adjunction.

8.20. If $f : B\widetilde{\mathcal{G}}_{\mathrm{dR}} \to B\mathcal{G}_{\mathrm{dR}}$ denotes the morphism of simplicial presheaves induced by the natural morphism of gerbes $\widetilde{\mathcal{G}}_{\mathrm{dR}} \to \mathcal{G}_{\mathrm{dR}}$ sending ω to $\omega|_{\mathcal{D}_{\mathrm{dR}}}$, then there are adjoint functors

$$f_! : \mathrm{SPr}(K)|_{B\widetilde{\mathcal{G}}_{\mathrm{dR}}} \to \mathrm{SPr}(K)|_{B\mathcal{G}_{\mathrm{dR}}}, \quad (F \to B\widetilde{\mathcal{G}}_{\mathrm{dR}}) \mapsto (F \to B\widetilde{\mathcal{G}}_{\mathrm{dR}} \to B\mathcal{G}_{\mathrm{dR}}), \tag{8.20.1}$$

$$f^* : \mathrm{SPr}(K)|_{B\mathcal{G}_{\mathrm{dR}}} \to \mathrm{SPr}(K)|_{B\widetilde{\mathcal{G}}_{\mathrm{dR}}}, \quad (F \to B\mathcal{G}_{\mathrm{dR}}) \mapsto (F \times_{B\mathcal{G}_{\mathrm{dR}}} B\widetilde{\mathcal{G}}_{\mathrm{dR}} \to B\widetilde{\mathcal{G}}_{\mathrm{dR}}). \tag{8.20.2}$$

By [Ho, 1.1.11], the functor f^* preserves fibrations and trivial fibrations and hence the pair $(f_!, f^*)$ is a Quillen adjunction. It follows from the various constructions that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Ho}(\mathcal{G}_{\mathrm{dR}} - \mathrm{dga}_K)^{\mathrm{op}} & \xrightarrow{2.21.2} & \mathrm{Ho}(\mathcal{G}_{\mathrm{dR}} - \mathrm{Alg}_K^{\Delta})^{\mathrm{op}} & \xrightarrow{[\mathbb{R}\mathrm{Spec}_{\mathcal{G}_{\mathrm{dR}}}(-)/\mathcal{G}_{\mathrm{dR}}]} & \mathrm{Ho}(\mathrm{SPr}(K)|_{B\mathcal{G}_{\mathrm{dR}}}) \\ {\scriptstyle r^*}\downarrow & & {\scriptstyle r^*}\downarrow & & \downarrow{\scriptstyle \mathbb{R}f^*} \\ \mathrm{Ho}(\widetilde{\mathcal{G}}_{\mathrm{dR}} - \mathrm{dga}_K)^{\mathrm{op}} & \xrightarrow{2.21.2} & \mathrm{Ho}(\widetilde{\mathcal{G}}_{\mathrm{dR}} - \mathrm{Alg}_K^{\Delta})^{\mathrm{op}} & \xrightarrow{[\mathbb{R}\mathrm{Spec}_{\widetilde{\mathcal{G}}_{\mathrm{dR}}}(-)/\widetilde{\mathcal{G}}_{\mathrm{dR}}]} & \mathrm{Ho}(\mathrm{SPr}(K)|_{B\widetilde{\mathcal{G}}_{\mathrm{dR}}}). \end{array} \tag{8.20.3}$$

Let $\widetilde{Y}_{\mathrm{dR}} \in \mathrm{Ho}(\mathrm{SPr}(K)_{B\widetilde{\mathcal{G}}_{\mathrm{dR}}})$ denote the stack obtained from the bottom row of 8.20.3 applied to $\widetilde{\mathcal{A}}$ (where $\widetilde{\mathcal{A}}$ is defined as in 8.18). The natural map $r^*\mathcal{A} \to \widetilde{\mathcal{A}}$ and the commutativity of 8.20.3 implies that there is a natural commutative diagram

(8.20.4)
$$\begin{array}{ccc} \widetilde{Y}_{\mathrm{dR}} & \xrightarrow{\tilde{\pi}} & B\widetilde{\mathcal{G}}_{\mathrm{dR}} \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y_{\mathrm{dR}} & \xrightarrow{\pi} & B\mathcal{G}_{\mathrm{dR}} \end{array}$$

in $\mathrm{Ho}(\mathrm{SPr}(K))$.

Proposition 8.21. *The map f is an equivalence and the map $\tilde{\pi}$ induces an equivalence $\tau_{\leq 1}\widetilde{Y}_{\mathrm{dR}} \simeq B\widetilde{\mathcal{G}}_{\mathrm{dR}}$. In particular, there is a natural isomorphism $\tau_{\leq 1}Y_{\mathrm{dR}} \simeq B\widetilde{\mathcal{G}}_{\mathrm{dR}}$ in* $\mathrm{Ho}(\mathrm{SPr}^{\leq 1}(K))$.

Proof. It suffices to prove the proposition after replacing K by a field extension. In particular, let $k \subset k'$ be a separable field extension (where k is the residue field of V) such that $X^o \otimes_V k$ has a k'–valued point, and let V' be the ring of Witt vectors of k'. Then the base change $X \otimes_V V'$ satisfies the assumptions of 8.13 and in addition there is a point $x \in X^o(V')$. Replacing V by V' and X by $X \otimes_V V'$ we may therefore assume that $\widetilde{\mathcal{G}}_{\mathrm{dR}}$ is trivial with trivialization defined by a point $x \in X^o(K)$. In this case the proposition follows from the proof of [Ol1, 2.27]. □

Construction of Y_{et}.

8.22. The construction of Y_{et} follows the same outline as the construction of Y_{dR}.

Fix a hypercover $U_\bullet \to X$ with each U_n a disjoint union of very small étale X–schemes, and let $E \to X_{\overline{K}}$ be a fixed choice of geometric generic points.

Consider the presheaf $\mathcal{B}^{\mathrm{ps}}$ of differential graded algebras on $\mathrm{Aff}_{\mathbb{Q}_p,\mathcal{G}_{\mathrm{et}}}$ which to any (R,ω) associates $GC(\mathbb{V}_\omega, E)$ (see 5.42 for the notation), and let $\mathcal{B}$ be the associated sheaf of differential graded algebras. By the same reasoning as in the proof of 8.16 $\mathcal{B}$ lies in $\mathcal{G}_{\mathrm{et}} - \widetilde{\mathrm{dga}}_{\mathbb{Q}_p}$. The stack Y_{et} is defined to be the image of $\mathcal{B}$ under the composite

(8.22.1)
$$\mathrm{Ho}(\mathcal{G}_{\mathrm{et}} - \widetilde{\mathrm{dga}}_{\mathbb{Q}_p}) \xrightarrow{\text{2.21.2 and 8.12.1}} \mathrm{Ho}(\mathcal{G}_{\mathrm{et}} - \mathrm{Alg}^{\Delta}_{\mathbb{Q}_p}) \xrightarrow{[\mathbb{R}\mathrm{Spec}_{\mathcal{G}_{\mathrm{et}}}(-)/\mathcal{G}_{\mathrm{et}}]} \mathrm{Ho}(\mathrm{SPr}(\mathbb{Q}_p)|_{B\mathcal{G}_{\mathrm{et}}}).$$

By the same reasoning as in the proof of 8.21, there is a natural isomorphism $\tau_{\leq 1}Y_{\mathrm{et}} \simeq B\widetilde{\mathcal{G}}_{\mathrm{et}}$ in $\mathrm{Ho}(\mathrm{SPr}^{\leq 1}(\mathbb{Q}_p))$.

The comparison isomorphism $\tilde{\theta}$.

8.23. Let $\mathcal{G}_{\widetilde{B}_{\mathrm{cris}}(V)}$ denote the gerbe $\mathcal{G}_{\mathrm{dR}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq \mathcal{G}_{\mathrm{et}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V)$.

To construct the desired equivalence $\tilde{\theta}$, it suffices in the notation of 8.15 and 8.22 to construct an equivalence of sheaves of differential graded algebras between $\mathcal{A}|_{\mathrm{Aff}_{\widetilde{B}_{\mathrm{cris}}(V),\mathcal{G}_{\widetilde{B}_{\mathrm{cris}}(V)}}}$ and $\mathcal{B}|_{\mathrm{Aff}_{\widetilde{B}_{\mathrm{cris}}(V),\mathcal{G}_{\widetilde{B}_{\mathrm{cris}}(V)}}}$.

Lemma 8.24. *For any object $(R,\omega) \in \mathrm{Aff}_{\widetilde{B}_{\mathrm{cris}}(V),\mathcal{G}_{\widetilde{B}_{\mathrm{cris}}(V)}}$ and very small étale $U \to X$ there is a natural isomorphism*

$$\mathbb{L}_\omega(B_{\mathrm{cris}}(U^\wedge)) \otimes_{B_{\mathrm{cris}}(V)\otimes_K B_{\mathrm{cris}}(V),\Delta} B_{\mathrm{cris}}(V) \simeq \mathbb{V}_{\omega,U^\wedge_{\overline{K}}} \otimes_{B_{\mathrm{cris}}(V)} B_{\mathrm{cris}}(U^\wedge) \tag{8.24.1}$$

of Galois modules on $U^\wedge_{\overline{K}}$. This isomorphism is functorial in U.

Proof. Choose a geometric generic point $\mathrm{Spec}(\Omega) \to U_{\overline{K}}$ and view $B_{\mathrm{cris}}(U^\wedge)$ (resp. $\mathbb{V}_{\omega,U^\wedge_{\overline{K}}}$) as a representation of $\pi_1(U^o_{\overline{K}}, \mathrm{Spec}(\Omega))$. Let $\omega^{\mathrm{dR}}_{B_{\mathrm{cris}}(U^\wedge)}$ and $\omega^{\mathrm{et}}_{B_{\mathrm{cris}}(U^\wedge)}$ be as in 7.11. By 7.12 there is a natural isomorphism between these two fiber functors for $\mathcal{D}_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V) \simeq \mathcal{D}_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$. Define schemes

$$I_{\mathrm{dR}} := \underline{\mathrm{Isom}}^\otimes(\omega^{\mathrm{dR}}_{B_{\mathrm{cris}}(U^\wedge)} \otimes_{B_{\mathrm{cris}}(U^\wedge)} (B_{\mathrm{cris}}(U^\wedge) \otimes_{B_{\mathrm{cris}}(V)} R), \omega \otimes_R (R \otimes_{B_{\mathrm{cris}}(V)} B_{\mathrm{cris}}(U^\wedge))), \tag{8.24.2}$$

$$I_{\mathrm{et}} := \underline{\mathrm{Isom}}^\otimes(\omega^{\mathrm{et}}_{B_{\mathrm{cris}}(U^\wedge)} \otimes_{B_{\mathrm{cris}}(U^\wedge)} (B_{\mathrm{cris}}(U^\wedge) \otimes_{B_{\mathrm{cris}}(V)} R), \omega \otimes_R (R \otimes_{B_{\mathrm{cris}}(V)} B_{\mathrm{cris}}(U^\wedge))). \tag{8.24.3}$$

The isomorphism in 7.12 induces an isomorphism $I_{\mathrm{dR}} \simeq I_{\mathrm{et}}$ of affine schemes over R. To prove the lemma it suffices to show that the coordinate ring of I_{dR} (resp. I_{et}) is canonically isomorphic to the left hand side (resp. right hand side) of 8.24.1.

By 7.10, the coordinate ring of I_{dR} is isomorphic to

$$(\omega^{\mathrm{dR}}_{B_{\mathrm{cris}}(U^\wedge)} \otimes_{B_{\mathrm{cris}}(U^\wedge)} (B_{\mathrm{cris}}(U^\wedge) \otimes_{B_{\mathrm{cris}}(V)} R))(\mathbb{L}_\omega \otimes_R (R \otimes_{B_{\mathrm{cris}}(V)} B_{\mathrm{cris}}(U^\wedge))) \tag{8.24.4}$$

which equals
(8.24.5)

$$(\mathbb{L}_\omega(B_{\mathrm{cris}}(U^\wedge)) \otimes_{B_{\mathrm{cris}}(V)} (R \otimes_{B_{\mathrm{cris}}(V)} B_{\mathrm{cris}}(U^\wedge))) \otimes_{((R\otimes_{B_{\mathrm{cris}}(V)} B_{\mathrm{cris}}(U^\wedge))^{\otimes 2},\Delta} (R \otimes_{B_{\mathrm{cris}}(V)} B_{\mathrm{cris}}(U^\wedge)).$$

We leave to the reader the task of showing that this expression is canonically isomorphic to $\mathbb{L}_\omega(B_{\mathrm{cris}}(U^\wedge)) \otimes_{B_{\mathrm{cris}}(V)\otimes B_{\mathrm{cris}}(V)} B_{\mathrm{cris}}(V)$ as desired.

Similarly, the coordinate ring of I_{et} is isomorphic to

$$(\mathbb{V}_{\omega,\Omega} \otimes_{B_{\mathrm{cris}}(V)} (B_{\mathrm{cris}}(U^\wedge) \otimes_{B_{\mathrm{cris}}(V)} R)) \otimes_{(R\otimes_{B_{\mathrm{cris}}(V)} B_{\mathrm{cris}}(U^\wedge))^{\otimes 2}} (R \otimes_{B_{\mathrm{cris}}(V)} B_{\mathrm{cris}}(U^\wedge)) \tag{8.24.6}$$

which is canonically isomorphic to $\mathbb{V}_{\omega,\Omega} \otimes_{B_{\mathrm{cris}}(V)} B_{\mathrm{cris}}(U^\wedge)$. □

8.25. Let $\mathcal{A}^{\mathrm{ps}}_2$ (resp. $\mathcal{A}^{\mathrm{ps}}_3$) denote the presheaf of differential graded algebras on $\mathrm{Aff}_{K,\mathcal{G}_{\mathrm{dR}}}$ which to any (R,ω) associates the differential graded R–algebra (notation as in 4.37)

$$DR(\mathbb{R}^\bullet_\omega)_{U_\bullet} \quad (\text{resp. } \mathbb{R}^\bullet_\omega(((U_\bullet, M_{U_\bullet})/K)_{\mathrm{cris}})), \tag{8.25.1}$$

and let $\mathcal{A}_2$ (resp. $\mathcal{A}_3$) denote the associated sheaves of differential graded algebras. The formation of the diagram 4.37.1 is functorial in (R,ω) so there is a natural diagram of sheaves of differential graded algebras

$$\mathcal{A}_3 \longrightarrow \mathcal{A}_2 \longleftarrow \mathcal{A}. \tag{8.25.2}$$

Finally define $\mathcal{Q}^{\mathrm{ps}}$ to be the presheaf of differential graded algebras on $\mathrm{Aff}_{B_{\mathrm{cris}}(V),\mathcal{G}_{B_{\mathrm{cris}}(V)}}$ which to any (R,ω) associates the differential graded algebra obtained by applying the functor of Thom–Sullivan cochains to the cosimplicial differential graded algebra

$$[n] \mapsto GC(U^{\wedge o}_{n,\overline{K}}, \widehat{E}_U, \mathbb{R}^\bullet_\omega(B_{\mathrm{cris}}(U^\wedge))) \otimes_{B_{\mathrm{cris}}(V)\otimes B_{\mathrm{cris}}(V),\Delta} B_{\mathrm{cris}}(V), \tag{8.25.3}$$

and let $\mathcal{Q}$ be the associated sheaf of differential graded algebras. By 8.24 and the argument of 6.17 there is a natural diagram of presheaves of differential graded algebras on $\mathrm{Aff}_{B_{\mathrm{cris}}(V),\mathcal{G}_{B_{\mathrm{cris}}(V)}}$

$$\mathcal{B}^{\mathrm{ps}} \longrightarrow \mathcal{Q}^{\mathrm{ps}} \longleftarrow \mathcal{A}_3^{\mathrm{ps}} \longrightarrow \mathcal{A}_2^{\mathrm{ps}} \longleftarrow \mathcal{A}^{\mathrm{ps}}, \tag{8.25.4}$$

and hence also a diagram of sheaves of differential graded algebras on $\mathrm{Aff}_{B_{\mathrm{cris}}(V),\mathcal{G}_{B_{\mathrm{cris}}(V)}}$

$$\mathcal{B} \xrightarrow{a} \mathcal{Q} \xleftarrow{b} \mathcal{A}_3 \xrightarrow{c} \mathcal{A}_2 \xleftarrow{d} \mathcal{A}. \tag{8.25.5}$$

Lemma 8.26. *The morphisms a, b, c, and d restrict to equivalences on $\widetilde{\mathrm{Aff}}_{\widetilde{B}_{\mathrm{cris}}(V),\mathcal{G}_{\widetilde{B}_{\mathrm{cris}}(V)}}$. In particular, there is natural isomorphism*

$$\mathcal{B}|_{\widetilde{\mathrm{Aff}}_{\widetilde{B}_{\mathrm{cris}}(V),\mathcal{G}_{\widetilde{B}_{\mathrm{cris}}(V)}}} \simeq \mathcal{A}|_{\widetilde{\mathrm{Aff}}_{\widetilde{B}_{\mathrm{cris}}(V),\mathcal{G}_{\widetilde{B}_{\mathrm{cris}}(V)}}} \tag{8.26.1}$$

in $\mathrm{Ho}(\mathcal{G}_{\widetilde{B}_{\mathrm{cris}}(V)} - \widetilde{\mathrm{dga}}_{\widetilde{B}_{\mathrm{cris}}(V)})$.

Proof. First observe that if $k \to k'$ is a finite extension of fields, where k is the residue field V, and if V' is the ring of Witt vectors of k' then we obtain the same sheaves of algebras on $\mathrm{Aff}_{\widetilde{B}_{\mathrm{cris}}(V),\mathcal{G}_{\widetilde{B}_{\mathrm{cris}}(V)}}$ when we apply the above construction to $X \otimes_V V'$. We may therefore assume that there exists a point $x \in X^o(V)$ as in 7.1.

Let $\omega_x \in \mathcal{G}_{B_{\mathrm{cris}}(V)}(B_{\mathrm{cris}}(V))$ be the fiber functor defined by the point x and the isomorphism in 7.12. Any object (R, ω) fpqc–locally admits a flat morphism to $(B_{\mathrm{cris}}(V), \omega_x)$. Now it follows from the construction of the algebras that if $B_{\mathrm{cris}}(V) \to R$ is a flat morphism then the horizontal arrows in the following diagram

$$\begin{CD}
\mathcal{B}^{\mathrm{ps}}(B_{\mathrm{cris}}(V), \omega_x) \otimes_{B_{\mathrm{cris}}(V)} R @>>> \mathcal{B}^{\mathrm{ps}}(R, \omega_x|_R) \\
@VVV @VVV \\
\mathcal{Q}^{\mathrm{ps}}(B_{\mathrm{cris}}(V), \omega_x) \otimes_{B_{\mathrm{cris}}(V)} R @>>> \mathcal{Q}^{\mathrm{ps}}(R, \omega_x|_R) \\
@AAA @AAA \\
\mathcal{A}_3^{\mathrm{ps}}(B_{\mathrm{cris}}(V), \omega_x) \otimes_{B_{\mathrm{cris}}(V)} R @>>> \mathcal{A}_3^{\mathrm{ps}}(R, \omega_x|_R) \\
@VVV @VVV \\
\mathcal{A}_2^{\mathrm{ps}}(B_{\mathrm{cris}}(V), \omega_x) \otimes_{B_{\mathrm{cris}}(V)} R @>>> \mathcal{A}_2^{\mathrm{ps}}(R, \omega_x|_R) \\
@AAA @AAA \\
\mathcal{A}^{\mathrm{ps}}(B_{\mathrm{cris}}(V), \omega_x) \otimes_{B_{\mathrm{cris}}(V)} R @>>> \mathcal{A}^{\mathrm{ps}}(R, \omega_x|_R)
\end{CD} \tag{8.26.2}$$

are equivalences. Thus it suffices to show that the maps in the diagram

$$\begin{tikzcd}
\mathcal{B}^{\mathrm{ps}}(B_{\mathrm{cris}}(V),\omega_x) \arrow[r] & \mathcal{Q}^{\mathrm{ps}}(B_{\mathrm{cris}}(V),\omega_x) \\
& \mathcal{A}_3^{\mathrm{ps}}(B_{\mathrm{cris}}(V),\omega_x) \arrow[u] \arrow[d] \\
\mathcal{A}^{\mathrm{ps}}(B_{\mathrm{cris}}(V),\omega_x) \arrow[r] & \mathcal{A}_2^{\mathrm{ps}}(B_{\mathrm{cris}}(V),\omega_x)
\end{tikzcd} \tag{8.26.3}$$

induces equivalences after tensoring with $\widetilde{B}_{\mathrm{cris}}(V)$. This follows from the observation that this diagram is equal to the diagram obtained by applying the functor of Thom–Sullivan cochains to the diagram 6.17.6. $\square$

This completes the construction of $\tilde{\theta}$.

P–adic Hodge theory for spaces of paths

In the following 8.27–8.32 we prove 1.11.

8.27. Let $x, y \in X^o(K)$ be two points as in 7.1 giving rise to fiber functors ω_x^{dR}, ω_y^{dR}, and ω_x^{et}, ω_y^{et} and define

$$P_{x,y}^{\mathrm{dR}} := \underline{\mathrm{Isom}}^{\otimes}(\omega_x^{\mathrm{dR}}, \omega_y^{\mathrm{dR}}), \quad P_{x,y}^{\mathrm{et}} := \underline{\mathrm{Isom}}^{\otimes}(\omega_x^{\mathrm{et}}, \omega_y^{\mathrm{et}}). \tag{8.27.1}$$

The scheme $P_{x,y}^{\mathrm{dR}}$ (resp. $P_{x,y}^{\mathrm{et}}$) is a torsor under $\pi_1(\mathcal{C}_{\mathrm{dR}}, \omega_x)$ (resp. $\pi_1(\mathcal{C}_{\mathrm{et}}, \omega_x)$) and the natural action of $\mathrm{Gal}(\overline{K}/K)$ on ω_x^{et} and ω_y^{et} induces an action of $\mathrm{Gal}(\overline{K}/K)$ on $P_{x,y}^{\mathrm{et}}$ compatible with the action on $\pi_1(\mathcal{C}_{\mathrm{et}}, x)$. Similarly, there is a natural semi–linear Frobenius automorphism $\varphi_{P_{x,y}^{\mathrm{dR}}}$ of $P_{x,y}^{\mathrm{dR}}$ compatible with the Frobenius automorphism on $\pi_1(\mathcal{C}_{\mathrm{dR}}, \omega_x)$.

8.28. For a site C let $\mathrm{SPr}_{*\coprod *}(C)$ denote the category of simplicial presheaves F with a map $*\coprod * \to F$. By the same argument as in [Ho, 1.1.8] the model category structure on $\mathrm{SPr}(C)$ induces a natural model category structure on $\mathrm{SPr}_{*\coprod *}(C)$. For $F \in \mathrm{SPr}_{*\coprod *}(C)$ we denote by $x_F : * \to F$ (resp. $y_F : * \to F$) the point obtained from the first (resp. second) inclusion $* \hookrightarrow * \coprod *$.

Define P_{x_F, y_F} to be the sheaf associated to the presheaf which to any $U \in C$ associates the set of homotopy classes of paths in $|F(U)|$ between x_F and y_F. If $\pi_0(F) = \{*\}$, then the sheaf P_{x_F, y_F} is naturally a torsor under $\pi_1(F, x_F)$. In particular, if $F \to F'$ is an equivalence in $\mathrm{SPr}_{*\coprod *}(C)$ then the induced map $P_{x_F, y_F} \to P_{x_{F'}, y_{F'}}$ is an isomorphism. Thus the association $F \mapsto P_{x_F, y_F}$ passes to the homotopy category $\mathrm{Ho}(\mathrm{SPr}_{*\coprod *}(C))$. Note also that P_{x_F, y_F} depends only on $\tau_{\leq 1} F$.

If $\mathcal{G}$ is a gerbe on C and $x, y \in \mathcal{G}$ are two global objects, then $B\mathcal{G}$ is naturally an object of $\mathrm{SPr}_{*\coprod *}(C)$, and there is a natural isomorphism

$$P_{x_{B\mathcal{G}}, y_{B\mathcal{G}}} \simeq \underline{\mathrm{Isom}}_{\mathcal{G}}(x, y). \tag{8.28.1}$$

8.29. The points $x, y \in X^o(K)$ give Y_{dR} (resp. Y_{et}) the structure of an object of $\mathrm{Ho}(\mathrm{SPr}_{*\coprod *}(K))$ (resp. $\mathrm{Ho}(\mathrm{SPr}_{*\coprod *}(\mathbb{Q}_p))$) as follows.

Let G^{dR} (resp. G^{et}) denote the pro–reductive completion of $\pi_1(\mathcal{C}_{\mathrm{dR}}, x)$ (resp. $\pi_1(\mathcal{C}_{\mathrm{et}}, x)$), and let $\mathbb{L}(\mathcal{O}_{G^{\mathrm{dR}}})$ (resp. $\mathbb{V}(\mathcal{O}_{G^{\mathrm{et}}})$) be the ind–$F$–isocrystal (resp. ind–smooth sheaf) obtained from x as in 4.35 (resp. 5.31). Denote by $\overline{P}^{\mathrm{dR}}_{x,y}$ (resp. $\overline{P}^{\mathrm{et}}_{x,y}$) the scheme

$$\underline{\mathrm{Isom}}^{\otimes}(\omega_x^{\mathrm{dR}}|_{\mathcal{D}_{\mathrm{dR}}}, \omega_y^{\mathrm{dR}}|_{\mathcal{D}_{\mathrm{dR}}}), \quad (\text{resp. } \underline{\mathrm{Isom}}^{\otimes}(\omega_x^{\mathrm{et}}|_{\mathcal{D}_{\mathrm{et}}}, \omega_y^{\mathrm{et}}|_{\mathcal{D}_{\mathrm{et}}})). \tag{8.29.1}$$

As in 7.10, there is a natural isomorphism

$$\omega_y^{\mathrm{dR}}(\mathbb{L}(\mathcal{O}_{G^{\mathrm{dR}}})) \simeq \mathcal{O}_{\overline{P}^{\mathrm{dR}}_{x,y}} \quad (\text{resp. } \omega_y^{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G^{\mathrm{et}}})) \simeq \mathcal{O}_{\overline{P}^{\mathrm{et}}_{x,y}}) \tag{8.29.2}$$

compatible with the action of G^{dR} (resp. G^{et}).

It follows that $\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_G))$ (resp. $\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_G))$) has the map to K (resp. $\mathbb{Q}_p$) induced by x and also an equivariant map to $\mathcal{O}_{\overline{P}^{\mathrm{dR}}_{x,y}}$ (resp. $\mathcal{O}^{\mathrm{et}}_{\overline{P}_{x,y}}$). Applying the functor $[\mathbb{R}\mathrm{Spec}_{G^{\mathrm{dR}}}(-)/G^{\mathrm{dR}}]$ (resp. $[\mathbb{R}\mathrm{Spec}_{G^{\mathrm{et}}}(-)/G^{\mathrm{et}}]$) and noting that $[\mathbb{R}\mathrm{Spec}_{G^{\mathrm{dR}}}(\mathcal{O}_{\overline{P}^{\mathrm{dR}}_{x,y}})/G^{\mathrm{dR}}]$ (resp. $[\mathbb{R}\mathrm{Spec}_{G^{\mathrm{et}}}(\mathcal{O}_{\overline{P}^{\mathrm{et}}_{x,y}})/G^{\mathrm{et}}]$) is isomorphic to $*$ we see that Y_{dR} (resp. Y_{et}) is naturally an object of $\mathrm{Ho}(\mathrm{SPr}_{*\coprod *}(K))$ (resp. $\mathrm{Ho}(\mathrm{SPr}_{*\coprod *}(\mathbb{Q}_p))$).

Note also that the F–isocrystal structure on Y_{dR} extends naturally to an F–isocrystal structure in the category $\mathrm{Ho}(\mathrm{SPr}_{*\coprod *}(K))$. Similarly, there is a natural action of $\mathrm{Gal}(\overline{K}/K)$ on Y_{et} in $\mathrm{Ho}(\mathrm{SPr}_{*\coprod *}(\mathbb{Q}_p))$.

Proposition 8.30. *There is a natural isomorphism*

$$P_{x_{Y_{\mathrm{dR}}}, y_{Y_{\mathrm{dR}}}} \simeq P^{\mathrm{dR}}_{x,y} \quad (\text{resp. } P_{x_{Y_{\mathrm{et}}}, y_{Y_{\mathrm{et}}}} \simeq P^{\mathrm{et}}_{x,y}) \tag{8.30.1}$$

compatible with the action of Frobenius (resp. $\mathrm{Gal}(\overline{K}/K)$*).*

Proof. We give the proof of the isomorphism $P_{x_{Y_{\mathrm{dR}}}, y_{Y_{\mathrm{dR}}}} \simeq P^{\mathrm{dR}}_{x,y}$ leaving the proof of the other isomorphism to the reader (using the same argument).

Let $\widetilde{Y}_{\mathrm{dR}}$ be as in 8.18. The same argument as in 8.29 gives $\widetilde{Y}_{\mathrm{dR}}$ the structure of an object of $\mathrm{Ho}(\mathrm{SPr}_{*\coprod *}(K))$ such that the equivalence $f : \widetilde{Y}_{\mathrm{dR}} \to Y_{\mathrm{dR}}$ of 8.21 is an equivalence in $\mathrm{Ho}(\mathrm{SPr}_{*\coprod *}(K))$. Furthermore, $\tau_{\leq 1}\widetilde{Y}_{\mathrm{dR}}$ is by 8.21 isomorphic in $\mathrm{Ho}(\mathrm{SPr}_{*\coprod *}(K))$ to $B\pi_1(\mathcal{C}_{\mathrm{dR}}, x)$ with the second point given by the quotient map

$$[P^{\mathrm{dR}}_{x,y}/\pi_1(\mathcal{C}_{\mathrm{dR}}, x)] \longrightarrow B\pi_1(\mathcal{C}_{\mathrm{dR}}, x). \tag{8.30.2}$$

This is exactly the point defined by y. The proposition follows from this and the discussion in 8.28. □

8.31. The equivalence 7.4.1 together with the isomorphisms in 7.12 induces a natural isomorphism

$$\overline{P}^{\mathrm{dR}}_{x,y} \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq \overline{P}^{\mathrm{et}}_{x,y} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \tag{8.31.1}$$

compatible with the Frobenius and Galois actions. We leave to the reader the verification that the comparison isomorphism 7.16 is compatible with this second point defined by $\overline{P}^{\mathrm{dR}}_{x,y}$ and $\overline{P}^{\mathrm{et}}_{x,y}$ (this essentially amounts to verifying that the comparison isomorphism is functorial). We thus obtain an isomorphism $Y_{\mathrm{dR}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq Y_{\mathrm{et}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V)$ in $\mathrm{Ho}(\mathrm{SPr}_{*\coprod *}(\widetilde{B}_{\mathrm{cris}}(V)))$ compatible with the action of Frobenius and Galois. Combining this with 8.30 we obtain an isomorphism

$$P^{\mathrm{dR}}_{x,y} \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq P^{\mathrm{et}}_{x,y} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \tag{8.31.2}$$

compatible with the action of Frobenius and Galois. By [Ol3, 15.2] applied to the coordinate rings of $P^{\mathrm{dR}}_{x,y}$ and $P^{\mathrm{et}}_{x,y}$ this isomorphism is induced by an isomorphism

$$P^{\mathrm{dR}}_{x,y} \otimes_K B_{\mathrm{cris}}(V) \simeq P^{\mathrm{et}}_{x,y} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \tag{8.31.3}$$

compatible with Frobenius and Galois actions. The following lemma 8.32 combined with 7.19 now proves 1.11. □

Lemma 8.32. *The Galois representation $\mathcal{O}_{P^{\mathrm{et}}_{x,y}}$ is a direct limit of finite–dimensional sub–Galois representations.*

Proof. Write $\mathcal{O}_{P^{\mathrm{dR}}_{x,y}}$ as a direct limit $\varinjlim_j W_j$ of finite dimensional subspaces, and let $V_j := (W_j \otimes_K B_{\mathrm{cris}}(V)) \cap \mathcal{O}_{P^{\mathrm{et}}_{x,y}}$. Then as in the proof of 7.18, V_j is finite dimensional, Galois stable, and $\mathcal{O}_{P^{\mathrm{et}}_{x,y}} = \varinjlim_j V_j$. □

9. Tangential base points

In this section we explain how the point $x \in X^o(V)$ used in the constructions of this paper can be replaced by a tangential base point [De2, §15]. We assume the reader is familiar with the basics of the Kummer étale topology (an excellent summary is [Il2]).

9.1. Let X/V and $D \subset X$ be as in 1.1, and fix a point $b \in D(V)$. We write M_b for the log structure on $b = \mathrm{Spec}(V)$ obtained from the log structure M_X on X by pullback. Let (b_k, M_{b_k}) denote the reduction of (b, M_b) modulo p (so $b_k = \mathrm{Spec}(k)$).

Throughout this section we make the following assumption.

Assumption 9.2. *The sheaf $\overline{M}_b$ on the étale site of b is trivial.*

Remark 9.3. In general, this assumption holds after making a finite étale extension $V \to V'$. For if $\bar{b} \to b$ is a geometric point, then the action of $\pi_1(b, \bar{b})$ on $\overline{M}_{b,\bar{b}} \simeq \mathbb{N}^r$ must preserve the irreducible elements. It follows that the action of $\pi_1(b, \bar{b})$ on $\overline{M}_{b,\bar{b}}$ factors through a finite quotient which gives the desired extension $V \to V'$.

De Rham tangential base point.

9.4. The closed immersion $(b_k, M_{b_k}) \hookrightarrow (b, M_b)$ defines an object of the convergent site of $(b_k, M_{b_k})/V$ and hence for any isocrystal E on $(b_k, M_{b_k})/K$ we obtain a module with integrable connection $(\mathcal{E}_b, N)$ on the generic fiber $(\mathrm{Spec}(K), M_K)/K$ of (b, M_b) (note that since (b, M_b) is not smooth over V the association $E \mapsto (\mathcal{E}_b, N)$ does not define an equivalence of categories). The natural map $d\log : M_b \to \Omega^1_{(b,M_b)/V}$ defines a canonical isomorphism $\Omega^1_{(b,M_n)/V} \simeq \overline{M}^{gp}_b \otimes_{\mathbb{Z}} V$, and the differential $d : V \to \Omega^1_{(b,M_b)/V}$ is zero. Thus the connection N is simply a K–linear map $\mathcal{E}_b \to \mathcal{E}_b \otimes_{\mathbb{Z}} \overline{M}^{gp}_b$ and the integrablity condition on N amounts to the condition that for any two elements $\rho, \rho' \in \mathrm{Hom}(\overline{M}^{gp}_b, \mathbb{Z})$ the induced endomorphisms

$$N_\rho, N_{\rho'} : \mathcal{E}_b \longrightarrow \mathcal{E}_b \tag{9.4.1}$$

commute. We say that E is *unipotent* if the endomorphisms N_ρ are nilpotent for all $\rho \in \mathrm{Hom}(\overline{M}^{gp}_b, \mathbb{Z})$ (see 9.8 below for a different interpretation of this condition).

Let $V_{\text{nilp}}((b_K, M_{b_K})/K)$ denote the category of pairs $(\mathcal{E}, N)$ consisting of a K–vector space $\mathcal{E}$ with a linear map $\mathcal{E} \to \mathcal{E} \otimes_{\mathbb{Z}} \overline{M}_b^{gp}$ such that the endormorphisms N_ρ defined above are all nilpotent and commute.

Proposition 9.5. *The functor sending E to $(\mathcal{E}_b, N)$ induces an equivalence between the category of unipotent isocrystals on $(b_k, M_{b_k})/K$ and the category $V_{\text{nilp}}((b_K, M_{b_K})/K)$*

Proof. Choose an isomorphism $M_b \simeq V^* \oplus \mathbb{N}^r$. Let T_b denote the completion at the origin of the affine space $\text{Spec}(\text{Sym}^\bullet\, \Omega^1_{(b,M_b)/\text{Spec}(V)})$ and let M_{T_b} be the log structure on T_b obtained from the map $\mathbb{N}^r \to \mathcal{O}_{T_b}$ sending the i–th standard generator e_i to $d\log(e_i) \in \Omega^1_{(b,M_b)/\text{Spec}(V)}$, so that there is a closed immersion $(b, M_b) \hookrightarrow (T_b, M_{T_b})$.

Since (T_b, M_{T_b}) is formally log smooth over V, evaluation on the widening $(b_k, M_{b_k}) \hookrightarrow (T_b, M_{T_b})$ induces a functor from the category of unipotent isocrystals on $(b_k, M_{b_k})/K$ to the category of modules with integrable connection $(\mathcal{E}, \nabla)$ on $(T_{b,K}, M_{T_{b,K}})$. We call a module with integrable connection $(\mathcal{E}, \nabla)$ on $(T_{b,K}, M_{T_{b,K}})$ *unipotent* if for any $1 \leq i \leq r$ the endomorphism

$$\nabla_i : \mathcal{E} \to \mathcal{E} \tag{9.5.1}$$

defined by the dual of $d\log(e_i)$ induces a nilpotent endomorphism of $\mathcal{E} \otimes k(b_K)$.

For any unipotent isocrystal E on $(b_k, M_{b_k})/K$ with associated module with integrable connection $(\mathcal{E}, \nabla)$ on $(T_{b,K}, M_{T_{b,K}})$, the comparison between crystalline and de Rham cohomology gives an isomorphism

$$H^*_{\text{cris}}((b_k, M_{b_k})/K, E) \simeq H^*_{\text{dR}}((T_{b,K}, M_{T_{b,K}}), (\mathcal{E}, \nabla)). \tag{9.5.2}$$

Looking at H^0 and H^1, it follows that the functor $E \mapsto (\mathcal{E}, \nabla)$ is fully faithful with essential image closed under extensions. In particular, we obtain an equivalence of categories between the category of unipotent isocrystals on $(b, M_b)/K$ and the category of unipotent modules with integrable connection on $(T_{b,K}, M_{T_{b,K}})$.

A pair $(\mathcal{E}, N)$ as in the proposition defines a module with connection $(\mathcal{E} \otimes \mathcal{O}_{T_{b,K}}, \nabla_N)$ by taking $\mathcal{V}$ to be the trivial vector bundle $\mathcal{E} \otimes_K \mathcal{O}_{T_{b,K}}$ and ∇ the connection defined by

$$\nabla(e \otimes 1) = N(e) \otimes 1 \in \mathcal{E} \otimes_{\mathbb{Z}} \overline{M}_b^{gp} \otimes_{\mathbb{Z}} \mathcal{O}_{T_{b,K}} \simeq \mathcal{E} \otimes_K \Omega^1_{T_{b,K}/K}. \tag{9.5.3}$$

Here we use the isomorphism $\Omega^1_{T_{b,K}/K} \simeq \mathcal{O}_{T_{b,K}} \otimes_{\mathbb{Z}} \overline{M}_b^{gp}$ provided by the map $d\log$.

Lemma 9.6. *The functor $(\mathcal{E}, N) \mapsto (\mathcal{E} \otimes \mathcal{O}_{T_{b,K}}, \nabla_N)$ is fully faithful.*

Proof. Since the functor is compatible with tensor products and duals it suffices to show that for $(\mathcal{E}, N) \in V_{\text{nilp}}((b_K, M_{b_K})/K)$ the natural reduction map

$$H^0_{\text{dR}}(\mathcal{E} \otimes \mathcal{O}_{T_{b,K}}, \nabla_N) \to \text{Ker}(N) \tag{9.6.1}$$

is an isomorphism.

If $e \in \text{Ker}(N)$ then $\nabla(e \otimes 1) = 0$, so 9.6.1 is clearly surjective.

To see that 9.6.1 is injective proceed as follows. Since the operators N_ρ commute and are nilpotent, there exists a basis $e_1, \ldots, e_n$ for $\mathcal{E}$ such that for each i we have $N(e_i) = \sum_{j<i} \omega_{ij} e_j$

for some $\omega_{ij} \in \Omega^1_{(b_K, M_{b_K})/K}$. If $\sum_i ge_i$ is a horizontal vector, then we find that

$$0 = \nabla(\sum_i g_i e_i) = \sum_i (dg_i e_i + g_i \sum_{j<i} \omega_{ij} e_j). \tag{9.6.2}$$

Using descending induction in j one sees from this formula that dg_i lies in $K \otimes_{\mathbb{Z}} \overline{M}^{\mathrm{gp}}_{b_K} \subset \Omega^1_{(T_{b,K}, M_{T_{b,K}})/K}$ for every i. On the other hand, if $I \subset \mathcal{O}_{T_{b,K}}$ denotes the ideal of the origin, then the image of $d : \mathcal{O}_{T_{b,K}} \rightarrow \Omega^1_{(T_{b,K}, M_{T_{b,K}})/K}$ is contained in $I \cdot \Omega^1_{(T_{b,K}, M_{T_{b,K}})/K}$. This implies that all the functions g_i are constant, and hence the map 9.6.1 is injective. $\square$

Let $(\mathcal{E}, \nabla)$ be a module with integrable connection on $(T_{b,K}, M_{T_{b,K}})/K$ with nilpotent residue, and fix a basis $e_1, \ldots, e_n$ for $\mathcal{E}_b$ such that for every $i \in [1, n]$ we have $N(e_i) = \sum_{j<i} \omega_{ij} e_j$ for some $\omega_{ij} \in K \otimes_{\mathbb{Z}} \overline{M}^{gp}_b$.

Lemma 9.7. *There exists a unique basis $\{\tilde{e}_i\}$ for $\mathcal{E}$ reducing to the basis $\{e_i\}$ for $\mathcal{E}_b$ such that $\nabla(\tilde{e}_i) = \sum_{j<i} \omega_{ij} \tilde{e}_j \in \mathcal{E} \otimes_{\mathbb{Z}} \overline{M}^{gp}_b \simeq \mathcal{E} \otimes \Omega^1_{(T_{b,K}, M_{T_{b,K}})/K}$ for every $i \in [1, n]$.*

Proof. Let $I \subset \mathcal{O}_{T_{b,K}}$ be the ideal defining b_K and observe that the differential sends I to $I \otimes \Omega^1_{(T_{b,K}, M_{T_{b,K}})/K}$. It follows that the connection ∇ induces for all $m \geq 0$ a map, which we denote by the same letter

$$\nabla : \mathcal{E}/I^m\mathcal{E} \longrightarrow (\mathcal{E}/I^m\mathcal{E}) \otimes \Omega^1_{(T_{b,K}, M_{T_{b,K}})/K}. \tag{9.7.1}$$

To prove the existence of the desired basis $\{\tilde{e}_i\}$ it suffices to show that if a basis $\{\tilde{e}_i\}$ exists with the desired properties modulo I^m then after adding suitable elements of $I^m\mathcal{E}$ to this basis we obtain a basis that works also modulo I^{m+1}. For then passing to the limit in m yields the desired basis for $\mathcal{E}$.

So assume given a basis $\{\tilde{e}_i\}$ with the desired properties modulo I^m. We show by induction on $i = 1, \ldots, n$ that we can modify $\tilde{e}_i$ so that $\nabla(\tilde{e}_i)$ is congruent to $\sum_{j<i} \omega_{ij} \tilde{e}_j$ modulo $I^{m+1}\mathcal{E}$.

For the case $i = 1$, note that since $e_1 \in \mathrm{Ker}(N)$ we can write $\nabla(\tilde{e}_1) = \sum_i \lambda_i e_i$ with $\lambda_i \in I^m \Omega^1_{(T_{b,K}, M_{T_{b,K}})/K}$. Since ∇ is integrable, we have

$$0 = \nabla^2(\tilde{e}_1) = \sum_i (d\lambda_i \tilde{e}_i + \lambda_i \wedge (\sum_{j<i} \omega_{ij} \tilde{e}_j)). \tag{9.7.2}$$

The coefficient of $\tilde{e}_j$ in this expression is equal to

$$d\lambda_j + \sum_{i>j} \lambda_i \wedge \omega_{ij}. \tag{9.7.3}$$

In particular if $\lambda_i = 0$ for $i > j_0$ then $d\lambda_{j_0} = 0$. In this case, λ_{j_0} defines a closed form in $\Omega^1_{T_{b,K}/K}$ (without log poles) which is exact by the usual Poincare lemma [B-O, proof of 6.12]. Let $\eta_{j_0} \in \mathcal{O}_{T_{b,K}}$ be a function with $d\eta_{j_0} = \lambda_{j_0}$. Note that since the differential d preserves the ideal I we can choose $\eta_{j_0} \in I^m$. Set $\tilde{e}'_1 := \tilde{e}_1 - \eta_{j_0} \tilde{e}_{j_0}$. Then if $\nabla(\tilde{e}'_1) = \lambda'_1 \tilde{e}'_1 + \sum_{i>1} \lambda'_i \tilde{e}_i$ we have $\lambda'_i = 0$ for $i > j_0 - 1$. Proceeding by descending induction on j_0 we obtain the case $i = 1$.

Next we modify $\tilde{e}_i$ assuming that $\nabla(\tilde{e}_s) = \sum_{j<s} \omega_{sj}\tilde{e}_j$ for $s < i$. Write

$$\nabla(\tilde{e}_i) = \sum_{j<i} \omega_{ij}\tilde{e}_j + \sum_j \lambda_j \tilde{e}_j \tag{9.7.4}$$

with $\lambda_j \in I^m\Omega^1_{(T_{b,K}, M_{T_{b,K}})/K}$. Then by induction and using the fact that the operators N_ρ commute, we have $\nabla(\sum_{j<i} \omega_{ij}\tilde{e}_j) = 0$, so again we find that

$$d\lambda_j + \sum_{i>j} \lambda_j \wedge \omega_{ij} = 0. \tag{9.7.5}$$

Repeating the inductive argument used in the case $i = 1$, we obtain the desired modification of $\tilde{e}_i$.

Finally we prove the uniqueness of the basis $\{\tilde{e}_i\}$. For this assume $\{\tilde{e}'_i\}$ is another basis with the same properties. Then $\tilde{e}_1 - \tilde{e}'_1$ is closed and in $I\mathcal{E}$. Write $\tilde{e}_1 - \tilde{e}'_1 = \sum_i g_i\tilde{e}_i$ with $g_i \in I$. Then

$$\nabla(\tilde{e}_1 - \tilde{e}'_1) = \sum_i (dg_i\tilde{e}_i + g_i \sum_{j<i} \omega_{ij}\tilde{e}_j). \tag{9.7.6}$$

If not all $g_i = 0$, then there exists a largest integer i_0 for which $g_{i_0} \neq 0$. The above formula then implies that $dg_{i_0} = 0$, which is impossible since d is injective on I. Consequently all $g_i = 0$ and $\tilde{e}_1 = \tilde{e}'_1$.

Looking at the module with integrable connection $\mathcal{E}/(\tilde{e}_1)$ and using induction on the rank, it follows that for each $i \geq 2$ there exists $g_i \in I$ so that $\tilde{e}_i = \tilde{e}'_i + g_i\tilde{e}_1$. By induction on k we then show that $\tilde{e}_i = \tilde{e}'_k$. If the result holds for $k < k_0$, then we have

$$\nabla(\tilde{e}_{k_0}) = \sum_{j<k_0} \omega_{k_0 j}\tilde{e}_j = \nabla(\tilde{e}'_{k_0}), \tag{9.7.7}$$

and hence $\nabla(g_{k_0}\tilde{e}_1) = dg_{k_0}\tilde{e}_1 = 0$. Again since d is injective on I it follows that $g_{k_0} = 0$ and hence $\tilde{e}_{k_0} = \tilde{e}'_{k_0}$. This completes the proof of the uniqueness and the lemma. □

Combining 9.6 and 9.7 we obtain 9.5. □

Corollary 9.8. *If E is a unipotent isocrystal on $(b_k, M_{b_k})/K$, then there exists a canonical filtration* Fil *on E by sub–isocrystals such that the associated graded* $\mathrm{gr}_{\mathrm{Fil}}E$ *is a direct sum of trivial isocrystals.*

Proof. If $(\mathcal{E}, N)$ is the object in $V_{\mathrm{nilp}}((b_K, M_{b_K})/K)$ associated to E, then since the operators N_ρ commute and are nilpotent there exists a canonical filtration $\mathrm{Fil}_{\mathcal{E}}$ on $\mathcal{E}$ defined inductively by setting $\mathrm{Fil}^0_{\mathcal{E}} := \mathrm{Ker}(N)$ and $\mathrm{Fil}^i_{\mathcal{E}}$ equal to the inverse image of $\mathrm{Fil}^0_{\mathcal{E}/\mathrm{Fil}^{i-1}}$. This filtration on $\mathcal{E}$ combined with 9.5 induces a canonical filtration on E with the desired properties. □

Corollary 9.9. *The category of unipotent isocrystals on $(b_k, M_{b_k})/K$ is Tannakian with fiber functor given by sending an isocrystal E to its value on (b, M_b). The fundamental group of this Tannakian category is canonically isomorphic to the vector group scheme $\mathbb{G}_{\mathrm{dR}}$ over K sending a K–algebra R to $R \otimes_{\mathbb{Z}} \mathrm{Hom}(\overline{M}_b^{gp}, \mathbb{Z})$.*

Proof. The first statement follows from 9.5 since the category $V_{\mathrm{nilp}}((b_K, M_{b_K})/K)$ is clearly Tannakian.

To calculate the fundamental group π_1, choose a basis $e_1, \ldots, e_r$ for $\overline{M}_b^{gp}$ and let $\rho_1, \ldots, \rho_r$ be the corresponding dual basis. The category $V_{\mathrm{nilp}}((b_K, M_{b_K})/K)$ is then identified with the category of K–vector spaces $\mathcal{E}$ with commuting nilpotent operators $N_{\rho_1}, \ldots, N_{\rho_r}$. Exponentiating these operators we see that the category $V_{\mathrm{nilp}}((b_K, M_{b_K})/K)$ is equivalent to the category of vector spaces $\mathcal{E}$ with commuting unipotent automorphisms $U_{\rho_1}, \ldots, U_{\rho_r}$. This category is in turn equivalent to the category of representations of $\mathbb{G}_a^r \simeq \mathbb{G}_{\mathrm{dR}}$. We leave to the reader the verification that this isomorphism $\pi_1 \simeq \mathbb{G}_{\mathrm{dR}}$ is independent of the choice of basis for $\overline{M}_b^{gp}$. □

9.10. Multiplication by p on M_b induces a lift of Frobenius $\tilde{F} : (b, M_b) \to (b, M_b)$. If E is an isocrystal on $(b_k, M_{b_k})/K$ with corresponding object $(\mathcal{E}, N) \in V_{\mathrm{nilp}}((b_K, M_{b_K})/K)$, then F^*E corresponds to the pair $(\mathcal{E} \otimes_{K,\sigma} K, pN)$. In particular, pullback by Frobenius induces an auto–equivalence of the category of unipotent isocrystals on $(b_k, M_{b_k})/K$. If $\mathbb{G}_{\mathrm{dR}} \simeq K \otimes_{\mathbb{Z}} \mathrm{Hom}(\overline{M}_b^{gp}, \mathbb{Z})$ denotes the fundamental group of this category, then it follows from the proof of 9.9 that the isomorphism $F_{\mathbb{G}_{\mathrm{dR}}} : \mathbb{G}_{\mathrm{dR}} \to \mathbb{G}_{\mathrm{dR}} \otimes_{K,\sigma} K$ induced by Frobenius is equal to the map induced by multiplication by p on $\mathrm{Hom}(\overline{M}_b^{gp}, \mathbb{Z})$, or equivalently the semi–linear automorphism $p\sigma$ on K.

Denote by $K(1)$ the F–isocrystal with underlying vector space K and semi–linear automorphism $p\sigma$. Then the above discussion implies that $\mathbb{G}_{\mathrm{dR}}$ is isomorphic as a vector group scheme with semi–linear automorphism to $K(1) \otimes_{\mathbb{Z}} \mathrm{Hom}(\overline{M}_{b_K}^{gp}, \mathbb{Z})$.

Corollary 9.11. *The functor*

$$\omega_b^{\mathrm{dR}} : V_{\mathrm{nilp}}(X_K, M_{X_K}) \longrightarrow \mathrm{Vec}_K \tag{9.11.1}$$

sending a module with integrable log connection $(\mathcal{E}, \nabla)$ to $\mathcal{E}(b)$ is a fiber functor.

Proof. It suffices to verify that the functor is faithful and exact. Let X_K^* be the completion of X_K along b. The functor sending $(\mathcal{E}, \nabla)$ to the pullback $(\mathcal{E}^*, \nabla^*)$ of $(\mathcal{E}, \nabla)$ to X_K^* is exact and faithful, and the residue of ∇^* is nilpotent. The result therefore follows from 9.7. □

9.12. For a Tannakian subcategory $\mathcal{C} \subset V_{\mathrm{nilp}}(X_K, M_{X_K})$, we write $\pi_1(\mathcal{C}, b)$ for the Tannaka dual of $\mathcal{C}$ with respect to the fiber functor 9.11.1.

If (E, φ) is an F–isocrystal, and $\mathcal{C}_{\mathrm{dR}}$ is as in 7.1, we can by 4.37 construct a natural pointed stack $X_{\mathcal{C}_{\mathrm{dR}}} \in \mathrm{Ho}(\mathrm{SPr}_*(K))$ with semi–linear Frobenius automorphism using the fiber functor 9.11.1. Moreover, the fundamental group of $X_{\mathcal{C}_{\mathrm{dR}}}$ is naturally isomorphic to $\pi_1(\mathcal{C}_{\mathrm{dR}}, b)$ and the cohomology of local systems on $X_{\mathcal{C}_{\mathrm{dR}}}$ is isomorphic to crystalline cohomology (this follows from the same reasoning used in [Ol1].

Étale tangential base point.

9.13. Let $(b_{\overline{K}}, M_{b_{\overline{K}}})$ be the base change of (b, M_b) to $\mathrm{Spec}(\overline{K})$.

Choose an isomorphism $M_{b_K} \simeq K^* \oplus \overline{M}_{b_K}$ corresponding to a section of the projection $M_{b_K} \to \overline{M}_{b_K}$ (note that here we are using 9.2), and define a log geometric point $(\bar{b}, M_{\bar{b}}) \to (b_{\overline{K}}, M_{b_{\overline{K}}})$ in the sense of [Il2, 4.1] as follows. The scheme $\bar{b}$ is equal to $\mathrm{Spec}(\overline{K})$. Let

$P \subset \overline{M}^{gp}_{b_{\overline{K}}} \otimes_{\mathbb{Z}} \mathbb{Q}$ be the submonoid of elements $m \in \overline{M}^{gp}_{b_{\overline{K}}} \otimes \mathbb{Q}$ for which there exists an integer n so that $n \cdot m$ is in $\overline{M}_{b_K}$, and let $M_{\bar{b}}$ be the monoid $\overline{K}^* \oplus P$ with map to $\overline{K}$ sending all nonzero elements of P to 0. The map to $(b_{\overline{K}}, M_{b_{\overline{K}}})$ is induced by the natural map $\overline{M}_{b_{\overline{K}}} \to P$ and the above splitting of M_b.

Note that since the splitting $M_{b_K} \simeq K^* \oplus \overline{M}_{b_K}$ is defined over K and not $\overline{K}$, for any $\sigma \in \mathrm{Gal}(\overline{K}/K)$ there is a natural commutative diagram

$$\begin{array}{ccc} (\bar{b}, M_{\bar{b}}) & \xrightarrow{\tilde{\sigma}} & (\bar{b}, M_{\bar{b}}) \\ \downarrow & & \downarrow \\ (b_{\overline{K}}, M_{b_{\overline{K}}}) & \xrightarrow{\sigma} & (b_{\overline{K}}, M_{b_{\overline{K}}}), \end{array} \tag{9.13.1}$$

where the map σ is the natural action on $(b_{\overline{K}}, M_{b_{\overline{K}}})$. In particular, the group $\mathrm{Gal}(\overline{K}/K)$ acts on the log étale fundamental group $\pi_1((b_{\overline{K}}, M_{b_{\overline{K}}}), (\bar{b}, M_{\bar{b}}))$ [Il2, §4].

Lemma 9.14. *There is a canonical isomorphism between* $\pi_1((b_{\overline{K}}, M_{b_{\overline{K}}}), (\bar{b}, M_{\bar{b}}))$ *and*

$$\mathrm{Hom}(\overline{M}_{b_K}, \mathbb{Z}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}(1) := (\varprojlim_n \mathrm{Hom}(\overline{M}_{b_K}, \mu_n(\overline{K}))), \tag{9.14.1}$$

compatible with the action of $\mathrm{Gal}(\overline{K}/K)$*, where* $\mathrm{Gal}(\overline{K}/K)$ *acts on the right hand side of* 9.14.1 *via the natural action on* $\mu_n(\overline{K})$*.*

Proof. This is discussed in [Il2, 4.7]. □

9.15. For an integer $n \geq 1$ let $LC_{\mathbb{Z}/(n)}(b_{\overline{K}}, M_{b_{\overline{K}}})$ denote the category of locally constant sheaves of $\mathbb{Z}/(n)$–modules of finite type on the Kummer étale site of $(b_{\overline{K}}, M_{b_{\overline{K}}})$, and define the category of *smooth* $\mathbb{Q}_p$*–sheaves* on $(b_{\overline{K}}, M_{b_{\overline{K}}})$ to be the category

$$Sm_{\mathbb{Q}_p}(b_{\overline{K}}, M_{b_{\overline{K}}}) := (\varprojlim_r LC_{\mathbb{Z}/(p^r)}(b_{\overline{K}}, M_{b_{\overline{K}}})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \tag{9.15.1}$$

Taking the stalk at $(\bar{b}, M_{\bar{b}})$ identifies the category $Sm_{\mathbb{Q}_p}(b_{\overline{K}}, M_{b_{\overline{K}}})$ with the category of continuous $\mathbb{Q}_p$–representations of $\pi_1((b_{\overline{K}}, M_{b_{\overline{K}}}), (\bar{b}_{\overline{K}}, M_{\bar{b}_{\overline{K}}}))$.

Let $Sm^{\mathrm{unip}}_{\mathbb{Q}_p}(b_{\overline{K}}, M_{b_{\overline{K}}})$ be the category of smooth $\mathbb{Q}_p$–sheaves L which admit an exhaustive filtration $F^\bullet$ such that the successive quotients F^i/F^{i+1} are trivial sheaves. The category $Sm^{\mathrm{unip}}_{\mathbb{Q}_p}(b_{\overline{K}}, M_{b_{\overline{K}}})$ is equivalent to the category of unipotent representations of the group $\pi_1((b_{\overline{K}}, M_{b_{\overline{K}}}), (\bar{b}, M_{\bar{b}}))$. It follows from this that $Sm^{\mathrm{unip}}_{\mathbb{Q}_p}(b_{\overline{K}}, M_{b_{\overline{K}}})$ is Tannakian with fiber functor sending a sheaf L to $L_{(\bar{b}, M_{\bar{b}})}$. Furthermore, the Tannaka dual $\mathbb{G}_{\mathrm{et}}$ is canonically isomorphic to the vector group scheme over $\mathbb{Q}_p$ defined by the $\mathbb{Q}_p$–vector space $\mathrm{Hom}(\overline{M}^{gp}_{b_K}, \mathbb{Z}) \otimes \mathbb{Q}_p(1)$.

For any $\sigma \in \mathrm{Gal}(\overline{K}/K)$, pullback by $\sigma : (b_{\overline{K}}, M_{b_{\overline{K}}}) \to (b_{\overline{K}}, M_{b_{\overline{K}}})$ induces an auto–equivalence of $Sm^{\mathrm{unip}}_{\mathbb{Q}_p}(b_{\overline{K}}, M_{b_{\overline{K}}})$ compatible with the fiber functor defined by $(\bar{b}, M_{\bar{b}})$. Under the isomorphism $\mathbb{G}_{\mathrm{et}} \simeq \mathrm{Hom}(\overline{M}^{gp}_{b_K}, \mathbb{Z}) \otimes \mathbb{Q}_p(1)$ the induced action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathbb{G}_{\mathrm{et}}$ is simply the action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathrm{Hom}(\overline{M}^{gp}_{b_K}, \mathbb{Z}) \otimes \mathbb{Q}_p(1)$ described in 9.14.

Corollary 9.16. *There is a natural isomorphism* $\iota_{\mathbb{G}} : \mathbb{G}_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V) \simeq \mathbb{G}_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$ *compatible with the action of Galois and Frobenius.*

Proof. By the discussion in 9.10 and 9.15 it suffices to exhibit a natural association between $K(1)$ and $\mathbb{Q}_p(1)$. This can be done as follows. Let $\beta : \mathbb{Q}_p(1) \hookrightarrow B_{\text{cris}}(V)$ be the map defined in [Fo4, 2.3.4]. Then the Galois invariants of $\mathbb{Q}_p(1) \otimes_{\mathbb{Q}_p} B_{\text{cris}}(V)$ are equal to $\mathbb{Q}_p(1) \otimes_{\mathbb{Q}_p} K \cdot \beta(\mathbb{Q}_p(1)) \subset \mathbb{Q}_p(1) \otimes_{\mathbb{Q}_p} B_{\text{cris}}(V)$ with Frobenius action given by $\xi \otimes \eta \mapsto \xi \otimes p\eta$. If $\xi \in \mathbb{Q}_p(1)$ is a generator, we therefor obtain an isomorphism

$$(\mathbb{Q}_p(1) \otimes_{\mathbb{Q}_p} B_{\text{cris}}(V))^{\text{Gal}(\overline{K}/K)} \simeq \mathbb{Q}_p(1) \otimes_{\mathbb{Q}_p} K \cdot \beta(\mathbb{Q}_p(1)) \simeq K(1) \tag{9.16.1}$$

by sending $\xi \otimes \beta(\xi^{-1})$ to $1 \in K$. Moreover, this isomorphism is independent of the choice of the generator ξ. □

9.17. By [Il2, 4.7 (c)], restriction induces an equivalence of categories between the category of finite Kummer étale log schemes over $(X_{\overline{K}}, M_{X_{\overline{K}}})$ and the category of finite étale schemes over $X^o_{\overline{K}}$. By passing to the limit it follows that the category of smooth $\mathbb{Q}_p$–sheaves (defined as in 9.15) on the log étale site of $(X_{\overline{K}}, M_{X_{\overline{K}}})$ is equivalent via the restriction functor to the category of smooth $\mathbb{Q}_p$–sheaves on $X^o_{\overline{K}}$. In particular, for a Tannakian category $\mathcal{C}$ of smooth $\mathbb{Q}_p$–sheaves on $X^o_{\overline{K}}$, there is a natural fiber functor obtained from the composite

$$\omega^{\text{et}}_b : (\text{smooth } \mathbb{Q}_p\text{–sheaves on } X^o_{\overline{K}}) \simeq Sm_{\mathbb{Q}_p}(X_{\overline{K}}, M_{X_{\overline{K}}}) \xrightarrow{L \mapsto L_{(\bar{b}, M_{\bar{b}})}} \text{Vec}_{\mathbb{Q}_p}. \tag{9.17.1}$$

We denote the corresponding group scheme by $\pi_1(\mathcal{C}, \bar{b})$.

9.18. We can also modify the construction of the stack $X_{\mathcal{C}_{\text{et}}}$ in 5.31 to the situation of the fiber functor 9.17.1 instead of that defined by a point in X^o.

For this let $E \to X_{\overline{K}}$ be a finite collection of geometric generic points whose image meets every connected component of $X_{\overline{K}}$. For each point $e \in E$, let $\widetilde{X}^o_e \to X_{\overline{K}}$ be the normalization of $X_{\overline{K}}$ in the maximal subfield of $k(E)$ unramified over $X^o_{\overline{K}}$. The scheme $\widetilde{X}^o_e$ is naturally a projective limit of finite étale $X^o_{\overline{K}}$–schemes, and hence by the equivalence [Il2, 4.7 (c)] is obtained from a projective system $(\widetilde{X}_e, M_{\widetilde{X}_e})$ of Kummer étale coverings of $(X_{\overline{K}}, M_{X_{\overline{K}}})$. In particular, the pullback of $(\widetilde{X}_e, M_{\widetilde{X}_e})$ to $(\bar{b}, M_{\bar{b}})$ is a disjoint union of log schemes isomorphic to $(\bar{b}, M_{\bar{b}})$. Define *specialization data for E relative to b* to be a collection of sections of the maps

$$(\widetilde{X}_e, M_{\widetilde{X}_e}) \times_{(X_{\overline{K}}, M_{X_{\overline{K}}})} (\bar{b}, M_{\bar{b}}) \to (\bar{b}, M_{\bar{b}}), \tag{9.18.1}$$

one for each $e \in E$.

Note that by [Il2, 4.6], any two choices of specialization data differ by the action of an element in $\prod_{e \in E} \pi_1(X^o_{\overline{K}}, e)$ on the log schemes $\{(\widetilde{X}_e, M_{\widetilde{X}_e})\}$.

Using the same method discussed in 5.31, we then obtain a pointed stack $X_{\mathcal{C}_{\text{et}}} \in \text{Ho}(\text{SPr}_*(\mathbb{Q}_p))$ which is independent of the choice of specialization data, whose fundamental group is $\pi_1(\mathcal{C}_{\text{et}}, b)$, and whose cohomology groups compute étale cohomology.

Pullback of associations and comparison.

9.19. Let

$$(\text{Spec}(\overline{K}), M_{\overline{K}}) \to (b_K, M_{b_K}) \tag{9.19.1}$$

be a log geometric point, and define $M_{\overline{V}}$ to be the log structure on $\mathrm{Spec}(\overline{V})$ given by

$$M_{\overline{V}} := \{m \in M_{\overline{K}} | nm \in \overline{V}^* \oplus_{V^*} M_b \text{ for some positive } n \in \mathbb{Z}\}. \tag{9.19.2}$$

Here we view $M_b \subset M_{b_K}$ as a submonoid of $M_{\overline{K}}$. If we choose an isomorphism $M_{\overline{K}} \simeq \overline{K}^* \oplus \mathbb{Q}^r_{\geq 0}$ for some r, where $\mathbb{Q}_{\geq 0} \subset \mathbb{Q}$ denotes the submonoid of nonnegative rational numbers, then it follows from the definition that $M_{\overline{V}} \simeq \overline{V}^* \oplus \mathbb{Q}^r_{\geq 0}$ with the map to $\overline{V}$ obtained by sending all nonzero elements of $\mathbb{Q}^r_{\geq 0}$ to 0.

Denote by $(\mathrm{Spec}(\overline{V}/p\overline{V}), M_{\overline{V}/p\overline{V}})$ the reduction modulo p. For each positive integer n define

$$\frac{1}{p^n} M_{V/pV} := \{m \in M_{\overline{V}/p\overline{V}} | p^n m \in M_{V/pV}\}, \tag{9.19.3}$$

where we view $M_{V/pV}$ as a submonoid of $M_{\overline{V}/p\overline{V}}$. Denote by N the inverse limit

$$N := \varprojlim_n \frac{1}{p^n} M_{V/pV}, \tag{9.19.4}$$

where the inverse limit is taken with respect to the multiplication by p maps $\frac{1}{p^{n+1}} M_{V/pV} \to \frac{1}{p^n} M_{V/pV}$. The maps

$$\frac{1}{p^n} M_{V/pV} \longrightarrow \overline{V}/p\overline{V} \tag{9.19.5}$$

induced by the map $M_{\overline{V}} \to \overline{V}$ define a map of monoids

$$N \to S := \varprojlim \overline{V}/p\overline{V}, \tag{9.19.6}$$

where the maps $\overline{V}/p\overline{V} \to \overline{V}/p\overline{V}$ on the right hand side are the Frobenius maps. Let $M_{A_{\mathrm{cris}}(V)}$ be the log structure on $\mathrm{Spec}(A_{\mathrm{cris}}(V))$ associated to the prelog structure

$$N \longrightarrow S \xrightarrow{t} W(S) \longrightarrow A_{\mathrm{cris}}(V), \tag{9.19.7}$$

where t denotes the Teichmuller lifting. It follows from the construction that the diagram

$$\begin{array}{ccc} M_{A_{\mathrm{cris}}(V)} & \xrightarrow{d} & M_b \\ \downarrow & & \downarrow \\ A_{\mathrm{cris}}(V) & \xrightarrow{\theta} & \overline{V} \end{array} \tag{9.19.8}$$

commutes, where d is the map induced by the projection of N onto M_b.

9.20. If we choose an isomorphism $M_{\overline{V}} \simeq \overline{V}^* \oplus \mathbb{Q}^r_{\geq 0}$, then N is isomorphic to

$$(\varprojlim (\overline{V}/p\overline{V})^*) \oplus \mathbb{N}^r. \tag{9.20.1}$$

From this it follows that the log structure $M_{A_{\mathrm{cris}}(V)}$ admits a chart by $\mathbb{N}^r$ and that the morphism of log schemes

$$(\mathrm{Spec}(\overline{V}), M_b|_{\overline{V}}) \to (\mathrm{Spec}(A_{\mathrm{cris}}(V)), M_{A_{\mathrm{cris}}(V)}) \tag{9.20.2}$$

induced by 9.19.8 is a log closed immersion.

9.21. The construction of the log structure $M_{A_{\mathrm{cris}}(V)}$ is functorial in $(\mathrm{Spec}(\overline{K}), M_{\overline{K}})$. This implies that for any $g \in \pi_1((b_K, M_{b_K}), (\mathrm{Spec}(\overline{K}), M_{\overline{K}}))$ there is a natural commutative diagram

$$\begin{tikzcd}
(\mathrm{Spec}(\overline{V}), M_b|_{\overline{V}}) \arrow[r] \arrow[d, "g"'] & (\mathrm{Spec}(A_{\mathrm{cris}}(V)), M_{A_{\mathrm{cris}}(V)}) \arrow[d, "g"] \\
(\mathrm{Spec}(\overline{V}), M_b|_{\overline{V}}) \arrow[r] & (\mathrm{Spec}(A_{\mathrm{cris}}(V)), M_{A_{\mathrm{cris}}(V)}),
\end{tikzcd} \tag{9.21.1}$$

where the action of g on the rings $A_{\mathrm{cris}}(V)$ and $\overline{V}$ is given by the action of the image of g in $\mathrm{Gal}(\overline{K}/K)$. The usual action of Frobenius on $A_{\mathrm{cris}}(V)$ also extends to an action on the log scheme $(\mathrm{Spec}(A_{\mathrm{cris}}(V)), M_{A_{\mathrm{cris}}(V)})$ by considering the multiplication by p map on N.

It follows that for any log geometric point $(\bar{b}, M_{\bar{b}}) \to (b_K, M_{b_K})$ we obtain an enlargement

$$(\mathrm{Spec}(\overline{V}/p\overline{V}), M_b|_{\overline{V}/p\overline{V}}) \hookrightarrow (\mathrm{Spec}(A_{\mathrm{cris}}(V)), M_{A_{\mathrm{cris}}(V)}) \tag{9.21.2}$$

with action of the group $\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}}))$ and a Frobenius automorphism. In particular, for any F–isocrystal E on the convergent topos of $(b_k, M_{b_k})/K$ we can evaluate E on this enlargement to obtain a $B_{\mathrm{cris}}(V)$–module with continuous action of $\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}}))$ and a semi–linear Frobenius automorphism. We will denote this data simply by $E(B_{\mathrm{cris}}(V), M_{B_{\mathrm{cris}}(V)})$.

Definition 9.22. Let (E, φ) be an F–isocrystal on $(b_k, M_{b_k})/K$, and let L be a smooth log–étale sheaf on (b_K, M_{b_K}). An *association* between (E, φ) and L is the data of an isomorphism

$$\iota_{(\bar{b}, M_{\bar{b}})} : E(B_{\mathrm{cris}}(V), M_{B_{\mathrm{cris}}(V)}) \simeq L \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \tag{9.22.1}$$

of $\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}}))$–modules compatible with the action of Frobenius for every log geometric point $(\bar{b}, M_{\bar{b}})$. These isomorphisms are furthermore required to be compatible with morphisms of log geometric points over (b_K, M_{b_K}) as in 6.13.

Remark 9.23. Note that as in 5.3 the data of an association is equivalent to the data of the isomorphism 9.22.1 for the choice of a single log geometric point. However, the above definition makes the notion independent of the choice of such a point.

9.24. If we fix a log geometric point $(\bar{b}, M_{\bar{b}}) \to (b_K, M_{b_K})$ and isomorphisms $M_{\bar{b}} \simeq \overline{K}^* \oplus \mathbb{Q}^r_{\geq 0}$ and $M_b \simeq V^* \oplus \mathbb{N}^r$ such that the map $M_b \to M_{\bar{b}}$ sends $\mathbb{N}^r$ to $\mathbb{Q}^r_{\geq 0}$, then we can describe everything explicitly as follows.

These choices induce an isomorphism

$$\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}})) \simeq \widehat{\mathbb{Z}}(1)^r \rtimes \mathrm{Gal}(\overline{K}/K), \tag{9.24.1}$$

where the semi–direct product is taken with respect to the natural action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathbb{Z}(1)$, as well as an isomorphism

$$\Omega^1_{(b, M_b)/V} \simeq \oplus_{i=1}^r V \cdot d\log(e_i), \tag{9.24.2}$$

where e_i denotes the i–th standard generator of $\mathbb{N}^r$.

For a unipotent F–isocrystal (E, φ) on $(b_k, M_{b_k})/K$ with associated object $(\mathcal{E}, N) \in V_{\mathrm{nilp}}(b_K, M_{b_K})$, define endomorphisms $N_i : \mathcal{E} \to \mathcal{E}$ by the formula

$$N(e) = \sum_{i=1}^r N_i(e) d\log(e_i), \tag{9.24.3}$$

and for an element $\sigma \in \pi_1((b_{\overline{K}}, M_{b_{\overline{K}}})$ let $\sigma_i \in \widehat{\mathbb{Z}}(1)$ be the i–th component of σ. Denote by $\bar{\sigma}_i$ the image of σ_i in $\mathbb{Z}_p(1)$.

There is a map

$$(\mathrm{Spec}(A_{\mathrm{cris}}(V)), M_{A_{\mathrm{cris}}(V)}) \to (b, M_b) \tag{9.24.4}$$

obtained from the map $\mathbb{N}^r \to M_{A_{\mathrm{cris}}(V)}$ which sends the i–th standard basis element $e_i \in \mathbb{N}^r$ to the Teichmuller lift of the element

$$\{\frac{1}{p^n} \in \mathbb{Q}^r_{\geq 0}\} \in \varprojlim \frac{1}{p^n} M_{V/p^nV} = N. \tag{9.24.5}$$

This retraction and the fact that E is an isocrystal gives an isomorphism

$$E(B_{\mathrm{cris}}(V), M_{B_{\mathrm{cris}}(V)}) \simeq \mathcal{E} \otimes_K B_{\mathrm{cris}}(V). \tag{9.24.6}$$

With this identification, the action of $\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}})) \simeq \widehat{\mathbb{Z}}(1) \rtimes \mathrm{Gal}(\overline{K}/K)$ on $\mathcal{E} \otimes_K B_{\mathrm{cris}}(V)$ is as follows: The group $\mathrm{Gal}(\overline{K}/K)$ acts via the action on the second factor $B_{\mathrm{cris}}(V)$ and an element σ_i of the i–th component $\widehat{\mathbb{Z}}(1) \subset \widehat{\mathbb{Z}}(1)^r$ acts by the formula

$$m \otimes 1 \mapsto \sum_{s \geq 0} N^s(m) \otimes \beta(\bar{\sigma}_i)^s, \tag{9.24.7}$$

where $\beta : \mathbb{Z}_p(1) \hookrightarrow B_{\mathrm{cris}}(V)$ is the map defined in [Ts1, p. 396]. The validity of this formula follows from the same reasoning used in [Fa1, top of p. 37].

9.25. If $(E, \varphi, \mathrm{Fil}_E) \in MF^\nabla_X(\Phi)$ is associated via ι to a smooth sheaf L on X^o_K, which we view as a smooth sheaf on the log étale site of (X_K, M_{X_K}), then the pullbacks $b^*(E, \varphi)$ and b^*L are naturally associated. This can be seen as follows.

Assume first that there exists a morphism $(b, M_b) \to U = \mathrm{Spec}(R)$ over X where U is a very small étale X–scheme. Let $(\bar{b}, M_{\bar{b}}) \to (b, M_b)$ be a log geometric point. Let $\overline{R}/R$ denote the normalization of R in the maximal extension of $\mathrm{Frac}(R)$ unramified over U^o_K.

Let (T, M_T) be the completion of U along b, with M_T the pullback of the log structure M_U. Let S be the coordinate ring of T, and let $\overline{S}$ be the normalization of S in the maximal extension of $\mathrm{Frac}(S)$ which is unramified over $S[1/pt_1 \cdots t_r]$.

Also choose specialization data as in 9.18. That is, a morphism s filling in the following diagram

$$\begin{tikzcd} (\bar{b}, M_{\bar{b}}) \arrow[r, "s"] \arrow[d] & (\mathrm{Spec}(\overline{S} \otimes K), M_{\overline{S} \otimes K}) \arrow[d] \\ (b, M_b) \arrow[r] & (T, M_T). \end{tikzcd} \tag{9.25.1}$$

Here the log structure on $\mathrm{Spec}(\overline{S} \otimes K)$ is defined as in 9.18.

Note that s also defines a commutative diagram

$$\begin{tikzcd} (\bar{b}, M_{\bar{b}}) \arrow[r, "s"] \arrow[d] & (\mathrm{Spec}(\overline{R} \otimes K), M_{\overline{R} \otimes K}) \arrow[d] \\ (b, M_b) \arrow[r] & (U, M_U), \end{tikzcd} \tag{9.25.2}$$

and hence by [Il2, 4.6], the specialization data s defines an isomorphism

$$\mathrm{Gal}(\overline{R}/R) \simeq \pi_1((U_K, M_{U_K}), (\bar{b}, M_{\bar{b}})). \tag{9.25.3}$$

The specialization data s includes a map $\overline{S} \to \overline{V}$, which in turn defines a morphism $A_{\mathrm{cris}}(S) \to A_{\mathrm{cris}}(V)$. This map can be extended to a morphism of enlargements

$$\begin{tikzcd} (\mathrm{Spec}(\overline{V}/p\overline{V}), M_V|_{\overline{V}/p\overline{V}}) \arrow[r] \arrow[d] & (\mathrm{Spec}(A_{\mathrm{cris}}(V)), M_{A_{\mathrm{cris}}(V)}) \arrow[d] \\ (\mathrm{Spec}(\overline{S}/p\overline{S}), M_S|_{\overline{S}/p\overline{S}}) \arrow[r] & (\mathrm{Spec}(A_{\mathrm{cris}}(S)), M_{A_{\mathrm{cris}}(S)}) \end{tikzcd} \tag{9.25.4}$$

as follows. The log structure $M_{A_{\mathrm{cris}}(S)}$ defined in 6.7 admits the following alternate description in this case. Define

$$\frac{1}{p^n}\Gamma(S, M_S) := \{m \in \Gamma(\overline{S}, M_{\overline{S}}) | p^n m \in \Gamma(S, M_S)\}, \tag{9.25.5}$$

and define

$$N_S := \varprojlim_n \frac{1}{p^n}\Gamma(S, M_S), \tag{9.25.6}$$

where the inverse limit is taken with respect to the multiplication by p maps. There is then a natural map

$$N_S \to \varprojlim(\overline{S}/p\overline{S}) \tag{9.25.7}$$

which when composed with the Teichmuller map defines a monoid map $N_S \to A_{\mathrm{cris}}(S)$. The log structure $M_{A_{\mathrm{cris}}(S)}$ is canonically isomorphic to the log structure associated to this prelog structure.

This alternate description of $M_{A_{\mathrm{cris}}(S)}$ shows in particular that the natural map $A_{\mathrm{cris}}(S) \to A_{\mathrm{cris}}(V)$ extends to a morphism of logarithmic enlargements as in 9.25.4, where the map on log structures is induced by the natural map $N_S \to N$ (where N is as in 9.19.4).

9.26. The choice of the specialization map s induces an action of $\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}}))$ on $E(B_{\mathrm{cris}}(S))$. It follows from the construction that the morphism of enlargements 9.25.4 is compatible with this action, and in addition is compatible with the Frobenius automorphisms. It follows that there is a canonical map

$$E(B_{\mathrm{cris}}(S)) \to E(B_{\mathrm{cris}}(V), M_{B_{\mathrm{cris}}(V)}) \tag{9.26.1}$$

compatible with the action of $\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}}))$ and Frobenius. In other words, there is a canonical isomorphism

$$E(B_{\mathrm{cris}}(V), M_{B_{\mathrm{cris}}(V)}) \simeq E(B_{\mathrm{cris}}(S)) \otimes_{B_{\mathrm{cris}}(S)} B_{\mathrm{cris}}(V) \tag{9.26.2}$$

compatible with the actions (note that a priori the action of $\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}}))$ need not descend to this quotient but the above shows that in fact it does descend).

On the other hand, the given association ι between $(E, \varphi, \mathrm{Fil}_E)$ and L defines an isomorphism

$$E(B_{\mathrm{cris}}(S)) \simeq L \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(S). \tag{9.26.3}$$

Tensoring with $B_{\text{cris}}(V)$ we obtain an isomorphism of $\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}}))$-modules

$$b^*E(B_{\text{cris}}(V), M_{B_{\text{cris}}(V)}) \simeq b^*L \otimes_{\mathbb{Q}_p} B_{\text{cris}}(V) \tag{9.26.4}$$

compatible with Frobenius. This is the desired pullback of the association ι. We leave to the reader the verification that it is independent of the choices in the above construction.

Remark 9.27. In the above we assume that there was a very small étale X–scheme U and a morphism $(b, M_b) \to (U, M_U)$ over (X, M_X). By [Fa2, II.2.1], such a neighborhood always exists when X is a scheme. To deal with the situation of X an algebraic space or Deligne–Mumford stack, one can proceed as follows. First note that after making a finite étale extension $V \to V'$ we do have such a morphism ρ, and hence we obtain an isomorphism

$$E(B_{\text{cris}}(V), M_{B_{\text{cris}}(V)}) \simeq L \otimes_{\mathbb{Q}_p} B_{\text{cris}}(V) \tag{9.27.1}$$

compatible with Frobenius and the action of the subgroup

$$\pi_1((b_{K'}, M_{b_{K'}}), (\bar{b}, M_{\bar{b}})) \subset \pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}})), \tag{9.27.2}$$

where K' denotes the field of fractions of V'. To define the pullback of ι it is therefore enough to prove that the isomorphism 9.27.1) is also compatible with the action of $\text{Gal}(K'/K)$. If $g \in \text{Gal}(K'/K)$, then the conjugate $g \circ 9.27.1 \circ g^{-1}$ is the isomorphism obtained from the map

$$(b_{K'}, M_{b_{K'}}) \xrightarrow{\ g\ } (b_{K'}, M_{b_{K'}}) \xrightarrow{\ \rho\ } (U, M_U), \tag{9.27.3}$$

and hence the desired extension of the action to $\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}}))$ follows from the independence of 9.27.1 on the choice of the map ρ.

Comparison theorem.

Let $\mathcal{C}_{\text{dR}}$ and $\mathcal{C}_{\text{et}}$ be as in 7.1.

Theorem 9.28. *With notation as in* 9.12 *and* 9.18*, there is a natural isomorphism*

$$X_{\mathcal{C}_{\text{dR}}} \otimes_K \widetilde{B}_{\text{cris}}(V) \simeq X_{\mathcal{C}_{\text{et}}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\text{cris}}(V) \tag{9.28.1}$$

in $\text{Ho}(\text{SPr}_*(\widetilde{B}_{\text{cris}}(V)))$ *compatible with the actions of Frobenius and Galois.*

Proof. Let $\widetilde{X}_{\mathcal{C}_{\text{dR}}}$ denotes the pointed stack obtained from carrying out the construction of 4.35 (see also 4.37) using the fiber functor

$$\tilde{\omega} : \mathcal{C}_{\text{dR}} \longrightarrow \text{Mod}_{\widetilde{B}_{\text{cris}}(V)}, \tag{9.28.2}$$

sending an isocrystal to its value on the enlargement 9.21.2. Then the proof of 1.7 carries over to give an isomorphism

$$\widetilde{X}_{\mathcal{C}_{\text{dR}}} \simeq X_{\mathcal{C}_{\text{et}}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\text{cris}}(V) \tag{9.28.3}$$

compatible with the Galois action. In fact, there is even a natural action of $\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}}))$ on both sides of 9.28.3 induced by the action on the enlargement 9.21.2 and the natural action on the fiber functor defined by $(\bar{b}, M_{\bar{b}})$. Chasing through the proof of 1.7 one sees that the isomorphism 9.28.3 is compatible with this action. On the other hand, the argument of 5.24 shows that the action of $\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}}))$ on $X_{\mathcal{C}_{\text{et}}}$ factors through $\text{Gal}(\overline{K}/K)$.

To prove the theorem, it therefore suffices to define an isomorphism $X_{\mathcal{C}_{\text{dR}}} \otimes_K \widetilde{B}_{\text{cris}}(V) \simeq \widetilde{X}_{\mathcal{C}_{\text{dR}}}$ compatible with Frobenius and $\text{Gal}(\overline{K}/K)$–action.

For this choose an isomorphism $M_b \simeq V^* \oplus \mathbb{N}^r$ for some r and for each $i = 1, \ldots, r$ choose a sequence of elements $m_n^i \in M_{\bar{b}}$ with $p^n \cdot m_n^i = e_i$, where e_i denotes the i–th standard generator of $\mathbb{N}^r$. The choice of the m_n^i defines a map $M_b \to N$, where N is as in 9.24.5, and hence we obtain a morphism of log schemes

$$r : (\mathrm{Spec}(A_{\mathrm{cris}}(V)), M_{A_{\mathrm{cris}}(V)}) \to (b, M_b). \tag{9.28.4}$$

Note that the choice of the m_n^i define an isomorphism $M_{\bar{b}} \simeq \overline{K}^* \oplus \mathbb{Q}^r_{\geq 0}$. Furthermore, the description in 9.14 of the group $\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}}))$ shows that there is a unique isomorphism

$$\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}})) \simeq \widehat{\mathbb{Z}}(1)^r \rtimes \mathrm{Gal}(\overline{K}/K) \tag{9.28.5}$$

such that the elements m_n^i are invariant under the action of $\mathrm{Gal}(\overline{K}/K)$. It follows that the retraction r is $\mathrm{Gal}(\overline{K}/K)$–equivariant. We therefore obtain an isomorphism of fiber functors

$$\omega_b \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq \tilde{\omega} \tag{9.28.6}$$

compatible with Frobenius and the action of $\mathrm{Gal}(\overline{K}/K)$. This isomorphism then induces an isomorphism of stacks $X_{\mathcal{C}_{\mathrm{dR}}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq \widetilde{X}_{\mathcal{C}_{\mathrm{dR}}}$ over $\widetilde{B}_{\mathrm{cris}}(V)$ compatible with the action of $\mathrm{Gal}(\overline{K}/K)$ and Frobenius.

To complete the proof of 9.28 it remains only to see that the isomorphism $X_{\mathcal{C}_{\mathrm{dR}}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq \widetilde{X}_{\mathcal{C}_{\mathrm{dR}}}$ is independent of the choice of r above. For this observe that a second choice of elements $\{m_n^{i\prime}\}$ differs from $\{m_n^i\}$ by the action of an element σ in

$$I := \mathrm{Ker}(\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}})) \to \mathrm{Gal}(\overline{K}/K)). \tag{9.28.7}$$

It follows that the isomorphism $X_{\mathcal{C}_{\mathrm{dR}}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq \widetilde{X}_{\mathcal{C}_{\mathrm{dR}}}$ obtained from the collection $\{m_n^{i\prime}\}$ differs from the one obtained from $\{m_n^i\}$ by the action on $\widetilde{X}_{\mathcal{C}_{\mathrm{dR}}}$ of an element in I. But as we have already shown that the action of I on $\widetilde{X}_{\mathcal{C}_{\mathrm{dR}}}$ is trivial it follows that the isomorphism $X_{\mathcal{C}_{\mathrm{dR}}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq X_{\mathcal{C}_{\mathrm{et}}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V)$ constructed above is independent of the choices. □

Corollary 9.29. *There is a natural isomorphism*

$$\pi_1(\mathcal{C}_{\mathrm{dR}}, b) \otimes_K B_{\mathrm{cris}}(V) \simeq \pi_1(\mathcal{C}_{\mathrm{et}}, \bar{b}) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \tag{9.29.1}$$

compatible with Frobenius and Galois actions.

Proof. This follows from the same argument proving 1.8. □

The log scheme (b, M_b) is a $K(\pi, 1)$.

Proposition 9.30. *If E is a unipotent isocrystal on $(b_k, M_{b_k})/K$, then there is a natural isomorphism*

$$H^*_{\mathrm{cris}}((b_k, M_{b_k})/K, E) \simeq H^*(\mathbb{G}_{\mathrm{dR}}, \mathcal{E}), \tag{9.30.1}$$

where the right hand side denotes group cohomology of the representation $\mathcal{E}$ obtained from E by applying the functor in 9.5.

Proof. Choose an embedding $(b, M_b) \hookrightarrow (T_b, M_{T_b})$ as in the proof of 9.7. We first construct the isomorphism 9.30.1 using this choice, and then prove that it is independent of the choice.

The crystalline cohomology of $(b_k, M_{b_k})/K$ is computed by the de Rham complex of the module with connection $E_{(T_{b,K}, M_{T_{b,K}})}$ on $(T_{b,K}, M_{T_{b,K}})$ over K. On the other hand, the de Rham complex of $(\mathcal{E}, N)$ on (b_K, M_{b_K}) over K is naturally isomorphic to the Hochschild complex of the representation $\mathcal{E}$ of $\mathrm{Lie}(\mathbb{G}_{\mathrm{dR}})$ corresponding to $(\mathcal{E}, N)$ [K-T, 4.27]. The restriction map from the de Rham cohomology of $E_{(T_{b,K}, M_{T_{b,K}})}$ to the de Rham cohomology of $(\mathcal{E}, N)$ therefore defines a map

$$H^*_{\mathrm{cris}}((b_k, M_{b_k})/K, E) \to H^*(\mathbb{G}_{\mathrm{dR}}, \mathcal{E}). \tag{9.30.2}$$

Filtering E by sub–isocrystals such that the successive quotients are trivial isocrystals and consideration of the associated long exact sequences shows that to prove that the restriction map 9.30.2 is an isomorphism it suffices to consider the case when E is the trivial isocrystal.

Let $K\{\{t_1, \ldots, t_r\}\}$ denote the value of $\mathcal{K}_{(b_k, M_{b_k})/K}$ on the widening

$$(b_k, M_{b_k}) \hookrightarrow (\mathrm{Spf}(V[[t_1, \ldots, t_r]]), M_{V[[t_1,\ldots,t_r]]}), \tag{9.30.3}$$

where the log structure $M_{V[[t_1,\ldots,t_r]]}$ is induced by the natural map $\mathbb{N}^r \to V[[t_1, \ldots, t_r]]$.

Then the de Rham complex of $\mathcal{K}_{(b_k, M_{b_k})/K}$ is given by the de Rham complex of

$$d : K\{\{t_1, \ldots, t_r\}\} \to K\{\{t_1, \ldots, t_r\}\} \otimes_K \Omega(\log), \quad t_i \mapsto t_i \otimes d\log(t_i). \tag{9.30.4}$$

where $\Omega(\log)$ denotes the free K-vector space with basis $d\log(t_i)$ $(i = 1, \ldots, r)$. The de Rham complex of $(\mathcal{E}, N)$ is the complex with zero differential and i–th term

$$\Omega^i(\log) := \bigwedge^i \Omega(\log). \tag{9.30.5}$$

Let $J \subset K\{\{t_1, \ldots, t_r\}\}$ be the ideal defined by $(t_1, \ldots, t_r)$ so that setting J to zero defines a map of complexes

$$K\{\{t_1, \ldots, t_r\}\} \otimes \Omega^{\cdot}(\log) \to \Omega^{\cdot}(\log). \tag{9.30.6}$$

We claim that this reduction map is a quasi-isomorphism.

For this let Ω denote the free K-vector space of rank r and basis $\{dt_i\}$. Then the kernel of 9.30.6 is the complex

$$J \to K\{\{t_1, \ldots, t_r\}\} \otimes_K \Omega \to K\{\{t_1, \ldots, t_r\}\} \otimes_K \Omega^2 \to \cdots, \tag{9.30.7}$$

which agrees with the usual de Rham complex computing the convergent cohomology of the point $\mathrm{Spec}(k)$ (with no log structure) except in degree 0. Since $\mathrm{Spec}(k)$ has trivial cohomology, the map

$$K \to (K\{\{t_1, \ldots, t_r\}\} \to K\{\{t_1, \ldots, t_r\}\} \otimes_K \Omega \to K\{\{t_1, \ldots, t_r\}\} \otimes_K \Omega^2 \to \cdots) \tag{9.30.8}$$

is a quasi-isomorphism, which implies that 9.30.7 is acyclic so 9.30.6 is a quasi-isomorphism.

It remains only to see that the functor 9.30.2 is independent of the choices. For this note that the functor which to any representation V of $\mathbb{G}_{\mathrm{dR}}$ associates the corresponding isocrystal $\mathcal{V}$ defines a functor

$$j : \mathrm{Rep}(\mathbb{G}_{\mathrm{dR}}) \to (\text{sheaves on convergent site of } (b_k, M_{b_k})/K). \tag{9.30.9}$$

Furthermore, there is a natural isomorphism of functors

$$(V \mapsto V^{\mathbb{G}_{\mathrm{dR}}}) \to \Gamma_{\mathrm{cris}} \circ j. \tag{9.30.10}$$

By the universality of the δ-functor $H^*(\mathbb{G}_{\mathrm{dR}}, -)$ we therefore obtain a map of δ–functors

$$H^*_{\mathrm{cris}}((b_k, M_{b_k})/K, j(-)) \leftarrow H^*(\mathbb{G}_{\mathrm{dR}}, -). \tag{9.30.11}$$

This gives a canonical description of the map 9.30.2 and hence completes the proof of the Proposition. □

Corollary 9.31. *Let $\mathbb{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}})$ denote the ind–isocrystal on $(b_k, M_{b_k})/K$ obtained from the coordinate ring $\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}}$ with its natural action of $\mathbb{G}_{\mathrm{dR}}$ coming from right translation. Then $H^i_{\mathrm{cris}}((b_k, M_{b_k})/K, \mathbb{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}})) = 0$ for $i > 0$ and $H^0_{\mathrm{cris}}((b_k, M_{b_k})/K, \mathbb{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}})) = K$.*

Proof. This follows from [Ol1, 2.18 (i)] and the above which shows that $\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}}$ is injective in $\mathrm{Rep}(\mathbb{G}_{\mathrm{dR}})$ and has $\mathbb{G}_{\mathrm{dR}}$–invariants equal to K. □

9.32. Let E be a unipotent isocrystal on $(b_k, M_{b_k})/K$. We can also compute cohomology of E using a resolution as in 4.31. Choose an inclusion $i : (b, M_b) \hookrightarrow (T_b, M_{T_b})$ as in the proof of 9.5. By the same construction used in the proof of 4.31, we obtain a complex $\mathbb{R}^\bullet$ of sheaves on $(b_k, M_{b_k})/K$ with a morphism $E \to \mathbb{R}^\bullet$.

Recall that $\mathbb{R}^i$ is constructed as follows. Let

$$j : ((b_k, M_{b_k})/V)_{\mathrm{conv}}|_{(T_b, M_{T_b})} \to ((b_k, M_{b_k})/V)_{\mathrm{conv}} \tag{9.32.1}$$

be the localization morphism. Let

$$\phi^*\gamma^* : T_{b,\mathrm{et}} \to ((b_k, M_{b_k})/V)_{\mathrm{conv}}|_{(T_b, M_{T_b})} \tag{9.32.2}$$

be the functor defined in 3.17. If $\Omega^i_{(T_b, M_{T_b})/K}$ denotes the sheaf of logarithmic differentials of (T_b, M_{T_b}) tensored with K then by definition we have

$$\mathbb{R}^i = j_*\phi^*\gamma^*\Omega^i_{(T_b, M_{T_b})/K}. \tag{9.32.3}$$

The inclusion $i : (b, M_b) \hookrightarrow (T_b, M_{T_b})$ induces a morphism of topoi

$$\epsilon : ((b_k, M_{b_k})/V)_{\mathrm{conv}}|_{(b, M_b)} \to ((b_k, M_{b_k})/V)_{\mathrm{conv}}|_{(T_b, M_{T_b})} \tag{9.32.4}$$

which sits in a commutative diagram

$$\begin{tikzcd} ((b_k, M_{b_k})/V)_{\mathrm{conv}} & ((b_k, M_{b_k})/V)_{\mathrm{conv}}|_{(b, M_b)} \arrow[l, "\tilde{j}"'] \arrow[d, "\epsilon"] \\ & ((b_k, M_{b_k})/V)_{\mathrm{conv}}|_{(T_b, M_{T_b})}. \arrow[ul, "j"'] \end{tikzcd} \tag{9.32.5}$$

Observe that there is also a commutative diagram of functors

$$\begin{tikzcd} T_{b,\mathrm{et}} \arrow[r, "\phi^*_{T_b}\gamma^*"] \arrow[d, "i^*"] & ((b_k, M_{b_k})/V)_{\mathrm{conv}}|_{(T_b, M_{T_b})} \arrow[d, "\epsilon^*"] \\ (\mathrm{Spec}(V))_{\mathrm{et}} \arrow[r, "\phi^*_b\gamma^*"] & ((b_k, M_{b_k})/V)_{\mathrm{conv}}|_{(b, M_b)}. \end{tikzcd} \tag{9.32.6}$$

9.33. Let

$$T=((b_k,M_k)\hookrightarrow(\mathrm{Spf}(A),M_A)),\quad T'=((b_k,M_k)\hookrightarrow(\mathrm{Spf}(B),M_B)) \tag{9.33.1}$$

be two widenings. Let

$$(b,M_b)\hookrightarrow(I_{T,T'},M_{T,T'})$$

denote the product in the category of widenings of T and T'. By definition this object represents the functor

$$(\text{widenings }(b_k,M_k)\hookrightarrow(\mathrm{Spf}(C),M_C))^{\mathrm{op}}\to\mathrm{Set} \tag{9.33.2}$$

sending $(b_k,M_k)\hookrightarrow(\mathrm{Spf}(C),M_C)$ to the set of pairs (f,g), where

$$f:((b_k,M_k)\hookrightarrow(\mathrm{Spf}(C),M_C))\to T \tag{9.33.3}$$

and

$$g:((b_k,M_k)\hookrightarrow(\mathrm{Spf}(C),M_C))\to T' \tag{9.33.4}$$

are morphisms of widenings.

Remark 9.34. Note that $I_{T,T'}$ comes equipped with two morphisms of widenings

$$\mathrm{pr}_1:I_{T,T'}\to T,\quad \mathrm{pr}_2:I_{T,T'}\to T'. \tag{9.34.1}$$

Remark 9.35. Since (b_k,M_{b_k}) is hollow, if $(b_k,M_k)\hookrightarrow(\mathrm{Spf}(C),M_C)$ is a widening then the nonunits of M_C map to topologically nilpotent elements of C. The images of the nonunits in M_C therefore define an ideal. We let C_0 denote the quotient of C by this ideal, and let T_0 denote the resulting widening

$$(b_k,M_{b_k})\hookrightarrow(\mathrm{Spf}(C_0),M_{C_0}). \tag{9.35.1}$$

Observe that if T' is hollow then the natural map

$$I_{T_0,T'}\to I_{T,T'} \tag{9.35.2}$$

is an isomorphism.

9.36. By definition of the localized topos, for any $F\in((b_k,M_{b_k})/V)_{\mathrm{conv}}|_{(T_b,M_{T_b})}$ the sheaf j_*F associates to any enlargement $T:(b_k,M_{b_k})\hookrightarrow(\mathrm{Spf}(A),M_A)$ the set $F(I_{(T_b,M_{T_b}),T})$. If T is hollow, then by 9.35 this is equal to $F(I_{(b,M_b),T})$.

9.37. Let $\widehat{\mathbb{G}}^r_m$ denote the formal completion of $\mathbb{G}^r_{m,V}$ along the closed immersion $\mathrm{Spec}(k)\hookrightarrow\mathbb{G}^r_{m,V}$ defined by the identity section. Then there is a canonical isomorphism

$$I_{(b,M_b),(b,M_b)}\simeq\widehat{\mathbb{G}}^r_m \tag{9.37.1}$$

with the log structure on $\widehat{\mathbb{G}}^r_m$ given by the map $\mathbb{N}^r\to\mathcal{O}_{\widehat{\mathbb{G}}^r_m}$ sending all nonzero elements to 0. If $u_1,\dots,u_r$ are the standard coordinates on $\mathbb{G}^r_m$ then the two projections

$$\mathrm{pr}_1,\mathrm{pr}_2:(\widehat{\mathbb{G}}^r_m,M_{\widehat{\mathbb{G}}^r_m})\to(b,M_b) \tag{9.37.2}$$

are induced by the two maps

$$\tau_1,\tau_2:\mathbb{N}^r\to\mathcal{O}^*_{\widehat{\mathbb{G}}^r_m}\oplus\mathbb{N}^r, \tag{9.37.3}$$

where $\tau_1(e_i)=(1,e_i)$ and $\tau_2(e_i)=(u_i,e_i)$.

This implies that for any morphism of widenings $T \to (b, M_b)$ there is a canonical isomorphism

$$I_{T,(b,M_b)} \simeq T \widehat{\times} \widehat{\mathbb{G}}_m^r, \tag{9.37.4}$$

where the right side denotes the product in the category of formal V-schemes.

By a similar argument one sees that if $Z \to (T_b, M_{T_b})$ is a morphism of widenings, then

$$I_{Z,(T_b,M_{T_b})} \simeq Z \widehat{\times} \widehat{\mathbb{G}}_m^r. \tag{9.37.5}$$

Lemma 9.38. *Let $\mathbb{R}^i_{(b,M_b)}$ denote the value of $\mathbb{R}^i$ on the enlargement $(b_k, M_{b_k}) \hookrightarrow (b, M_b)$. Let ∇ denote the connection on $\mathbb{R}^i_{(b,M_b)}$ and let $DR(\mathbb{R}^i_{(b,M_b)})$ denote the associated de Rham complex. Then the natural map*

$$(\mathbb{R}^i_{(b,M_b)})^{\nabla} \to DR(\mathbb{R}^i_{(b,M_b)}) \tag{9.38.1}$$

is a quasi-isomorphism.

Proof. Note first that it follows from the construction of the complex $\mathbb{R}^\bullet$ that if

$$0 \to E_1 \to E \to E_2 \to 0 \tag{9.38.2}$$

is an exact sequence of isocrystals on $(b_k, M_{b_k})/K$ then there is a natural exact sequence of complexes

$$0 \to \mathbb{R}^\bullet_1 \to \mathbb{R}^\bullet \to \mathbb{R}^\bullet_2 \to 0 \tag{9.38.3}$$

where $\mathbb{R}^\bullet_i$ denotes the complex obtained from E_i. By consideration of a filtration of E and the corresponding long exact sequences of cohomology groups, it therefore suffices to prove the lemma when E is the trivial isocrystal $\mathcal{K}$.

In this case, we can compute $\mathbb{R}^i_{(b,M_b)}$ with its connection as follows. Let $\Omega(\log)$ denote $i^*\Omega^1_{(T_b,M_{T_b})/K}$. By the discussion in 9.36, and using the isomorphism 9.37.1, we see that

$$\mathbb{R}^i_{(b,M_b)} = K\{\{u_1 - 1, \ldots, u_r - 1\}\} \otimes_K \Omega^i(\log), \tag{9.38.4}$$

where $\Omega^i(\log) = \bigwedge^i \Omega(\log)$ and $K\{\{u_1 - 1, \ldots, u_r - 1\}\}$ is defined to be the value of $\mathcal{K}$ on the widening

$$(b_k, M_{b_k}) \hookrightarrow (\widehat{\mathbb{G}}_m^r, M_{\widehat{\mathbb{G}}_m^r}). \tag{9.38.5}$$

Moreover, it follows from the construction of the connection that

$$\nabla((u_j - 1)^k \otimes \omega) = (k(u_j - 1)^k + (u_j - 1)^{k-1}) \otimes \omega)\mathrm{dlog}(t_j), \tag{9.38.6}$$

where we write $d\log(t_j)$ for the image of the j-th standard generator of $\mathbb{N}^r$ under the map

$$\mathbb{N}^r \longrightarrow M_b \xrightarrow{d\log} \Omega^1_{(b,M_b)/V}. \tag{9.38.7}$$

Note that it follows from 9.38.6 that it further suffices to prove the lemma in the case when $i = 0$, which we assume henceforth.

Let $\Omega^1_{\widehat{\mathbb{G}}_m^r/V}$ denote the free $K\{\{u_1 - 1, \ldots, u_r - 1\}\}$ module on generators du_i/u_i ($i = 1, \ldots, r$). If we identify $K\{\{u_1 - 1, \ldots, u_r - 1\}\} \otimes_K \Omega^1_{(b,M_b)/V}$ with $\Omega^1_{\widehat{\mathbb{G}}_m^r/V}$ via the map sending $d\log(t_j)$ to du_j/u_j, then the connection 9.38.6 becomes identified with the standard connection

$$d : K\{\{u_1 - 1, \dots, u_r - 1\}\} \to K\{\{u_1 - 1, \dots, u_r - 1\}\} \otimes \Omega^1_{\widehat{\mathbb{G}}^r_m/V}. \tag{9.38.8}$$

Now the de Rham complex of this connection computes the convergent cohomology of Spec(k) over K (with no log structures), and hence the de Rham complex is acyclic. □

Lemma 9.39. *The map $E \to \mathbb{R}^\bullet$ in 9.32 is a quasi-isomorphism.*

Proof. It suffices to show that the map

$$j^*E \to j^*\mathbb{R}^\bullet \tag{9.39.1}$$

is a quasi-isomorphism in $((b_k, M_{b_k})/V)_{\rm conv}|_{(T_b, M_{T_b})}$. For any enlargment $Z : (b_k, M_k) \hookrightarrow (Z, M_Z)$ with a morphism $Z \to (T_b, M_{T_b})$ the value of $j^*\mathbb{R}^i$ on Z is by the isomorphism 9.37.5 equal to

$$\mathcal{O}_Z\{\{u_1 - 1, \dots, u_r - 1\}\} \otimes_{\mathcal{O}_{T_b}} \Omega^i_{(T_b, M_{T_b})/V}, \tag{9.39.2}$$

where $\mathcal{O}_Z\{\{u_1, \dots, u_r - 1\}\}$ denotes the value of $\mathcal{K}$ on the widening

$$(b_k, M_{b_k}) \hookrightarrow Z \widehat{\times} \widehat{\mathbb{G}}^r_m. \tag{9.39.3}$$

It follows from the construction of the map $D^i : \mathbb{R}^i \to \mathbb{R}^{i+1}$ that it restricts over Z to the unique $\mathcal{O}_Z$–linear map sending $(\prod_j (u_j - 1)^{a_j}) \otimes \omega$ to

$$(\sum_j a_j u_j (u_1 - 1)^{a_1} \cdots (u_j - 1)^{a_j - 1} \cdots (u_r - 1)^{a_r}) \otimes d\log(t'_j) \wedge \omega, \tag{9.39.4}$$

where $\{d\log(t'_j)\}$ denotes the basis for $\Omega^1_{(T_b, M_{T_b})/V}$ defined by the chart $\mathbb{N}^r \to M_{T_b}$. Observe that $D^i(u_j)/u_j = \mathrm{dlog}(t'_j)$.

Let $\widetilde{\mathbb{R}}^\bullet$ denote the resolution of $\mathcal{K}$ in $(\mathrm{Spec}(k)/V)_{\rm conv}$ given by the embedding $\mathrm{Spec}(k) \hookrightarrow \widehat{\mathbb{G}}^r_m$. If

$$\sigma : (\mathrm{Spec}(k)/V)_{\rm conv}|_{\widehat{\mathbb{G}}^r_m} \to (\mathrm{Spec}(k)/V)_{\rm conv} \tag{9.39.5}$$

is the localization morphism, then

$$\widetilde{\mathbb{R}}^i := \sigma_* \phi^*_{\widehat{\mathbb{G}}^r_m} \gamma^* \Omega^i_{\widehat{\mathbb{G}}^r_m}. \tag{9.39.6}$$

Now from the explicit formula 9.39.4 and the observation that $D^i(u_j)/u_j = \mathrm{dlog}(t'_j)$, we see that under the canonical isomorphism of topoi (where the right side is the convergent topos with no log structures)

$$((b_k, M_{b_k})/V)_{\rm conv}|_{(T_b, M_{T_b})} \simeq (\mathrm{Spec}(k)/T_b)_{\rm conv} \tag{9.39.7}$$

the complex $j^*\mathbb{R}^\bullet$ becomes identified with the restriction of $\widetilde{\mathbb{R}}^\bullet$ to $(\mathrm{Spec}(k)/T_b)_{\rm conv}$. Since $\widetilde{\mathbb{R}}^\bullet$ is a resolution of $\mathcal{K}$ by [Og2, 0.5.4] this implies the lemma. □

Corollary 9.40. *Let $E_{(b,M_b)}$ denote value of E on the enlargement $(b_k, M_{b_k}) \hookrightarrow (b, M_b)$. Then the natural map $E_{(b,M_b)} \to \mathbb{R}^\bullet_{(b,M_b)}$ is a quasi-isomorphism. In particular, by 9.38 we have a diagram of quasi-isomorphisms*

(9.40.1)
$$\begin{tikzcd} & \mathbb{R}^{\bullet,\nabla}_{(b,M_b)} \arrow[d] \\ DR(E_{(b,M_b)}) \arrow[r] & DR(\mathbb{R}^\bullet_{(b,M_b)}). \end{tikzcd}$$

Lemma 9.41. *The reduction map*

(9.41.1)
$$DR(\mathbb{R}^\bullet_{(T_b,M_{T_b})}) \to DR(\mathbb{R}^\bullet_{(b,M_b)})$$

is a quasi-isomorphism.

Proof. As in the proof of 9.38 it suffices to consider the case when E is the trivial isocrystal. Since $\mathcal{K} \to \mathbb{R}^\bullet$ is a quasi-isomorphism, it suffices to show that the reduction map

(9.41.2)
$$DR(\mathcal{K}_{(T_b,M_{T_b})}) \to DR(\mathcal{K}_{(b,M_b)})$$

is a quasi-isomorphism. This follows from the same argument used in the proof of 9.30. $\square$

Summary 9.42. *The map $E \to \mathbb{R}^\bullet$ is a quasi–isomorphism, and the maps*

(9.42.1)
$$DR(\mathbb{R}^\bullet_{(T_b,M_{T_b})}) \to DR(\mathbb{R}^\bullet_{(b,M_b)}), \quad \mathbb{R}^{\bullet,\nabla}_{(b,M_b)} \to DR(\mathbb{R}^\bullet_{(b,M_b)})$$

are quasi–isomorphisms, where $\mathbb{R}^\bullet_{(T_b,M_{T_b})}$ (resp. $\mathbb{R}^\bullet_{(b,M_b)}$) is the restriction of $\mathbb{R}^\bullet$ to (T_b, M_{T_b}) (resp. (b, M_b)) with the natural connection defined by the isocrystal structure, and $DR(-)$ denotes the de Rham complex of $(-)$. In particular, the de Rham complex of $E_{(T_b,M_{T_b})}$ is naturally isomorphic in the derived category to $\mathbb{R}^{\bullet,\nabla}_{(b,M_b)}$.

There is also an étale version of 9.30.

Proposition 9.43. *For any unipotent $\mathbb{Q}_p$–sheaf L on $(b_{\overline{K}}, M_{b_{\overline{K}}})$, there is a natural isomorphism*

(9.43.1)
$$H^*((b_{\overline{K}}, M_{b_{\overline{K}}}), L) \simeq H^*(\mathbb{G}_{\mathrm{et}}, L_{(\bar{b},M_{\bar{b}})}),$$

where the right hand side denotes group cohomology.

Proof. Since any Kummer étale map $V \to (b_{\overline{K}}, M_{b_{\overline{K}}})$ is a Kummer covering [Il2, 3.11], the cohomology $H^*((b_{\overline{K}}, M_{b_{\overline{K}}}), L)$ is isomorphic to the continuous group cohomology of the $\pi_1((b_{\overline{K}}, M_{b_{\overline{K}}}), (\bar{b}, M_{\bar{b}}))$–module $L_{(\bar{b},M_{\bar{b}})}$. The functor which sends a representation L of $\mathbb{G}_{\mathrm{et}}$ to $H^*((b_{\overline{K}}, M_{b_{\overline{K}}}), L)$ defines a cohomological δ–functor on the category $\mathrm{Rep}(\mathbb{G}_{\mathrm{et}})$, and hence the natural isomorphism $H^0(\mathbb{G}_{\mathrm{et}}, -) \simeq H^0((b_{\overline{K}}, M_{b_{\overline{K}}}), -)$ induces a natural morphism of cohomological δ–functors

(9.43.2)
$$H^*(\mathbb{G}_{\mathrm{et}}, -) \to H^*((b_{\overline{K}}, M_{b_{\overline{K}}}), -).$$

To prove that this map is an isomorphism, choose an isomorphism $M_b \simeq K^* \oplus \mathbb{N}^r$ defining isomorphisms

(9.43.3)
$$\pi_1((b_{\overline{K}}, M_{b_{\overline{K}}}), (\bar{b}, M_{\bar{b}})) \simeq \widehat{\mathbb{Z}}(1)^r, \quad \mathbb{G}_{\mathrm{et}} \simeq \widehat{\mathbb{Z}}(1)^r \otimes_{\widehat{\mathbb{Z}}} \mathbb{G}_a.$$

The exact sequence

$$0 \longrightarrow \widehat{\mathbb{Z}}(1)^{r-1} \xrightarrow{\underline{a}\mapsto(\underline{a},0)} \widehat{\mathbb{Z}}(1)^{r} \longrightarrow \widehat{\mathbb{Z}}(1) \longrightarrow 0 \tag{9.43.4}$$

induces for every representation L of $\mathbb{G}_{\mathrm{et}}$ spectral sequences [Se, I.2.6]

$$E_2^{pq} = H^p(\widehat{\mathbb{Z}}(1), H^q(\widehat{\mathbb{Z}}(1)^{r-1}, L)) \implies H^{p+q}(\widehat{\mathbb{Z}}(1)^r, L) \tag{9.43.5}$$

$$E_2^{pq} = H^p(\widehat{\mathbb{Z}}(1)\otimes_{\widehat{\mathbb{Z}}}\mathbb{G}_a, H^q((\widehat{\mathbb{Z}}(1)\otimes_{\mathbb{Z}}\mathbb{G}_a)^{r-1}, L)) \implies H^{p+q}(\mathbb{G}_{\mathrm{et}}, L). \tag{9.43.6}$$

These spectral sequences are compatible with the morphisms of δ–functors 9.43.2. This reduces the problem to the case when $r = 1$.

When $r = 1$, the groups $H^i(\widehat{\mathbb{Z}}(1), L)$ are zero for $i > 1$ by [Se, p. I-19, exemple 1]. Also the groups $H^i(\mathbb{G}_{\mathrm{et}}, L)$ are zero for $i > 1$ (this can be seen for example by noting that the Hochschild complex computing this cohomology has no terms in degrees ≥ 2). Thus in this special case it suffices to consider $i = 0$ and $i = 1$ in which case the result is clear (for $i = 0$ both the groups compute invariants, and for $i = 1$ they compute extensions of $\mathbb{Q}_p$ by L). □

Corollary 9.44. *Let $\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}})$ denote the ind–sheaf corresponding to the coordinate ring $\mathcal{O}_{\mathbb{G}_{\mathrm{et}}}$ with $\mathbb{G}_{\mathrm{et}}$–action induced by right translation. Then $H^i((b_{\overline{K}}, M_{b_{\overline{K}}}), \mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}})) = 0$ for $i > 0$ and $H^i((b_{\overline{K}}, M_{b_{\overline{K}}}), \mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}})) = \mathbb{Q}_p$.*

Proof. As in 9.31, this follows from [Ol1, 2.18 (i)] which shows that $\mathcal{O}_{\mathbb{G}_{\mathrm{et}}}$ is injective in the category $\mathrm{Rep}(\mathbb{G}_{\mathrm{et}})$ and has $\mathbb{G}_{\mathrm{et}}$–invariants equal to $\mathbb{Q}_p$. □

Compatibility with $\iota_{\mathbb{G}}$.

9.45. Let $\mathcal{C}_{\mathrm{dR}}$ and $\mathcal{C}_{\mathrm{et}}$ be as in 7.1. The functors 9.11.1 and 9.17.1

$$\omega_b^{\mathrm{dR}} : \mathcal{C}_{\mathrm{dR}} \to V_{\mathrm{nilp}}(b, M_b), \quad \omega_b^{\mathrm{et}} : \mathcal{C}_{\mathrm{et}} \to (\text{unipotent smooth sheaves on } (b_{\overline{K}}, M_{b_{\overline{K}}})) \tag{9.45.1}$$

induce by Tannaka duality morphisms of group schemes

$$\ell_{\mathrm{dR}} : \mathbb{G}_{\mathrm{dR}} \to \pi_1(\mathcal{C}_{\mathrm{dR}}, b), \quad \ell_{\mathrm{et}} : \mathbb{G}_{\mathrm{et}} \to \pi_1(\mathcal{C}_{\mathrm{et}}, b). \tag{9.45.2}$$

Theorem 9.46. *The diagram*

$$\begin{tikzcd} \mathbb{G}_{\mathrm{dR}}\otimes_K B_{\mathrm{cris}}(V) \arrow[r, "\ell_{\mathrm{dR}}"] \arrow[d, "\iota_{\mathbb{G}}"'] & \pi_1(\mathcal{C}_{\mathrm{dR}}, b)\otimes_K B_{\mathrm{cris}}(V) \arrow[d, "\iota_{\mathcal{C}}"] \\ \mathbb{G}_{\mathrm{et}}\otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \arrow[r, "\ell_{\mathrm{et}}"] & \pi_1(\mathcal{C}_{\mathrm{et}}, b)\otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \end{tikzcd} \tag{9.46.1}$$

commutes.

The proof is in several steps 9.47–9.58.

9.47. Let $\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}}$ (resp. $\mathcal{O}_{\mathbb{G}_{\mathrm{et}}}$) denote the coordinate ring of $\mathbb{G}_{\mathrm{dR}}$ (resp. $\mathbb{G}_{\mathrm{et}}$) which we view as a $\mathbb{G}_{\mathrm{dR}}$–bimodule (resp. $\mathbb{G}_{\mathrm{et}}$–bimodule) with the action coming from left and right translation. Denote by $\mathbb{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}})$ (resp. $\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}})$) the unipotent isocrystal on $(b_k, M_{b_k})/K$ (resp. smooth log-étale sheaf on $(b_{\overline{K}}, M_{b_{\overline{K}}})$) with right $\mathbb{G}_{\mathrm{dR}}$–action (resp. $\mathbb{G}_{\mathrm{et}}$–action) induced by Tannaka duality and the left action.

Let $\mathcal{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}})$ denote the ind–object in $V_{\mathrm{nilp}}(b_K, M_{b_K})$ obtained from $\mathbb{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}})$, and denote by $\mathbb{R}\Gamma_{\mathrm{dR}}(\mathcal{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}}))$ its de Rham complex. The ring structure on $\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}}$ gives this the structure of a differential graded algebra and the right action of $\mathbb{G}_{\mathrm{dR}}$ makes it an object of $\mathbb{G}_{\mathrm{dR}}-\mathrm{dga}_K$. Denote by Y_{dR} the stack obtained from this equivariant differential graded algebra:

$$Y_{\mathrm{dR}} := [\mathbb{R}\mathrm{Spec}_{\mathbb{G}_{\mathrm{dR}}}(\mathbb{R}\Gamma_{\mathrm{dR}}(\mathcal{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}})))/\mathbb{G}_{\mathrm{dR}}]. \tag{9.47.1}$$

Similarly, let $\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}}))$ denote the group cohomology of the representation $\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}})$. The cohomology $\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}}))$ is a $\mathbb{G}_{\mathrm{et}}$–equivariant differential graded algebra, and we define

$$Y_{\mathrm{et}} := [\mathbb{R}\mathrm{Spec}_{\mathbb{G}_{\mathrm{et}}}(\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}})))/\mathbb{G}_{\mathrm{et}}]. \tag{9.47.2}$$

As in 4.34 and 5.21, the identity elements of $\mathbb{G}_{\mathrm{dR}}$ and $\mathbb{G}_{\mathrm{et}}$ give Y_{dR} and Y_{et} natural structures of pointed stacks.

Lemma 9.48. *The projections $Y_{\mathrm{dR}} \to B\mathbb{G}_{\mathrm{dR}}$ and $Y_{\mathrm{et}} \to B\mathbb{G}_{\mathrm{et}}$ are isomorphisms in $\mathrm{Ho}(\mathrm{SPr}_*(K))$ and $\mathrm{Ho}(\mathrm{SPr}_*(\mathbb{Q}_p))$ respectively.*

Proof. As in [KPT, 1.3.10], the homotopy fiber of $Y_{\mathrm{dR}} \to B\mathbb{G}_{\mathrm{dR}}$ (resp. $Y_{\mathrm{et}} \to B\mathbb{G}_{\mathrm{et}}$) is isomorphic to $\mathbb{R}\mathrm{Spec}(\mathbb{R}\Gamma_{\mathrm{dR}}(\mathcal{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}})))$ (resp. $\mathbb{R}\mathrm{Spec}(\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}}))))$. The result therefore follows from 9.31 and 9.44. □

9.49. More generally, if $U_\bullet \to b$ is an étale hypercover, we can define $\mathbb{R}\Gamma_{\mathrm{dR}}(\mathcal{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}}))_{U_\bullet}$ to be the $\mathbb{G}_{\mathrm{dR}}$–equivariant differential graded algebra obtained by applying the functor of Thom–Sullivan cochains to the $\mathbb{G}_{\mathrm{dR}}$–equivariant cosimplicial algebra which associates to $[n] \in \Delta$ the de Rham complex of $\mathcal{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}})$ restricted to $(U_n, M_b|_{U_n})$. There is a natural map $\mathbb{R}\Gamma_{\mathrm{dR}}(\mathcal{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}})) \to \mathbb{R}\Gamma_{\mathrm{dR}}(\mathcal{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}}))_{U_\bullet}$ which cohomological descent implies is an isomorphism.

Let $U_\bullet \to X$ be an étale hypercover with each U_n an affine scheme, and let $U_{b,\bullet}$ be the pullback to b. Denote by $\mathbb{R}\Gamma_{\mathrm{dR}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}))_{U_\bullet}$ the G_{dR}–equivariant differential graded algebra obtained from the cosimplicial differential graded algebra which to $[n] \in \Delta$ associates the de Rham complex of the module with connection $\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})$ restricted to (U_n, M_{U_n}). There is a natural map $\mathbb{G}_{\mathrm{dR}} \to G_{\mathrm{dR}}$ which induces a morphism $\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}})|_{U_{b,\bullet}} \to \mathcal{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}})$ of modules with integrable connection on $(U_{b,\bullet}, M_B|_{U_{b,\bullet}})$. We therefore obtain a map of differential graded algebras

$$\mathbb{R}\Gamma_{\mathrm{dR}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}))_{U_\bullet} \to \mathbb{R}\Gamma_{\mathrm{dR}}(\mathcal{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}}))_{U_{b,\bullet}}. \tag{9.49.1}$$

If we view $\mathbb{R}\Gamma_{\mathrm{dR}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}))_{U_\bullet}$ as a $\mathbb{G}_{\mathrm{dR}}$–equivariant algebra via the map $\mathbb{G}_{\mathrm{dR}} \hookrightarrow G_{\mathrm{dR}}$, then this map is even $\mathbb{G}_{\mathrm{dR}}$–equivariant. Here G_{dR} denotes the Tannaka dual of the category $\mathcal{D}_{\mathrm{dR}} := \langle E \rangle_\otimes$ with respect to the fiber functor defined by (b, M_b).

Applying the functor $\mathbb{R}\mathrm{Spec}_{\mathbb{G}_{\mathrm{dR}}}$ we obtain a diagram of pointed stacks

$$Y_{\mathrm{dR}} \to [\mathbb{R}\mathrm{Spec}_{\mathbb{G}_{\mathrm{dR}}}(\mathbb{R}\Gamma_{\mathrm{dR}}(\mathbb{L}(\mathcal{O}_{G_{\mathrm{dR}}}))_{U_\bullet})/\mathbb{G}_{\mathrm{dR}}] \to X_{\mathcal{C}_{\mathrm{dR}}}. \tag{9.49.2}$$

Lemma 9.50. *The induced map*

$$\mathbb{G}_{\mathrm{dR}} \simeq \pi_1(Y_{\mathrm{dR}}) \to \pi_1(X_{\mathcal{C}_{\mathrm{dR}}}) \simeq \pi_1(\mathcal{C}_{\mathrm{dR}}, b) \tag{9.50.1}$$

is the map ℓ_{dR}.

Proof. Let $\widetilde{X}$ be as in 5.33. Recall that if $\widetilde{G} := \pi_1(\mathcal{C}_{\mathrm{dR}}, \omega_b^{\mathrm{dR}})$, then $\widetilde{X}$ is given by

$$\widetilde{X} := [\mathbb{R}\mathrm{Spec}_{\widetilde{G}}\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{\widetilde{G}}))/\widetilde{G}]. \tag{9.50.2}$$

The same construction used to define the map 9.49.2 gives a map $Y_{\mathrm{dR}} \to \widetilde{X}$ lifting the map to $X_{\mathcal{C}_{\mathrm{dR}}}$ such that the diagram

$$\begin{array}{ccc} Y_{\mathrm{dR}} & \xrightarrow{\pi} & B\mathbb{G}_{\mathrm{dR}} \\ \downarrow & & \downarrow{\scriptstyle \ell_{\mathrm{dR}}} \\ \widetilde{X} & \xrightarrow{\tilde{\pi}} & B\widetilde{G} \end{array} \tag{9.50.3}$$

commutes, where π and $\tilde{\pi}$ denote the natural projections. Applying the functor π_1 the Lemma follows. □

9.51. There is a similar description of the map ℓ_{et}.

If $(\bar{b}_i, M_{\bar{b}_i}) \to (b_K, M_{b_K})$ $(i = 1, 2)$ are two geometric points, define their product

$$(\bar{b}_1, M_{\bar{b}_1}) \tilde{\times}_{(b_K, M_{b_K})} (\bar{b}_2, M_{\bar{b}_2}) \tag{9.51.1}$$

as in the following lemma (note that a log geometric point is naturally a pro–log scheme):

Lemma 9.52. *Let $f : (X, M_X) \to (Y, M_Y)$ and $(Z, M_Z) \to (Y, M_Y)$ be morphisms of fine log algebraic spaces, and let $(P, M_P) := (X, M_X) \times_{(Y, M_Y)} (Z, M_Z)$ be the fiber product in the category of integral log schemes. Consider the category $\mathcal{C}$ whose objects are morphisms $t : (T, M_T) \to (P, M_P)$ of log algebraic spaces such that the maps $t^*\mathrm{pr}_1^* M_X \to M_T$ and $t^*\mathrm{pr}_2^* M_Y \to M_T$ are both isomorphisms. Then the category $\mathcal{C}$ has a final object which we denote by*

$$(X, M_X) \tilde{\times}_{(Y, M_Y)} (Z, M_Z). \tag{9.52.1}$$

If $(X, M_X) = \varprojlim (X_i, M_{X_i})$ and $(Z, M_Z) = \varprojlim (Z_j, M_{Z_j})$ are pro–objects in the category of fine log schemes over (Y, M_Y) then we also define

$$(X, M_X) \tilde{\times}_{(Y, M_Y)} (Z, M_Z) := \varprojlim_{i,j} (X_i, M_{X_i}) \tilde{\times}_{(Y, M_Y)} (Z_j, M_{Z_j}). \tag{9.52.2}$$

Proof. This follows from A.3. □

Remark 9.53. For a log geometric point $(\bar{b}, M_{\bar{b}}) \to (b_K, M_{b_K})$, the fiber product

$$(\bar{b}, M_{\bar{b}}) \tilde{\times}_{(b_K, M_{b_K})} (\bar{b}, M_{\bar{b}}) \tag{9.53.1}$$

is isomorphic to

$$\coprod_{g \in \pi} (\bar{b}, M_{\bar{b}}), \tag{9.53.2}$$

where π denotes the group $\pi_1((b_K, M_{b_K}), (\bar{b}, M_{\bar{b}}))$. Indeed there is a natural map

$$\coprod_{g \in \pi} (\bar{b}, M_{\bar{b}}) \to (\bar{b}, M_{\bar{b}}) \tilde{\times}_{(b_K, M_{b_K})} (\bar{b}, M_{\bar{b}}) \tag{9.53.3}$$

which on the g–th component is $1 \times g$. That this map is an isomorphism can be seen by noting that the universal property of $(\bar{b}, M_{\bar{b}}) \tilde{\times}_{(b_K, M_{b_K})} (\bar{b}, M_{\bar{b}})$ implies that this scheme represents the functor over $\overline{K}$ which to any $\overline{K}$–scheme T associates the set of pairs (ρ, α), where $\rho : T \to \bar{b}$ is a morphism over K and $\alpha : M_{\bar{b}}|_T \to \rho^* M_{\bar{b}}$ is an isomorphism over $M_{b_K}|_T$. Choose isomorphisms

$M_b \simeq V^* \oplus \mathbb{N}^r$ and $M_{\bar{b}} \simeq \overline{K}^* \oplus \mathbb{Q}^r_{\geq 0}$ such that the map $M_b \to M_{\bar{b}}$ sends $\mathbb{N}^r$ to $\mathbb{Q}^r_{\geq 0}$. As in 9.14, these choices induce an isomorphism

$$\pi \simeq \widehat{\mathbb{Z}}(1)^r \rtimes \mathrm{Gal}(\overline{K}/K), \tag{9.53.4}$$

where an element $g \in \mathrm{Gal}(\overline{K}/K)$ acts on the log structure $M_{\bar{b}}$ by the map

$$\overline{K}^* \oplus \mathbb{Q}^r_{\geq 0} \to \overline{K}^* \oplus \mathbb{Q}^r_{\geq 0}, \quad (u, q) \mapsto (g(u), q). \tag{9.53.5}$$

Now by Galois theory, if (ρ, α) is a pair as above, then ρ is obtained from the structure morphism $T \to \bar{b}$ by composing with an element $g \in \mathrm{Gal}(\overline{K}/K)$. The isomorphism α is then induced by a map of monoids $\mathbb{Q}^r_{\geq 0} \to \mathcal{O}_T^*$ sending all elements of $\mathbb{N}^r$ to 1. Such a map of monoids is precisely given by an element of $\widehat{\mathbb{Z}}(1)$. From this it follows that 9.53.3 is an isomorphism.

9.54. We can now carry out the construction of 5.21 replacing products of geometric points by $\tilde{\times}$ defined above. For any étale hypercover $U_{b,\bullet} \to b_{\overline{K}}$ and choice of log geometric points $E \to (b_{\overline{K}}, M_{b_{\overline{K}}})$, we obtain a $\mathbb{G}_{\mathrm{et}}$–equivariant differential graded algebra $GC(\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}}))_{U_{b,\bullet}}$ canonically isomorphic in $\mathrm{Ho}(\mathbb{G}_{\mathrm{et}} - \mathrm{dga}_{\mathbb{Q}_p})$ to the algebra $\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}}))$ obtained from the choice of a single log geometric point $(\bar{b}, M_{\bar{b}}) \to (b_K, M_{b_K})$. Moreover if $U_{b,\bullet}$ is obtained by base change from a hypercover of b_K, then there is a natural action in the homotopy category of $\mathrm{Gal}(\overline{K}/K)$ on this algebra.

9.55. Let $U_\bullet \to X$ be an étale hypercover, with each U_n a finite disjoint union of very small affine X–schemes. Denote by $U_{b,\bullet}$ the base change to b. Let $W_\bullet$ denote the simplicial formal scheme obtained by completing $U_\bullet$ along the ideal defined by b, and let $\widehat{U}_\bullet$ denote the simplicial affine scheme obtained by taking the spectrum in each degree of the cosimplicial algebra

$$[n] \mapsto \Gamma(W_n, \mathcal{O}_{W_n}). \tag{9.55.1}$$

Observe that for any morphism $[n] \to [m]$ in Δ the corresponding map $\widehat{U}_m \to \widehat{U}_n$ is finite and étale. Moreover, there is a natural diagram of simplicial log schemes

$$(U_{b,\bullet}, M_{U_{b,\bullet}}) \longrightarrow (\widehat{U}_\bullet, M_{\widehat{U}_\bullet}) \longrightarrow (U_\bullet, M_{U_\bullet}), \tag{9.55.2}$$

where the log structures are all obtained by pullback from M_X.

9.56. Let $E \to X_{\overline{K}}$ be a family of geometric generic points, and for each $e \in E$ choose a commutative diagram

$$\begin{array}{ccc} \hat{e} & \longrightarrow & e \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\widehat{\mathcal{O}}_{X,b_{\overline{K}}}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{X,b_{\overline{K}}}), \end{array} \tag{9.56.1}$$

where $\widehat{\mathcal{O}}_{X,b_{\overline{K}}}$ denotes the completion of the strict henselization $\mathcal{O}_{X,b_{\overline{K}}}$ and $\hat{e} \to \mathrm{Spec}(\widehat{\mathcal{O}}_{X,b_{\overline{K}}})$ is a geometric generic point. Denote by $\widehat{E} \to \mathrm{Spec}(\widehat{\mathcal{O}}_{X_{b,\overline{K}}})$ the resulting family of geometric generic points.

Observe that for each $[n] \in \Delta$, there is a natural commutative diagram

$$
\begin{tikzcd}
\coprod_{\mathrm{Hom}_{\mathrm{Spec}(\widehat{\mathcal{O}}_{X,b_{\overline{K}}})}(\widehat{E},\widehat{U}_{n,\overline{K}})} \widehat{E} \arrow[r] \arrow[d] & \coprod_{\mathrm{Hom}_X(E,U_{n,\overline{K}})} E \arrow[d] \\
\widehat{U}_n \arrow[r] & U_n.
\end{tikzcd}
\tag{9.56.2}
$$

This follows from observing that any lifting $\hat{e} \to U_n$ of the composite $\hat{e} \to e \to X$ necessarily factors through e.

It follows that if G is an algebraic group and L an ind–smooth sheaf of G–equivariant differential graded algebras on the log étale site of $(X_{\overline{K}}, M_{X_{\overline{K}}})$, then there is a natural map of cosimplicial G–equivariant differential graded algebras

$$
([n] \mapsto GC(U_{n,\overline{K}}, E, L)) \to ([n] \mapsto GC(\widehat{U}_{n,\overline{K}}, \widehat{E}, L|_{\widehat{U}_\bullet})).
\tag{9.56.3}
$$

Here the notation is as in 5.21. Let $GC(L|_{\widehat{U}_\bullet}, \widehat{E}) \in \mathrm{Ho}(G - \mathrm{dga}_{\mathbb{Q}_p})$ denote the G–equivariant differential graded algebra obtained by applying the functor of Thom–Sullivan cochains to the cosimplicial algebra $([n] \mapsto GC(\widehat{U}_{n,\overline{K}}, \widehat{E}, L|_{\widehat{U}_\bullet}))$. As in 5.24, the object $GC(L|_{\widehat{U}_\bullet}, \widehat{E})$ in the homotopy category is up to canonical isomorphism independent of the choices. Furthermore, the map 9.56.3 induces a canonical morphism in the homotopy category

$$
GC(L, E) \longrightarrow GC(L|_{\widehat{U}_\bullet}, \widehat{E}).
\tag{9.56.4}
$$

9.57. In particular, if G_{et} denotes $\pi_1(\langle L|_{X_{\overline{K}}}\rangle_\otimes, \omega_b^{\mathrm{et}})$ and $\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})$ denotes the ind–smooth sheaf with right G_{et}–action corresponding to the coordinate ring $\mathcal{O}_{G_{\mathrm{et}}}$, then there is a natural map

$$
GC(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}}), E) \longrightarrow GC(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})|_{\widehat{U}_\bullet}, \widehat{E})
\tag{9.57.1}
$$

in $\mathrm{Ho}(G_{\mathrm{et}} - \mathrm{dga}_{\mathbb{Q}_p})$.

There is also a natural map

$$
GC(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})|_{\widehat{U}_\bullet}, \widehat{E}) \to \mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}}))
\tag{9.57.2}
$$

in $\mathrm{Ho}(\mathbb{G}_{\mathrm{et}} - \mathrm{dga}_{\mathbb{Q}_p})$ defined as follows.

First view $\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})$ as a $\mathbb{G}_{\mathrm{et}}$–equivariant sheaf using the natural map $\mathbb{G}_{\mathrm{et}} \to G_{\mathrm{et}}$, and let $(\widehat{X}_b, M_{\widehat{X}_b})$ denote $\mathrm{Spec}(\widehat{\mathcal{O}}_{X,b_{\overline{K}}})$ with log structure that obtained by pullback from M_X.

Choose for each $e \in \widehat{E}$ a section s_e of the map $(\widehat{X}^{e\sim}_{(\bar{b},M_{\bar{b}})}, M_{\widehat{X}_{(\bar{b},M_{\bar{b}})^{e\sim}}}) \to (\bar{b}, M_{\bar{b}})$, where $(\widehat{X}^{e\sim}_{(\bar{b},M_{\bar{b}})}, M_{\widehat{X}_{(\bar{b},M_{\bar{b}})^{e\sim}}}) \to (\widehat{X}_{\overline{K}}, M_{\widehat{X}_{\overline{K}}})$ denotes the universal covering space (a pro–log scheme) obtained from the geometric point e (see [Il2, 4.6] for the definition of the universal covering space).

Now note that each connected component of $\widehat{U}_n$ maps isomorphically to $\widehat{X}_{\overline{K}}$. It follows that for each lifting $\rho : e \to \widehat{U}_n$ of the geometric point $e \to \widehat{X}_{\overline{K}}$, the section s_e induces a section of the map

$$
(\widehat{U}^{\rho\sim}_{n,\overline{K}}, M_{\widehat{U}^{\rho\sim}_{n,\overline{K}}}) \times_{(\widehat{U}_n, M_{\widehat{U}_n}),\rho} (\bar{b}, M_{\bar{b}}) \to (\bar{b}, M_{\bar{b}}).
\tag{9.57.3}
$$

Hence for any n there is a canonical bijection of sets

$$B : (\text{liftings } \hat{e} \text{ of } e \text{ to } \widehat{U}_{n,\overline{K}}) \rightarrow (\text{liftings of } (\bar{b}, M_{\bar{b}}) \rightarrow (\widehat{X}_{\overline{K}}, M_{\widehat{X}_{\overline{K}}}) \text{ to } (\widehat{U}_{n,\overline{K}}, M_{\widehat{U}_{n,\overline{K}}})). \tag{9.57.4}$$

Furthermore, for each lifting $\hat{e}$ there is a canonical isomorphism

$$\pi_1((\widehat{U}_{n,\overline{K}}, M_{\widehat{U}_{n,\overline{K}}}), \hat{e}) \simeq \pi_1((\widehat{U}_{n,\overline{K}}, M_{\widehat{U}_{n,\overline{K}}}), B(\hat{e})), \tag{9.57.5}$$

and for any smooth sheaf L on $(\widehat{X}_{\overline{K}}, M_{\widehat{X}_{\overline{K}}})$ and lifting $\hat{e}$ there is a natural isomorphism $L_{\hat{e}} \simeq L_{B(\hat{e})}$ compatible with the identification 9.57.5. On the other hand, for each lifting ρ of $(\bar{b}, M_{\bar{b}})$ to $\widehat{U}_{n,\overline{K}}$, the natural map

$$\pi_1((b_K, M_{b_K}), \rho) \rightarrow \pi_1((\widehat{U}_n, M_{\widehat{U}_n}), \rho) \tag{9.57.6}$$

is an isomorphism. From this it follows that there is a natural isomorphism in $\mathrm{Ho}(G_{\mathrm{et}}-\mathrm{dga}_{\mathbb{Q}_p})$

$$GC(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})|_{\widehat{U}_\bullet}, \widehat{E}) \simeq \mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})|_{(b_{\overline{K}}, M_{b_{\overline{K}}})}). \tag{9.57.7}$$

The map $\mathbb{G}_{\mathrm{et}} \rightarrow G_{\mathrm{et}}$ induces a natural map

$$\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})|_{(b_{\overline{K}}, M_{b_{\overline{K}}})} \rightarrow \mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}}) \tag{9.57.8}$$

compatible with the right $\mathbb{G}_{\mathrm{et}}$–action. We therefore obtain a map

$$\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})|_{(b_{\overline{K}}, M_{b_{\overline{K}}})}) \rightarrow \mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}})) \tag{9.57.9}$$

in $\mathrm{Ho}(\mathbb{G}_{\mathrm{et}} - \mathrm{dga}_{\mathbb{Q}_p})$. This map combined with 9.57.7 induces the map 9.57.2.

Applying the $\mathbb{R}\mathrm{Spec}_{\mathbb{G}_{\mathrm{et}}}(-)$–functor we then obtain a diagram of pointed stacks

$$Y_{\mathrm{et}} \rightarrow [\mathbb{R}\mathrm{Spec}_{\mathbb{G}_{\mathrm{et}}}(\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G_{\mathrm{et}}})))/\mathbb{G}_{\mathrm{et}}] \rightarrow X_{\mathcal{C}_{\mathrm{dR}}}. \tag{9.57.10}$$

From this we obtain a morphism

$$\mathbb{G}_{\mathrm{et}} \simeq \pi_1(Y_{\mathrm{et}}) \rightarrow \pi_1(X_{\mathcal{C}_{\mathrm{et}}}) \simeq \pi_1(\mathcal{C}_{\mathrm{et}}, b). \tag{9.57.11}$$

By an argument similar to the one used in 9.50 this map is equal to ℓ_{et}.

9.58. The comparison isomorphism $\iota_{\mathbb{G}}$ can be described as follows. Let $\mathbb{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}}) \rightarrow \mathbb{R}^\bullet$ be the resolution discussed in 9.42, and let $W_\bullet \rightarrow b_K$ be an étale hypercover. Choose also a family of log geometric points $E \rightarrow (b_{\overline{K}}, M_{b_{\overline{K}}})$.

Observe that by 9.16 and the discussion in 9.24, the sheaf $\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}})$ is naturally associated to $\mathbb{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}})$ by an association compatible with the right action of $\mathbb{G}_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \simeq \mathbb{G}_{\mathrm{dR}} \otimes_K B_{\mathrm{cris}}(V)$. Then as in 6.17.6 we obtain a commutative diagram

$$\begin{tikzcd} GC(\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\mathrm{et}}})|_{W_\bullet}, E) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \arrow[r, "a"] & GC(\mathbb{R}^\bullet(B_{\mathrm{cris}}(W_\bullet), M_{B_{\mathrm{cris}}(W_\bullet, b)}), E) \\ & \mathbb{R}^\bullet(W_\bullet, M_b|_{W_\bullet})^\nabla \otimes_K B_{\mathrm{cris}}(V) \arrow[u, "b"] \arrow[d, "c"] \\ DR((\mathcal{L}(\mathcal{O}_{\mathbb{G}_{\mathrm{dR}}})|_{W_\bullet}, N)) \otimes_K B_{\mathrm{cris}}(V) \arrow[r, "d"] & DR(\mathbb{R}|_{W_\bullet}) \otimes_K B_{\mathrm{cris}}(V). \end{tikzcd} \tag{9.58.1}$$

The map a is the composite

$$(9.58.2)\qquad \begin{array}{ccc} GC(\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\rm et}})|_{W_\bullet}, E)\otimes_{\mathbb{Q}_p} B_{\rm cris}(V) & \longrightarrow & GC(\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\rm et}})\otimes_{\mathbb{Q}_p} B_{\rm cris}(V)|_{W_\bullet}, E) \\ & & \downarrow{\scriptstyle \iota} \\ GC(\mathbb{R}^\bullet(B_{\rm cris}(W_\bullet), M_{B_{\rm cris}(W_\bullet)}), E) & \longleftarrow & GC(\mathbb{L}(\mathcal{O}_{\mathbb{G}_{\rm dR}})(B_{\rm cris}(W_\bullet), M_{B_{\rm cris}(W_\bullet)}), E), \end{array}$$

where ι denotes the association. Hence a becomes an equivalence after tensoring with $\widetilde{B}_{\rm cris}(V)$. The map c is an equivalence by 9.42, and the map d is an equivalence since $\mathbb{R}^\bullet$ is a resolution of $\mathbb{L}(\mathcal{O}_{\mathbb{G}_{\rm dR}})$.

Finally note that by 9.44 there is a natural quasi–isomorphism $B_{\rm cris}(V) \to GC(\mathbb{V}(\mathcal{O}_{\mathbb{G}_{\rm et}})|_{W_\bullet}, E)\otimes_{\mathbb{Q}_p} B_{\rm cris}(V)$, and similarly by 9.31 there is a natural quasi–isomorphism

$$(9.58.3)\qquad B_{\rm cris}(V) \to DR((\mathcal{L}(\mathcal{O}_{\mathbb{G}_{\rm dR}})|_{W_\bullet}, N)) \otimes_K B_{\rm cris}(V).$$

Furthermore, since all the morphisms in 9.58.1 are morphisms of differential graded $B_{\rm cris}(V)$–algebras, the two induced maps $B_{\rm cris}(V) \to DR(\mathbb{R}|_{W_\bullet}) \otimes_K B_{\rm cris}(V)$ are equal. It follows that the map b is also an equivalence.

From the diagram 9.58.1 we therefore obtain an isomorphism

$$(9.58.4)\qquad \iota_Y : Y_{\rm dR} \otimes_K \widetilde{B}_{\rm cris}(V) \to Y_{\rm et} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\rm cris}(V)$$

such that the diagram of isomorphisms

$$(9.58.5)\qquad \begin{array}{ccc} Y_{\rm dR} \otimes_K \widetilde{B}_{\rm cris}(V) & \xrightarrow{\iota_Y} & Y_{\rm et} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\rm cris}(V) \\ \downarrow & & \downarrow \\ B\mathbb{G}_{\rm dR} \otimes_K \widetilde{B}_{\rm cris}(V) & \xrightarrow{\iota_{\mathbb{G}}} & B\mathbb{G}_{\rm et} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\rm cris}(V) \end{array}$$

commutes. To complete the proof of 9.46, it now only remains to observe that there is a natural morphism of diagrams from 6.17.6 to 9.58.1, and hence the induced diagram

$$(9.58.6)\qquad \begin{array}{ccc} Y_{\rm dR} \otimes_K \widetilde{B}_{\rm cris}(V) & \xrightarrow{\delta_{\rm dR}} & X_{\mathcal{C}_{\rm dR}} \otimes_K \widetilde{B}_{\rm cris}(V) \\ {\scriptstyle \iota_Y}\downarrow & & \downarrow{\scriptstyle \iota_{\mathcal{C}}} \\ Y_{\rm et} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\rm cris}(V) & \xrightarrow{\delta_{\rm et}} & X_{\mathcal{C}_{\rm et}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\rm cris}(V) \end{array}$$

commutes. Applying the π_1–functor we obtain that 9.46 commutes after tensoring with $\widetilde{B}_{\rm cris}(V)$. But then by [Ol3, 15.2] the diagram 9.46 commutes already over $B_{\rm cris}(V)$. $\square$

10. A generalization

In this section we formulate a conjecture which we learned from some communications with Toen (though of course any mistakes are entirely due to the present author).

10.1. Let X, V, and K be as in 6.1, and assume for simplicity that the residue field k is separably closed.

As in 8.2, if Υ is any field and $\mathcal{G}$ is a gerbe over the category of affine Υ–schemes with the fpqc topology, one can associate to $\mathcal{G}$ a simplicial presheaf $B\mathcal{G} \in \mathrm{SPr}(\Upsilon)$ such that for any Υ–algebra R the simplicial set $B\mathcal{G}(R)$ is the nerve of the groupoid $\mathcal{G}(R)$.

10.2. Let $\mathcal{G}_{\mathrm{cris}}$ denote the gerbe of fiber functors for the category of crystalline representations of $\mathrm{Gal}(\overline{K}/K)$ and consider the category $\mathrm{Ho}(\mathrm{SPr}_*(\mathbb{Q}_p)|_{B\mathcal{G}_{\mathrm{cris}}})$. Denote by

$$\omega_{\mathrm{et}} \in \mathcal{G}(\mathbb{Q}_p) \tag{10.2.1}$$

the fiber functor which associates to a crystalline representation the underlying $\mathbb{Q}_p$–vector space. If $\mathcal{H}_{\mathrm{cris}}$ denotes the gerbe of fiber functors for the category of F–isocrystals on k/K then there is also a natural functor

$$\omega_{\mathbf{D}} : \mathcal{H}_{\mathrm{cris}} \longrightarrow \mathcal{G}_{\mathrm{cris}} \tag{10.2.2}$$

which to a fiber functor η for $\mathrm{FIsoc}(k/K)$ associates the fiber functor for the category of crystalline representations sending V to $\eta(\mathbf{D}(V))$. Finally there is a fiber functor $\omega_{\mathrm{dR}} \in \mathcal{H}_{\mathrm{cris}}(K)$ which sends an F–isocrystal to its underlying K–vector space. Moreover, by definition of crystalline representation there is a natural isomorphism

$$\omega_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \simeq (\omega_{\mathrm{dR}} \circ \omega_{\mathbf{D}}) \otimes_K B_{\mathrm{cris}}(V). \tag{10.2.3}$$

We thus obtain a commutative diagram of simplicial presheaves

$$\begin{array}{ccc} \mathrm{Spec}(B_{\mathrm{cris}}(V)) & \longrightarrow & \mathrm{Spec}(\mathbb{Q}_p) \\ {\scriptstyle \omega_{\mathrm{dR}}\otimes B_{\mathrm{cris}}(V)}\downarrow & & \downarrow{\scriptstyle \omega_{\mathrm{et}}} \\ B\mathcal{H}_{\mathrm{cris}} & \longrightarrow & B\mathcal{G}_{\mathrm{cris}}. \end{array} \tag{10.2.4}$$

In particular, if $F \in \mathrm{Ho}(\mathrm{SPr}(K)|_{B\mathcal{G}_{\mathrm{cris}}})$ then by pulling back we obtain a stack $\mathbf{D}(F) \in \mathrm{Ho}(\mathrm{SPr}_*(K)|_{B\mathcal{H}_{\mathrm{cris}}})$ and $F_{\mathrm{et}} \in \mathrm{Ho}(\mathrm{SPr}_*(\mathbb{Q}_p))$ together with an isomorphism

$$F_{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \simeq \omega_{\mathrm{dR}}(\mathbf{D}(F)) \otimes_K B_{\mathrm{cris}}(V). \tag{10.2.5}$$

There is a natural action of $\mathrm{Gal}(\overline{K}/K)$ on ω_{et} which induces an action of $\mathrm{Gal}(\overline{K}/K)$ on F_{et}. Similarly there is a semi–linear Frobenius automorphism of ω_{dR} which induces a semi–linear automorphism $\varphi : \omega_{\mathrm{dR}}(\mathbf{D}(F))^\sigma \to \omega_{\mathrm{dR}}(\mathbf{D}(F))$. Because the isomorphism of fiber functors 10.2.3 is compatible with these structures it follows that the isomorphism 10.2.5 is compatible with the Frobenius structures and Galois actions.

10.3. Now let $(E, \mathrm{Fil}_E, \varphi_E)$ and L be associates sheaves as in 7.1. Let $\mathcal{C}_{\mathrm{dR}}$ and $\mathcal{C}_{\mathrm{et}}$ be as in 1.5 giving rise to stacks $X_{\mathcal{C}_{\mathrm{dR}}} \in \mathrm{Ho}(\mathrm{SPr}_*(K))$ and $X_{\mathcal{C}_{\mathrm{et}}} \in \mathrm{Ho}(\mathrm{SPr}_*(\mathbb{Q}_p))$. By 1.7 there is a natural isomorphism

$$\iota : X_{\mathcal{C}_{\mathrm{dR}}} \otimes_K \widetilde{B}_{\mathrm{cris}}(V) \simeq X_{\mathcal{C}_{\mathrm{et}}} \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V). \tag{10.3.1}$$

Finally we note that it is shown in [Ol1, 3.62] that there is a natural stack

$$F_{\mathcal{C}_{\mathrm{dR}}} \in \mathrm{Ho}(\mathrm{SPr}_*(\mathbb{Q}_p)|_{B\mathcal{H}_{\mathrm{cris}}}) \tag{10.3.2}$$

giving rise to $X_{\mathcal{C}_{\mathrm{dR}}}$ with its F–isocrystal structure.

The following conjecture would be a natural extension of the results of this paper.

Conjecture 10.4. *There exists a stack $F_{\mathcal{C}} \in \mathrm{Ho}(\mathrm{SPr}_*(\mathbb{Q}_p)|_{B\mathcal{G}_{\mathrm{cris}}})$ and isomorphisms $\mathbf{D}(F_{\mathcal{C}}) \simeq F_{\mathcal{C}_{\mathrm{dR}}}$, $\omega_{\mathrm{et}}(F) \simeq X_{\mathcal{C}_{\mathrm{et}}}$ identifying the isomorphism ι with the natural isomorphism $(\omega_{\mathrm{dR}} \circ \mathbf{D})(F_{\mathcal{C}}) \otimes B_{\mathrm{cris}}(V) \simeq \omega_{\mathrm{et}}(F_{\mathcal{C}}) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$.*

Appendix A. Exactification

A.1. Let $f : (X, M_X) \to (S, M_S)$ be a morphism of fine log schemes, let (D, M_D) denote the fiber product

$$(X, M_X) \times_{(S,M_S)} (X, M_X) \tag{A.1.1}$$

in the category of fine log schemes, and let

$$\Delta : (X, M_X) \to (D, M_D) \tag{A.1.2}$$

denote the diagonal morphism. Write $p_i : (D, M_D) \to (X, M_X)$ $(i = 1, 2)$ for the two projections.

A.2. Let $\mathscr{C}$ denote the category whose objects are morphisms of fine log algebraic spaces over (S, M_S)

$$(g, g^b) : (T, M_T) \to (D, M_D) \tag{A.2.1}$$

such that the two composite maps

$$(T, M_T) \xrightarrow{g} (D, M_D) \xrightarrow{p_i} (X, M_X) \tag{A.2.2}$$

are strict. Morphisms in $\mathscr{C}$ are (D, M_D)-morphisms.

Proposition A.3. *The category $\mathscr{C}$ has a final object*

$$\pi : (\widetilde{D}, M_{\widetilde{D}}) \to (D, M_D). \tag{A.3.1}$$

Proof. Let $\mathscr{C}'$ denote the category whose objects are pairs

$$(g : T \to X \times_S X, \iota), \tag{A.3.2}$$

where g is a morphism of S-spaces, and $\iota : g^*p_1^*M_X \to g^*p_2^*M_X$ is an isomorphism of log structures on T such that the diagram

$$\begin{tikzcd} & t^*M_S \arrow[dl, "f^b"'] \arrow[dr, "f^b"] & \\ g^*p_1^*M_X \arrow[rr, "\iota"] & & g^*p_2^*M_X \end{tikzcd} \tag{A.3.3}$$

commutes, where $t : T \to S$ is the structure morphism. A morphism

$$(g : T \to X \times_S X, \iota) \to (g' : T' \to X \times_S X, \iota') \tag{A.3.4}$$

is an $X \times_S X$-morphism $h : T \to T'$ such that the induced diagram

$$\begin{tikzcd} h^*g'^*p_1^*M_X \arrow[r, "h^*\iota'"] \arrow[d, "\simeq"] & h^*g'^*p_2^*M_X \arrow[d, "\simeq"] \\ g^*p_1^*M_X \arrow[r, "\iota"] & g^*p_2^*M_X \end{tikzcd} \tag{A.3.5}$$

commutes. Let

$$F : \mathscr{C} \to \mathscr{C}' \tag{A.3.6}$$

be the functor sending $(g, g^b) : (T, M_T) \to (D, M_D)$ to the underlying algebraic space T (which is a $X \times_S X$-space via the natural projection $D \to X \times_S X$) and the isomorphism

$$(p_1 g)^* M_X \xrightarrow{(p_1 g)^b} M_T \xrightarrow{(p_2 g)^{b-1}} (p_2 g)^* M_X. \tag{A.3.7}$$

Lemma A.4. *The functor A.3.6 is an equivalence of categories.*

Proof. For an object $(g : T \to X \times_S X, \iota) \in \mathscr{C}'$, let $g_i : T \to X$ $(i = 1, 2)$ be the composite $p_i \circ g$. We then obtain a commutative diagram of fine log algebraic spaces

$$\begin{tikzcd} (T, g_2^* M_X) \arrow[r, "(g_2, \mathrm{id})"] \arrow[d, "(g_1, \iota)"] & (X, M_X) \arrow[d, "f"] \\ (X, M_X) \arrow[r, "f"] & (S, M_S), \end{tikzcd} \tag{A.4.1}$$

and therefore also an object

$$((T, g_2^* M_X) \to (D, M_D)) \in \mathscr{C}. \tag{A.4.2}$$

This construction defines a functor

$$G : \mathscr{C}' \to \mathscr{C}. \tag{A.4.3}$$

It follows immediately from the construction that there are natural isomorphisms $FG \simeq \mathrm{id}_{\mathscr{C}'}$ and $GF \simeq \mathrm{id}_{\mathscr{C}}$. □

It follows that in order to prove A.3, it suffices to show that the category $\mathscr{C}'$ has a final object. Let

$$I : (\text{Algebraic spaces}/X \times_S X)^{\mathrm{op}} \to \mathrm{Set} \tag{A.4.4}$$

be the functor sending $g : T \to X \times_S X$ to the set of isomorphisms $\iota : g^* p_1^* M_X \to g^* p_2^* M_X$ such that the diagram A.3.3 commutes. Then by the very definition of $\mathscr{C}'$ to show that $\mathscr{C}'$ has a final object it suffices to show that I is an algebraic space.

This follows from the general theory in [Ol2]. Let $\mathcal{L}og_{(S, M_S)}$ denote the stack defined in [Ol2]. By [Ol2, 1.1] the stack $\mathcal{L}og_{(S, M_S)}$ is an algebraic stack, and in particular the diagonal morphism

$$\Delta : \mathcal{L}og_{(S, M_S)} \to \mathcal{L}og_{(S, M_S)} \times_S \mathcal{L}og_{(S, M_S)} \tag{A.4.5}$$

is representable. This implies that I is representable, as I is isomorphic to the fiber product of the diagram

$$\begin{tikzcd} & X \times_S X \arrow[d, "p_1^* M_X \times p_2^* M_X"] \\ \mathcal{L}og_{(S, M_S)} \arrow[r, "\Delta"] & \mathcal{L}og_{(S, M_S)} \times_S \mathcal{L}og_{(S, M_S)}. \end{tikzcd} \tag{A.4.6}$$

□

A.5. The diagonal map $\Delta : (X, M_X) \to (D, M_D)$ is in particular an object of $\mathscr{C}$, and therefore we obtain a factorization of Δ

(A.5.1)
$$\begin{tikzcd} (X, M_X) \arrow[r, "\widetilde{\Delta}"] \arrow[rr, bend left, "\Delta"] & (\widetilde{D}, M_{\widetilde{D}}) \arrow[r, "\pi"] & (D, M_D), \end{tikzcd}$$

where $\widetilde{\Delta}$ is strict. The underlying morphism of algebraic spaces of $\widetilde{\Delta}$ is an immersion since this is the case for Δ. The morphism $\widetilde{\Delta}$ is therefore an exact immersion of log schemes. We call $\widetilde{\Delta}$ the *exactification of* Δ.

Example A.6. Let $S = \mathrm{Spec}(R)$ for some ring R, and let M_S be the trivial log structure on S. Let $X = \mathbb{A}^2_R$ with log structure M_X induced by the map

(A.6.1)
$$\mathbb{N}^2 \to R[x, y], \quad (n, m) \mapsto x^n y^m.$$

Let $\mathcal{S}$ denote the stack theoretic quotient of $\mathbb{A}^2_R$ by the action of $\mathbb{G}^2_m$ given on scheme-valued points by

(A.6.2)
$$(u_1, u_2) * (a, b) := (u_1 a, u_2 b).$$

Then the map

(A.6.3)
$$M_X : X \to \mathcal{L}og_{(S, \mathcal{O}^*_S)}$$

factors through $\mathcal{S}$ as

(A.6.4)
$$X \xrightarrow{A} \mathcal{S} \xrightarrow{B} \mathcal{L}og_{(S, \mathcal{O}^*_S)},$$

where B is étale by [Ol2, 5.25] and A is the natural projection. Let J denote the fiber product of the diagram

(A.6.5)
$$\begin{tikzcd} & X \arrow[d, "A \times A"] \\ \mathcal{S} \arrow[r, "\Delta"] & \mathcal{S} \times_S \mathcal{S}. \end{tikzcd}$$

Then there is a commutative diagram

(A.6.6)
$$\begin{tikzcd} & J \arrow[d, "\gamma"] \\ X \arrow[ur, hook, "j"] \arrow[r, hook, "\widetilde{\Delta}"] & \widetilde{D}, \end{tikzcd}$$

where γ is étale since B is étale. On the other hand, there is a natural isomorphism (see for example the discussion in [Ol2, 3.11 and 5.14])

(A.6.7)
$$J \simeq \mathrm{Spec}(R[x, y, u_1^{\pm}, u_2^{\pm}]),$$

with the first (resp. second) projection $J \to X$ is given by the map

(A.6.8)
$$x \mapsto x, \quad y \mapsto y \quad (\text{resp.} \quad x \mapsto u_1 x, \quad y \mapsto u_2 y),$$

and the diagonal map j is defined by $u_1 = u_2 = 1$.

A.7. More generally, for any finite set K, let (D^K, M_{D^K}) denote the fiber product of the diagram

(A.7.1)
$$\begin{tikzcd} & \prod_{k\in K}(X, M_X) \arrow[d, "\prod_{k\in K} f"] \\ (S, M_S) \arrow[r, "\Delta_K"] & \prod_{k\in K}(S, M_S), \end{tikzcd}$$

where Δ_K denotes the diagonal, and define $\mathscr{C}_K$ to be the category with objects (S, M_S)-morphisms

(A.7.2)
$$g : (T, M_T) \to (D^K, M_{D^K})$$

such that for every $k \in K$ the composite morphism

(A.7.3)
$$(T, M_T) \xrightarrow{g} (D^K, M_{D^K}) \xrightarrow{p_k} (X, M_X)$$

is strict, where $p_k : (D^K, M_{D^K}) \to (X, M_X)$ denotes the projection to the k-th factor. Morphisms in $\mathscr{C}_K$ are (D^K, M_{D^K})-morphisms.

Proposition A.8. *The category $\mathscr{C}_K$ has a final object*

(A.8.1)
$$\pi_K : (\widetilde{D}^K, M_{\widetilde{D}^K}) \to (D^K, M_{D^K}).$$

Proof. Fix an isomorphism $K = \{0, \dots, n\}$ for $n+1 = |K|$. We may without loss of generality assume that $n \geq 2$, as the case $n = 1$ is A.3. Then

(A.8.2)
$$(D^K, M_{D^K}) \simeq \underbrace{(D, M_D) \times_{p_2,(X,M_X),p_1} (D, M_D) \times \cdots \times_{p_2,(X,M_X),p_1} (D, M_D)}_{n}.$$

From this and the definition of $\mathscr{C}_K$ it then follows that an initial object is given by

(A.8.3)
$$\underbrace{(\widetilde{D}, M_{\widetilde{D}}) \times_{p_2,(X,M_X),p_1} (\widetilde{D}, M_{\widetilde{D}}) \times \cdots \times_{p_2,(X,M_X),p_1} (\widetilde{D}, M_{\widetilde{D}})}_{n}$$

with the projection to (D^K, M_{D^K}) induced by the maps $\pi : (\widetilde{D}, M_{\widetilde{D}}) \to (D, M_D)$. $\square$

A.9. As before, the multidiagonal

(A.9.1)
$$\Delta_K : (X, M_X) \to (D^K, M_{D^K})$$

factors as

(A.9.2)
$$(X, M_X) \xrightarrow{\widetilde{\Delta}_K} (\widetilde{D}^K, M_{\widetilde{D}^K}) \xrightarrow{\pi_K} (D^K, M_{D^K}).$$

A.10. Finally note that if $h : K \to K'$ is a morphism of finite sets, then h induces a commutative diagram

(A.10.1)
$$\begin{tikzcd} & (\widetilde{D}^{K'}, M_{\widetilde{D}^{K'}}) \arrow[r, "\pi^{K'}"] \arrow[d, "h"] & (D^{K'}, M_{D^{K'}}) \arrow[d, "h"] \\ (X, M_X) \arrow[ur, "\widetilde{\Delta}_{K'}"] \arrow[r, "\widetilde{\Delta}_K"] & (\widetilde{D}^K, M_{\widetilde{D}^K}) \arrow[r, "\pi_K"] & (D^K, M_{D^K}). \end{tikzcd}$$

A.11. In the category of fine log formal schemes there is a more general notion of exactification generalizing the charted exactification discussed in [Sh2, 2.1.14]. We explain this notion of exactification after some preliminaries.

Lemma A.12. *Let (X, M_X) be a fine log scheme, and let $\pi : \overline{M}_X \to N$ be a surjection of constructible sheaves of monoids such that if $\bar{\eta} \to X$ is a generization* [SGA4, VIII.7.2] *of a geometric point $\bar{s} \to X$ then the diagram*

$$\begin{tikzcd} \overline{M}_{X,\bar{s}} \arrow[r, "\pi_{\bar{s}}"] \arrow[d] & N_{\bar{s}} \arrow[d] \\ \overline{M}_{X,\bar{\eta}} \arrow[r, "\pi_{\bar{\eta}}"] & N_{\bar{\eta}} \end{tikzcd}$$

is cocartesian, where the vertical arrows are the specialization maps. Let

$$I : (\mathrm{Sch}/X)^{\mathrm{op}} \to \mathrm{Set}$$

be the functor which to any X-scheme $f : T \to X$ associates the set of morphisms of log sctructures $f^*M_X \to M_T$ on T such that the induced map $f^{-1}\overline{M}_X \to \overline{M}_T$ factors through an isomorphism $f^{-1}N \to \overline{M}_T$. Then I is an algebraic space.

Proof. Note first that since $\pi : \overline{M}_X \to N$ is surjective if $\rho : f^*M_X \to M_T$ is an element of $I(T)$ then the map ρ is surjective. In particular, there are no nontrivial automorphisms of M_T compatible with ρ. Therefore I is naturally a substack of $\mathcal{L}og_{(X,M_X)}$, and to prove the lemma it suffices to show that I is algebraic.

In fact, I is an open substack of $\mathcal{L}og_{(X,M_X)}$. This is equivalent to saying that if $f : (T, M_T) \to (X, M_X)$ is a morphism of fine log schemes, then the condition that the map $f^{-1}\overline{M}_X \to \overline{M}_T$ factors through an isomorphism $f^{-1}N \to \overline{M}_T$ is representable by an open subset of T.

Let $U \subset T$ be the set of points $s \in T$ for which the map $\overline{M}_{X,f(\bar{s})} \to \overline{M}_{T,\bar{s}}$ factors through an isomorphism $N_{f(\bar{s})} \to \overline{M}_{T,\bar{s}}$. We claim that U is open. Since $f^{-1}\overline{M}_X$, $f^{-1}N$, and $\overline{M}_T$ are constructible sheaves on T, the set U is constructible. Therefore it suffices to show that U is closed under generization. For this let $\eta \in T$ be a generization of $s \in U$ and consider the diagram

(A.12.1)
$$\begin{tikzcd} \overline{M}_{X,f(\bar{s})} \arrow[r] \arrow[d] & N_{f(\bar{s})} \arrow[r, "\simeq"] \arrow[d] & \overline{M}_{T,\bar{s}} \arrow[d] \\ \overline{M}_{X,f(\bar{\eta})} \arrow[r] \arrow[rr, bend right] & N_{f(\bar{\eta})} \arrow[r, dashed] & \overline{M}_{T,\bar{\eta}}. \end{tikzcd}$$

Since the map $\overline{M}_{X,f(\bar{s})} \to \overline{M}_{T,\bar{s}}$ is surjective, the square

$$\begin{tikzcd} \overline{M}_{X,f(\bar{s})} \arrow[r] \arrow[d] & \overline{M}_{T,\bar{s}} \arrow[d] \\ \overline{M}_{X,f(\bar{\eta})} \arrow[r] & \overline{M}_{T,\bar{\eta}} \end{tikzcd}$$

is cocartesian. Since the left square in A.12.1 is also cocartesian by assumption, this implies that the dotted arrow in A.12.1 exists an is an isomorphisms.

This completes the proof that U is open, and it follows that the condition that $f^{-1}\overline{M}_X \rightarrow \overline{M}_T$ factors through an isomorphism $f^{-1}N \rightarrow \overline{M}_T$ is represented by $(U, M_T|_U)$. $\square$

A.13. Let $(Z, M_Z) \hookrightarrow (T, M_T)$ be a closed immersion of fine log formal schemes, where $Z \subset T$ is a subscheme of definition. Let $\mathcal{C}$ denote the category of commutative squares of fine log formal schemes

(A.13.1)
$$\begin{tikzcd} (U, M_U) \arrow[r, hook, "i"] \arrow[d, "a"] & (W, M_W) \arrow[d] \\ (Z, M_Z) \arrow[r, hook] & (T, M_T), \end{tikzcd}$$

where $U \subset W$ is a subscheme of definition. Also let $\mathcal{C}_0 \subset \mathcal{C}$ denote the full subcategory of squares for which the morphism a is strict.

Proposition A.14. *The category $\mathcal{C}$ has a final object, and this final object is in $\mathcal{C}_0$.*

Proof. Let $J \subset \mathscr{O}_T$ denote the ideal defining Z, and let (T_n, M_{T_n}) denote the reduction of (T, M_T) modulo J^{n+1}. Since J is an ideal of definition by assumption (T_n, M_{T_n}) is fine log scheme, and there is a closed immersion $(Z, M_Z) \hookrightarrow (T_n, M_{T_n})$ defined by a nilpotent ideal. In particular, the étale sites of T_n and Z are canonically isomorphic. We therefore obtain a surjection $\pi_n : \overline{M}_{T_n} \rightarrow \overline{M}_Z$ of constructible sheaves of monoids on $T_{n,\text{et}}$. Moreover, this surjection satisfies the assumptions of A.12. Let $(\widetilde{T}_n, M_{\widetilde{T}_n})$ denote the log algebraic space representing the functor in A.12 applied to $\pi_n : \overline{M}_{T_n} \rightarrow \overline{M}_Z$, so we have a commutative diagram

$$\begin{tikzcd} & (\widetilde{T}_n, M_{\widetilde{T}_n}) \arrow[d] \\ (Z, M_Z) \arrow[ur, hook] \arrow[r, hook] & (T_n, M_{T_n}). \end{tikzcd}$$

Lemma A.15. *The algebraic space $\widetilde{T}_n$ is affine over T_n. In particular, $\widetilde{T}_n$ is a scheme.*

Proof. We may clearly work étale locally on T_n. Let $\bar{z} \rightarrow T_n$ be a geometric point, and choose a finitely generated group G and a homomorphism

(A.15.1)
$$G \rightarrow M^{\text{gp}}_{T_n,\bar{z}}$$

such that the induced map $G \rightarrow \overline{M}^{\text{gp}}_{T_n,\bar{z}}$ is surjective. Then the composite map

$$G \rightarrow \overline{M}^{\text{gp}}_{T_n,\bar{z}} \rightarrow \overline{M}^{\text{gp}}_{Z,\bar{z}}$$

is also surjective. Let $P \subset G$ (resp. $Q \subset G$) denote the inverse image of $M_{T_n,\bar{z}}$ (resp. $M_{Z,\bar{z}}$) under A.15.1 (resp. the composite map $G \rightarrow M^{\text{gp}}_{T_n,\bar{z}} \rightarrow M^{\text{gp}}_{Z,\bar{z}}$). We then have a commutative diagram

$$\begin{tikzcd} P \arrow[r, hook] \arrow[d] & Q \arrow[d] \\ M_{T_n,\bar{z}} \arrow[r] & M_{Z,\bar{z}}. \end{tikzcd}$$

Observe that Q is the localization of P by a face $F \in P$. By [Ka, 2.10], after possible replacing T_n by an étale neighborhood of $\bar{z}$, we can extend this commutative diagram to a commutative diagram of fine log schemes

$$\begin{tikzcd} (Z, M_Z) \arrow[r] \arrow[d] & (T_n, M_{T_n}) \arrow[d] \\ \mathrm{Spec}(Q \to \mathbb{Z}[Q]) \arrow[r] & \mathrm{Spec}(P \to \mathbb{Z}[P]), \end{tikzcd}$$

where the vertical arrows are charts. Here for a fine monoid M we write $\mathrm{Spec}(M \to \mathbb{Z}[M])$ for the log scheme with underlying scheme $\mathrm{Spec}(\mathbb{Z}[M])$ and log structure induced by the map $M \to \mathbb{Z}[M]$. In this situation, the scheme $\widetilde{T}_n$ can be described explicitly as

$$\widetilde{T}_n = \mathrm{Spec}(\mathbb{Z}[Q]) \times_{\mathrm{Spec}(\mathbb{Z}[P])} T_n.$$

□

Observe that by definition there is an isomorphism

$$(\widetilde{T}_n, M_{\widetilde{T}_n}) \times_{(T_n, M_{T_n})} (T_{n-1}, M_{T_{n-1}}).$$

In particular, all the underlying topological spaces of the $\widetilde{T}_n$ are canonically identified with the topological space $|\widetilde{T}_0|$ of T_0. Let $\mathscr{O}_{\widetilde{T}}$ denote the sheaf on $|\widetilde{T}_0|$ given by

$$\mathscr{O}_{\widetilde{T}} := \varprojlim_n \mathscr{O}_{\widetilde{T}_n}.$$

Also define $M_{\widetilde{T}}$ on $\widetilde{T}_{0,\mathrm{et}}$ to be the sheaf of monoids

$$M_{\widetilde{T}} := \varprojlim_n M_{\widetilde{T}_n}.$$

We then have a commutative diagram of ringed spaces with log structures

$$\begin{tikzcd} & (\widetilde{T}, M_{\widetilde{T}}) \arrow[d] \\ (Z, M_Z) \arrow[ur, hook] \arrow[r, hook] & (T, M_T). \end{tikzcd}$$

Lemma A.16. *$(\widetilde{T}, M_{\widetilde{T}})$ is a fine formal log scheme.*

Proof. The assertion is étale local on T. We may therefore assume that $T = \mathrm{Spf}(A)$ for some ring A, and as in the proof of A.15 that there exists a commutative diagram

$$\begin{tikzcd}
(Z, M_Z) \arrow[r] \arrow[d] & (T, M_T) \arrow[d] \\
\mathrm{Spec}(Q \to \mathbb{Z}[Q]) \arrow[r] & \mathrm{Spec}(P \to \mathbb{Z}[P]),
\end{tikzcd}$$

where the vertical arrows are charts and the map $P \to Q$ is injective and induces an isomorphism $P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$. In this case, $\widetilde{T}$ is equal to the formal spectrum of the J-adic completion of $A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$, and the log structure M_T is induced by the natural map $Q \to \mathbb{Z}[Q]$. □

Let T^{ex} denote the completion of $\widetilde{T}$ along the closed subscheme Z, and let $M_{T^{\mathrm{ex}}}$ denote the pullback of $M_{\widetilde{T}}$ to T^{ex}. Then

$$\begin{tikzcd}
(Z, M_Z) \arrow[r, hook] \arrow[d, "\mathrm{id}"] & (T^{\mathrm{ex}}, M_{T^{\mathrm{ex}}}) \arrow[d] \\
(Z, M_Z) \arrow[r, hook] & (T, M_T)
\end{tikzcd}$$

is an object of $\mathcal{C}_0$. We claim that this is the final object of $\mathcal{C}$.

Lemma A.17. *The inclusion $\mathcal{C}_0$ has a left adjoint $L : \mathcal{C} \to \mathcal{C}_0$.*

Proof. Consider an object A.13.1 of $\mathcal{C}$. On $W_{\mathrm{et}} \simeq U_{\mathrm{et}}$ we then have a diagram of log structures

$$\begin{tikzcd}
& M_W \arrow[d, two heads] \\
a^* M_Z \arrow[r] & M_U.
\end{tikzcd}$$

Let M'_W denote the fiber product of this diagram. Then M'_W is a log structure with map to $\mathscr{O}_W$ given by the composite

$$M'_W \to M_W \to \mathscr{O}_W.$$

Moreover, the projection $M_W \to a^* M_Z$ induces an isomorphism $i^* M'_W \simeq a^* M_Z$ over U. Moreover, one sees easily that M'_W is fine, and that the diagram of fine log formal schemes

$$\begin{tikzcd}
(U, M_U) \arrow[r, hook] \arrow[d] & (W, M_W) \arrow[d] \\
(U, a^* M_Z) \arrow[r, hook] & (W, M'_W).
\end{tikzcd}$$

is cocartesian. In particular, there is a natural map $(W, M'_W) \to (T, M_T)$ (this also follows from the construction of M'_W). We define L by sending A.13.1 to the diagram

$$\begin{tikzcd}
(U, a^* M_Z) \arrow[r] \arrow[d] & (W, M'_W) \arrow[d] \\
(Z, M_Z) \arrow[r] & (T, M_T).
\end{tikzcd}$$

□

Since the inclusion $\mathcal{C}_0 \subset \mathcal{C}$ has a left adjoint, to prove that $(T^{\mathrm{ex}}, M_{T^{\mathrm{ex}}})$ is a final object in $\mathcal{C}$ it suffices to show that $(T^{\mathrm{ex}}, M_{T^{\mathrm{ex}}})$ is a final object in $\mathcal{C}_0$. So consider an object A.13.1 with a strict. Let $I \subset \mathscr{O}_W$ be the ideal of U in W, and for $n \geq 0$ let $W_n \subset W$ denote the closed subscheme defined by I^{n+1} (note that by assumption U is a subscheme of definition of W). Then to construct the desired arrow $(W, M_W) \to (T^{\mathrm{ex}}, M_{T^{\mathrm{ex}}})$ it suffices to construct a morphism $(W_n, M_{W_n}) \to (T^{\mathrm{ex}}, M_{T^{\mathrm{ex}}})$ for every n. We may therefore in addition assume that W is a scheme and that $i : U \hookrightarrow W$ is defined by a nilpotent ideal. In this case we get by the universal property of $(\widetilde{T}_n, M_{\widetilde{T}_n})$ a commutative diagram

$$\begin{tikzcd}
(U, M_U) \arrow[r, hook] \arrow[d] & (W, M_W) \arrow[d] \\
(Z, M_Z) \arrow[r, hook] & (\widetilde{T}_n, M_{\widetilde{T}_n}).
\end{tikzcd}$$

Composing with the map $(\widetilde{T}_n, M_{\widetilde{T}_n}) \to (\widetilde{T}, M_{\widetilde{T}})$ we obtain a commutative diagram

$$\begin{tikzcd}
(U, M_U) \arrow[r, hook] \arrow[d] & (W, M_W) \arrow[d] \\
(Z, M_Z) \arrow[r, hook] & (\widetilde{T}, M_{\widetilde{T}}).
\end{tikzcd}$$

Since the ideal of U in W is nilpotent the morphism $(W, M_W) \to (\widetilde{T}, M_{\widetilde{T}})$ factors uniquely through $(T^{\mathrm{ex}}, M_{T^{\mathrm{ex}}})$ and so we finally obtain a commutative diagram

$$\begin{tikzcd}
(U, M_U) \arrow[r, hook] \arrow[d] & (W, M_W) \arrow[d] \\
(Z, M_Z) \arrow[r, hook] & (T^{\mathrm{ex}}, M_{T^{\mathrm{ex}}}).
\end{tikzcd}$$

The uniqueness of this diagram also follows from the universal property of $(\widetilde{T}, M_{\widetilde{T}})$. This completes the proof of A.14. $\square$

Remark A.18. We call the final object $(Z, M_Z) \hookrightarrow (T^{\mathrm{ex}}, M_{T^{\mathrm{ex}}})$ in A.14 the *exactification* of $(Z, M_Z) \hookrightarrow (T, M_T)$.

Appendix B. Remarks on localization in model categories

B.1. Let C be a model category. For any object $S \in C$ the localized category $C|_S$ of objects over S has, by a similar argument to the one used in [Ho, 1.1.8], a model category structure in which a morphism

(B.1.1)
$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\
& S &
\end{tikzcd}$$

is a cofibration (resp. fibration, weak equivalence) if the underlying morphism $f : X \to Y$ is a cofibration (resp. fibration, weak equivalence) in C.

B.2. Let $h : S \to R$ be a morphism in C. Let

$$F : C|_S \to C|_R \tag{B.2.1}$$

be the functor sending $X \to S$ to the composite

$$X \to S \to R. \tag{B.2.2}$$

The functor F has a right adjoint

$$U : C|_R \to C|_S \tag{B.2.3}$$

which sends $Y \to R$ to

$$\mathrm{pr}_2 : Y \times_R S \to S. \tag{B.2.4}$$

Let $\varphi : FU \to \mathrm{id}_C$ be the adjunction map. Then since F clearly preserves cofibrations and trivial cofibrations the triple (F, U, φ) is a Quillen adjunction [Ho, 1.3.1].

B.3. Recall [Hi, 13.1.1] that a model category C is called *right proper* if for every cartesian diagram in C

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d] & X \arrow[d, "g"] \\ A \arrow[r, "h"] & B \end{tikzcd} \tag{B.3.1}$$

with g a fibration and h a weak equivalence, the map $p : P \to X$ is a weak equivalence.

Proposition B.4. *Suppose C is right proper, and that $h : S \to R$ is a weak equivalence in C. Then the Quillen adjunction*

$$(F, U, \varphi) : C|_S \to C|_R \tag{B.4.1}$$

is a Quillen equivalence.

Proof. By the definition of a Quillen equivalence [Ho, 1.3.12], it suffices to show that given a commutative diagram in C

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "k"] & Y \arrow[d, "g"] \\ S \arrow[r, "h"] & R \end{tikzcd} \tag{B.4.2}$$

with k a cofibration and g a fibration, the map f is a weak equivalence if and only if the induced map

$$f' : X \to S \times_R Y \tag{B.4.3}$$

is a weak equivalence. This follows from the 2-out-of-3 property for weak equivalences applied to the diagram

$$X \xrightarrow{f'} S \times_R Y \xrightarrow{\mathrm{pr}_2} Y, \tag{B.4.4}$$

and the fact that the map $\mathrm{pr}_2 : S \times_R Y \to Y$ is a weak equivalence since C is right proper. $\square$

Remark B.5. Similarly a model category is called *left proper* if for every pushout diagram

(B.5.1)
$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, "g"] & A \arrow[d] \\ X \arrow[r] & P \end{tikzcd}$$

with g a cofibration and h a weak equivalence, the map $X \to P$ is a weak equivalence.

If $\mathcal{C}$ is a left proper model category and $h : S \to R$ is a weak equivalence in $\mathcal{C}$, then by applying B.4 to the opposite category $\mathcal{C}^o$ (with the natural model category structure) one obtains that the functor

(B.5.2)
$$\mathcal{C}|_{\backslash R} \to \mathcal{C}|_{\backslash S}$$

between the model categories of objects under R and S respectively is a Quillen equivalence.

A model category which is both left and right proper is called *proper*.

B.6. Quillen adjunctions extend naturally to localized model categories. To explain this, let C and D be model categories, and let

(B.6.1)
$$F : C \to D, \quad U : D \to C, \quad \varphi : FU \to \mathrm{id}_C$$

be a Quillen adjunction (so F is left adjoint to U).

Fix an object $X \in D$. We then get model categories $D_{\backslash X}$ and $C_{\backslash U(X)}$ of object under X and $U(X)$ respectively. As noted in B.1 (applied to D^o and C^o) there are natural model category structures on $D_{\backslash X}$ and $C_{\backslash U(X)}$.

The functor U induces a functor

(B.6.2)
$$U_{\backslash X} : D_{\backslash X} \to C_{\backslash U(X)}, \quad (X \to Y) \mapsto (U(X) \to U(Y)).$$

This functor has a left adjoint

(B.6.3)
$$F_{\backslash X} : C_{\backslash U(X)} \to D_{\backslash X}$$

sending $U(X) \to Z$ to the pushout of the diagram

(B.6.4)
$$\begin{tikzcd} FU(X) \arrow[r] \arrow[d, "\varphi"'] & F(Z) \\ X. & \end{tikzcd}$$

The functor $U_{\backslash X}$ preserves fibrations and trivial fibrations, as this is true of U. The pair $(F_{\backslash X}, U_{\backslash X})$ is therefore a Quillen adjunction by [Ho, 1.3.4].

B.7. The forgetful functor

(B.7.1)
$$f_{X*} : D_{\backslash X} \to D, \quad (X \to Y) \mapsto Y$$

has a left adjoint f_X^* sending $Z \in D$ to $Z \coprod X$ with the natural map $X \to Z \coprod X$. The pair (f_X^*, f_{X*}) is a Quillen adjunction since f_{X*} clearly preserves fibrations and trivial fibrations.

We therefore obtain a commutative diagram of right Quillen functors

(B.7.2)
$$\begin{tikzcd} D_{\backslash X} \arrow[r, "U_{\backslash X}"] \arrow[d, "f_{X*}"'] & C_{\backslash U(X)} \arrow[d, "f_{U(X)*}"] \\ D \arrow[r, "U"] & C. \end{tikzcd}$$

Passing to the associated homotopy categories we obtain a commutative diagram of derived functors

(B.7.3)
$$\begin{tikzcd} \mathrm{Ho}(D_{\backslash X}) \arrow[r, "\mathbb{R}U_{\backslash X}"] \arrow[d, "\mathbb{R}f_{X*}"'] & \mathrm{Ho}(C_{\backslash U(X)}) \arrow[d, "\mathbb{R}f_{U(X)*}"] \\ \mathrm{Ho}(D) \arrow[r, "\mathbb{R}U"] & \mathrm{Ho}(C). \end{tikzcd}$$

Observe that since f_{X*} preserves arbitrary equivalences we have $\mathbb{R}f_{X*} \simeq f_{X*}$.

B.8. Dually, let $Y \in C$ be an object. We can then consider the localized categories $C_{\backslash Y}$ and $D_{\backslash F(Y)}$. Let

(B.8.1)
$$F^{\backslash Y} : C_{\backslash Y} \to D_{\backslash F(Y)}$$

be the functor sending $Y \to Z$ to $F(Y) \to F(Z)$. This functor has a right adjoint $U^{\backslash Y}$ sending $F(Y) \to X$ to the composite

(B.8.2)
$$Y \xrightarrow{\text{adjunction}} UF(Y) \longrightarrow F(X).$$

Since F takes cofibrations to cofibrations and trivial cofibrations to trivial cofibrations, the same is true of $F^{\backslash Y}$. Therefore $(F^{\backslash Y}, U^{\backslash Y})$ is a Quillen adjunction.

We have a commutative diagram of functors

(B.8.3)
$$\begin{tikzcd} C_{\backslash Y} \arrow[r, "F^{\backslash Y}"] \arrow[d, "f_{Y*}"] & D_{\backslash F(Y)} \arrow[d, "f_{F(Y)*}"] \\ C \arrow[r, "F"] & D. \end{tikzcd}$$

As above, this implies that there is a natural transformation

(B.8.4)
$$\eta : \mathbb{L}F \circ \mathbb{R}f_{Y*} \to \mathbb{R}f_{F(Y)*} \circ \mathbb{L}F^{\backslash Y}.$$

Proposition B.9. *If Y is a cofibrant object in C, then B.8.4 is an equivalence, so we have a commutative diagram*

(B.9.1)
$$\begin{tikzcd} \mathrm{Ho}(C_{\backslash Y}) \arrow[r, "F^{\backslash Y}"] \arrow[d, "f_{Y*}"] & \mathrm{Ho}(D_{\backslash F(Y)}) \arrow[d, "f_{F(Y)*}"] \\ \mathrm{Ho}(C) \arrow[r, "F"] & \mathrm{Ho}(D). \end{tikzcd}$$

Proof. If $Y \to Z$ is a cofibration, then we have

(B.9.2)
$$\mathbb{R}f_{F(Y)*}\mathbb{L}F^{\backslash Y}(Y \to Z) = F(Z).$$

On the other hand,

$$\mathbb{L}F(\mathbb{R}f_{Y*}(Y \to Z) = F(Z'), \tag{B.9.3}$$

where $Z' \to Z$ is a cofibrant replacement. Now if Y is cofibrant then Z is also cofibrant, so the map $F(Z') \to F(Z)$ is an equivalence. □

Appendix C. The coherator for algebraic stacks

In this section we discuss a version of the coherator for an algebraic stacks (see [T-T] for the case of schemes and [Jo, 10.1] for a result in the lisse-étale topology).

C.1. Let S be a scheme, and let $\mathscr{X}$ be a stack over the category of affine S-schemes Aff_S with the fpqc topology. Assume that the diagonal

$$\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X} \tag{C.1.1}$$

is representable and affine, and that there exists an fpqc surjection $\omega : \mathrm{Spec}(R) \to \mathscr{X}$.

Define $\mathscr{X}_{\mathrm{fpqc}}$ to be the topos associated to the *small fpqc site of* $\mathscr{X}$*:*

Objects: flat S-morphisms $t : T \to \mathscr{X}$, where $T \in \mathrm{Aff}_S$.

Morphisms: 2-commutative triangles over S

$$\begin{tikzcd} T' \arrow[rr] \arrow[dr] & & T \arrow[dl] \\ & \mathscr{X}. & \end{tikzcd} \tag{C.1.2}$$

Coverings: A collection of morphisms $\{T_i \to T\}_{i \in I}$ is a *covering* if the underlying collection of maps in Aff_S is an fpqc covering in the usual sense (see for example [Vi, page 30]).

The topos $\mathscr{X}_{\mathrm{fpqc}}$ is ringed with structure sheaf given by

$$\mathscr{O}_{\mathscr{X}}(T \to \mathscr{X}) := \Gamma(T, \mathscr{O}_T). \tag{C.1.3}$$

Remark C.2. If $f : \mathscr{X} \to \mathscr{Y}$ is a morphism of stacks satisfying the conditions of C.1, then just as in the case of the lisse-étale site [LM-B, 12.2] there is a functor

$$f_* : \mathscr{X}_{\mathrm{fpqc}} \to \mathscr{Y}_{\mathrm{fpqc}} \tag{C.2.1}$$

sending $F \in \mathscr{X}_{\mathrm{fpqc}}$ to the sheaf

$$(T \to \mathscr{Y}) \mapsto \Gamma((T \times_{\mathscr{Y}} \mathscr{X})_{\mathrm{fpqc}}, F). \tag{C.2.2}$$

This functor has a left adjoint f^* but this functor is not in general exact. If f is flat, however, then f^* is exact (it is just the restriction functor) and f induces a morphism of topoi

$$\mathscr{X}_{\mathrm{fpqc}} \to \mathscr{Y}_{\mathrm{fpqc}}. \tag{C.2.3}$$

Remark C.3. Recall that if T is a scheme, then there is a natural morphism of ringed topoi

$$\epsilon : T_{\mathrm{fpqc}} \to T_{\mathrm{Zar}}. \tag{C.3.1}$$

A sheaf of $\mathscr{O}_{T_{\mathrm{fpqc}}}$-modules E is called *quasi-coherent* if E is isomorphic to ϵ^*F for some quasi-coherent sheaf (in the usual sense) on T_{Zar}. Furthermore, the pullback functor ϵ^* induces an equivalence of categories between quasi-coherent sheaves on T_{Zar} and quasi-coherent sheaves of $\mathcal{O}_{T_{\mathrm{fpqc}}}$-modules (this follows from descent theory [Vi, 4.23]).

C.4. By standard considerations, the category of $\mathscr{O}_{\mathscr{X}}$-modules in $\mathscr{X}_{\mathrm{fpqc}}$ is equivalent to the category of collections of data $\{(E_{(T,t)}, \varphi)\}$ consisting of a sheaf $\mathscr{O}_T$-modules $E_{(T,t)}$ in T_{fpqc} for every flat morphism $t : T \to \mathscr{X}$, and for every morphism $f : (T', t') \to (T, t)$ a morphism of $\mathscr{O}_{T'}$-modules

(C.4.1) $$\varphi : f^* E_{(T,t)} \to E_{(T',t')}.$$

These morphisms are further required to satisfy a natural cocycle condition for compositions.

A sheaf of $\mathscr{O}_{\mathscr{X}}$-modules E is called *quasi-coherent* if each $E_{(T,t)}$ is quasi-coherent and if the transition morphisms C.4.1 are all isomorphisms. Let $\mathrm{Qcoh}(\mathscr{X})$ denote the category of quasi-coherent sheaves on $\mathscr{X}$, and let $\mathrm{Mod}(\mathscr{X})$ denote the category of all $\mathscr{O}_{\mathscr{X}}$-modules. Then there is a natural inclusion

(C.4.2) $$j : \mathrm{Qcoh}(\mathscr{X}) \hookrightarrow \mathrm{Mod}(\mathscr{X}).$$

The following lemma is the main reason we consider the small fpqc site as opposed to the big site.

Lemma C.5. *The essential image of j is closed under kernels, cokernels, and extensions.*

Proof. It is immediate that the essential image of j is closed under cokernels and extensions. For the statement about kernels, let $f : E \to F$ be a morphism of quasi-coherent sheaves, and let K denote the kernel in $\mathrm{Mod}(\mathscr{X})$. Then for any flat morphism $t : T \to \mathscr{X}$ with $T \in \mathrm{Aff}_S$, the restriction $K_{T,\mathrm{Zar}}$ of K to T_{Zar} is simply the kernel of the map of Zariski sheaves $E_{T,\mathrm{Zar}} \to F_{T,\mathrm{Zar}}$ induced by f. It therefore suffices to show that if $h : (T', t') \to (T, t)$ is a morphism in the small fpqc site of $\mathscr{X}$ then the induced map

(C.5.1) $$h^* K_{T,\mathrm{Zar}} \to K_{T',\mathrm{Zar}}$$

is an isomorphism. Since t and t' are flat, there exists a flat surjection $p : P \to T'$ and a commutative diagram

(C.5.2) $$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"] & T \arrow[d, "t"] \\ T' \arrow[r, "t'"] & \mathscr{X} \end{tikzcd}$$

with q also flat. To verify that C.5.1 is an isomorphism, it suffices to show that it becomes an isomorphism after applying p^*. It therefore suffices to show that the analogues of C.5.1 for the morphism $P \to T'$ and the morphism q are isomorphisms. This follows from the observation that C.5.1 is clearly an isomorphism if h is flat. $\square$

Lemma C.6. *The functor j has a right adjoint*

(C.6.1) $$u : \mathrm{Mod}(\mathscr{X}) \to \mathrm{Qcoh}(\mathscr{X}).$$

Moreover, the adjunction map $\mathrm{id} \to uj$ *is an isomorphism.*

Remark C.7. The functor u of C.6 is called the *coherator*.

Proof. Let $\omega : U_0 \to \mathscr{X}$ be an fpqc surjection with U affine, and let U_1 denote $U_0 \times_{\mathscr{X}} U_0$ so we have a diagram

$$\begin{tikzcd} U_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[rr, bend left, "\eta"] & U_0 \arrow[r, "\omega"] & \mathscr{X}. \end{tikzcd} \tag{C.7.1}$$

For $F \in \mathrm{Mod}(\mathscr{X})$, define

$$u(F) := \mathrm{Ker}(\omega_*\Gamma(U_0, F)\tilde{} \rightrightarrows \eta_*\Gamma(U_1, F)\tilde{}), \tag{C.7.2}$$

where $\Gamma(U_0, F)\tilde{}$ (resp. $\Gamma(U_1, F)\tilde{}$) denotes the quasi-coherent sheaf on U_0 (resp. U_1) associated to $\Gamma(U_0, F)$ (resp. $\Gamma(U_1, F)$). Since F is a sheaf for the fpqc topology the sequence

$$0 \to F \to \omega_*\omega^*F \rightrightarrows \eta_*\eta^*F \tag{C.7.3}$$

is exact. From the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] & \omega_*\omega^*F \arrow[r, shift left] \arrow[r, shift right] & \eta_*\eta^*F \\ 0 \arrow[r] & u(F) \arrow[r] \arrow[u, dashed] & \omega_*\Gamma(U_0, F)\tilde{} \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & \eta_*\Gamma(U_1, F)\tilde{} \arrow[u] \end{tikzcd} \tag{C.7.4}$$

we then obtain a map $ju(F) \to F$. From this definition of $u(F)$ it is clear that the adjunction map $uj(G) \to G$ is an isomorphism if $G \in \mathrm{Qcoh}(\mathscr{X})$.

If G is a quasi-coherent sheaf, and $\varphi : G \to F$ is a morphism of sheaves where $F \in \mathrm{Mod}(\mathscr{X})$ then the induced maps $\omega^*G \to \omega^*F$ and $\eta^*G \to \eta^*F$ factor uniquely through $\Gamma(U_0, F)\tilde{}$ and $\Gamma(U_1, F)\tilde{}$ respectively. It follows that φ also factors through a unique map $G \to u(F)$. This implies that u is a right adjoint to j. $\square$

C.8. Since j is exact, the functor u takes injectives to injectives, and for any $F \in D^+(\mathrm{Mod}(\mathscr{X}))$ there is a canonical map

$$jRu(F) \to F. \tag{C.8.1}$$

Theorem C.9. *The functor j induces an equivalence of categories*

$$j : D^+(\mathrm{Qcoh}(\mathscr{X})) \to D^+_{\mathrm{qcoh}}(\mathrm{Mod}(\mathscr{X})), \tag{C.9.1}$$

where $D^+_{\mathrm{qcoh}}(\mathrm{Mod}(\mathscr{X})) \subset D^+(\mathrm{Mod}(\mathscr{X}))$ denotes the full subcategory of complexes with quasi-coherent cohomology sheaves. A quasi-inverse to j is given by

$$Ru : D^+_{\mathrm{qcoh}}(\mathrm{Mod}(\mathscr{X})) \to D^+(\mathrm{Qcoh}(\mathscr{X})). \tag{C.9.2}$$

Proof. For a flat morphism $U \to \mathscr{X}$ with U an affine scheme, let $\mathscr{S}_U$ denote the following site:

Objects: Morphisms of affine schemes $V \to U$ such that the composite morphism $V \to U \to \mathscr{X}$ is flat.

Morphisms: Morphisms of U-schemes.

Coverings: A collection of maps $\{V_i \to V\}_{i \in I}$ in $\mathscr{S}_U$ is a covering if it is an fpqc covering in the usual sense.

Then $\mathscr{X}_{\text{fpqc}}|_U$ is equivalent to the topos associated to $\mathscr{S}_U$. As in C.4 the category of sheaves of $\mathscr{O}_{\mathscr{X}|_U}$-modules (where $\mathscr{O}_{\mathscr{X}|_U}$ is the restriction of $\mathscr{O}_{\mathscr{X}}$) is equivalent to the category of collections of data $\{E_{V\to U}, \varphi\}$ consisting of a sheaf of $\mathscr{O}_{V_{\text{fpqc}}}$-modules on V_{fpqc} for every $(V \to U) \in \mathscr{S}_U$, and a transition morphism

$$\varphi : f^*E_{V\to U} \to E_{V'\to U}, \tag{C.9.3}$$

for every morphism $(V \to U) \to (V' \to U)$ in $\mathcal{S}_U$. These transition morphisms are further required to satisfy a natural cocycle condition. A sheaf E of $\mathscr{O}_{\mathscr{X}|_U}$-modules is called *quasi-coherent* if each $E_{V\to U}$ is quasi-coherent, and if the transition maps C.9.3 are all isomorphisms.

Lemma C.10. *Let $U \to \mathscr{X}$ be a flat morphism with U an affine scheme, and let F be a quasi-coherent sheaf on $\mathscr{X}_{\text{fpqc}}|_U$.*

(i) *We have $H^i(\mathscr{X}_{\text{fpqc}}|_U, F) = 0$ for $i > 0$.*

(ii) *If $\omega : \mathscr{X}_{\text{fpqc}}|_U \to \mathscr{X}_{\text{fpqc}}$ is the projection, then we have $R^i\omega_*F = 0$ for $i > 0$.*

Proof. Note first that (ii) follows immediately from (i) as $R^i\omega_*F$ is equal to the sheaf associated to the presheaf which to any flat morphism $W \to \mathscr{X}$ associates $H^i(\mathscr{X}_{\text{fpqc}}|_{U\times_{\mathscr{X}}W}, F)$, where W is an affine scheme (and note that $U \times_{\mathscr{X}} W$ is affine since $\mathscr{X}$ has affine diagonal).

Statement (i) can be seen as follows (this is essentially the same as in the usual case of the big fpqc site). For any fpqc covering $P : V \to U$ with V an affine scheme, let $V_\cdot$ denote the 0-coskeleton of P (see for example [LM-B, 12.4]). We then have a spectral sequence [De4, 1.4.5]

$$E_1^{pq} = H^q(\mathscr{X}_{\text{fpqc}}|_{V_p}, F) \implies H^{p+q}(\mathscr{X}_{\text{fpqc}}|_U, F). \tag{C.10.1}$$

The $q = 0$ line in this spectral sequence is the complex

$$F(V_0) \to F(V_1) \to F(V_2) \to \cdots , \tag{C.10.2}$$

which is exact by the usual flat descent theory (see for example [Mi, I.2.18]).

This in turn implies that the natural map

$$H^1(\mathscr{X}_{\text{fpqc}}|_U, F) \to H^1(\mathscr{X}_{\text{fpqc}}|_V, F) \tag{C.10.3}$$

is injective. Since any cohomology class $\alpha \in H^1(\mathscr{X}_{\text{fpqc}}|_U, F)$ maps to zero in $H^1(\mathscr{X}_{\text{fpqc}}|_V, F)$ for some fpqc covering $V \to U$ this implies that $H^1(\mathscr{X}_{\text{fpqc}}|_U, F) = 0$. This proves (i) for $i = 1$.

For general i, we proceed by induction on $i \geq 1$. So fix an integer i and assume that for any flat morphism $W \to \mathscr{X}$ with W an affine scheme, and quasi-coherent sheaf G on $\mathscr{X}_{\text{fpqc}}|_W$ we have $H^j(\mathscr{X}_{\text{fpqc}}|_W, G) = 0$ for $1 \leq j \leq i-1$. Then the spectral sequence C.10.1 and the exactness of C.10.2 shows that for any fpqc covering $V \to U$ the pullback map

$$H^i(\mathscr{X}_{\text{fpqc}}|_U, F) \to H^i(\mathscr{X}_{\text{fpqc}}|_V, F) \tag{C.10.4}$$

is injective. Since for any class $\alpha \in H^i(\mathscr{X}_{\text{fpqc}}|_U, F)$ there exists an fpqc covering $V \to U$ such that α maps to zero in $H^i(\mathscr{X}_{\text{fpqc}}|_V, F)$, it follows that $H^i(\mathscr{X}_{\text{fpqc}}|_U, F) = 0$. □

Let $\omega : U_0 \to \mathscr{X}$ be an fpqc covering with U_0 affine, and let $A : U_\cdot \to \mathscr{X}$ be the associated simplicial space.

Lemma C.11. *Let $I \in \mathrm{Mod}(\mathscr{X})$ be an injective sheaf, and let $\widetilde{I}.$ be the sheaf in the localized topos $\mathscr{X}_{\mathrm{fpqc}}|_{U.}$ whose restriction to each $\mathscr{X}_{\mathrm{fpqc}}|_{U_n}$ is equal to the quasi-coherent sheaf associated to $\Gamma(U_n, I)$. Then $R^iA_*(\widetilde{I}.) = 0$ for $i > 0$, where*

$$A : \mathscr{X}_{\mathrm{fpqc}}|_{U.} \to \mathscr{X}_{\mathrm{fpqc}} \tag{C.11.1}$$

is the projection.

Proof. Since the natural map $I \to \omega_*\omega^*I$ is injective and hence a direct summand, it suffices to consider the case when $I = \omega_*J$ for an injective sheaf J in $\mathscr{X}|_{U_0}$.

Let $P.$ denote $U. \times_{\mathscr{X}} U_0$ and consider the commutative diagram of topoi

$$\begin{array}{ccc} \mathscr{X}_{\mathrm{fpqc}}|_{P.} & \xrightarrow{B} & \mathscr{X}_{\mathrm{fpqc}}|_{U_0} \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \omega} \\ \mathscr{X}_{\mathrm{fpqc}}|_{U.} & \xrightarrow{A} & \mathscr{X}_{\mathrm{fpqc}}. \end{array} \tag{C.11.2}$$

Let $\widetilde{J}.$ denote the sheaf in $\mathscr{X}_{\mathrm{fpqc}}|_{P.}$ whose restriction to $\mathscr{X}_{\mathrm{fpqc}}|_{P_n}$ is the quasi-coherent sheaf associated to $\Gamma(\mathscr{X}_{\mathrm{fpqc}}|_{P_n}, J)$. Then we have $\widetilde{I}. = \eta_*\widetilde{J}.$.

For any natural number n and $i \geq 0$, the restriction of $R^i\eta_*\widetilde{J}.$ to $\mathscr{X}_{\mathrm{fpqc}}|_{U_n}$ is equal to the sheaf associated to the presheaf on $\mathscr{S}_{U_n}$ which to any $V \to U_n$ associates $H^i(\mathscr{X}_{\mathrm{fpqc}}|_{V \times_{U_n} P_n}, \widetilde{J}_n)$. By C.10 it follows that $R^i\eta_*\widetilde{J}. = 0$ for $i > 0$. To prove C.11 it therefore suffices to show that

$$R^i(A_* \circ \eta_*)(\widetilde{J}.) = R^i(\omega_* \circ B_*)(\widetilde{J}.) = 0 \tag{C.11.3}$$

for $i > 0$.

Let $\widetilde{J}_{-1}$ denote the quasi-coherent sheaf on $\mathscr{X}_{\mathrm{fpqc}}|_{U_0}$ associated $\Gamma(\mathscr{X}_{\mathrm{fpqc}}|_{U_0}, J)$. We show that the natural map

$$\widetilde{J}_{-1} \to RB_*\widetilde{J}. \tag{C.11.4}$$

is an isomorphism. This will complete the proof C.11 for then we have

$$R^i(\omega_* \circ B_*)(\widetilde{J}.) \simeq R^i\omega_*\widetilde{J}_{-1} \tag{C.11.5}$$

and the right side is zero by C.10 (ii).

To see that C.11.4 is an isomorphism, consider the spectral sequence [De4, 1.4.5]

$$E_1^{pq} = R^qB_{p*}\widetilde{J}_p \implies R^{p+q}B_*\widetilde{J}., \tag{C.11.6}$$

where $B_p : \mathscr{X}_{\mathrm{fpqc}}|_{P_p} \to \mathscr{X}_{\mathrm{fpqc}}|_{U_0}$ is the projection. As in the proof of C.10 (ii), it follows from C.10 (i) that $R^qB_{p*}\widetilde{J}_p = 0$ for $q > 0$. Therefore $RB_*\widetilde{J}.$ is represented by the complex

$$C^{\cdot} : B_{0*}\widetilde{J}_0 \to B_{1*}\widetilde{J}_1 \to \cdots \tag{C.11.7}$$

which is the normalized complex of the cosimplical module

$$\mathscr{F}. : [n] \mapsto B_{n*}\widetilde{J}_n. \tag{C.11.8}$$

Let $\rho : \widetilde{J}_{-1} \to \mathscr{F}.$ be the natural map. The identity map $U_0 \to U_0$ over $\mathscr{X}$ induces a section of the projection $P. \to U_0$. This section in turn induces for every n a map

$$g_n : P_n \to P_{n+1} \tag{C.11.9}$$

given by the map

$$P_n = \underbrace{P_0 \times_{U_0} P_0 \cdots \times_{U_0} P_n}_{n+1} \longrightarrow \underbrace{P_0 \times_{U_0} P_0 \cdots \times_{U_0} P_n}_{n+2} = P_{n+2} \tag{C.11.10}$$

which on scheme-valued points is given by

$$(\alpha_0, \dots, \alpha_n) \mapsto (sB_0(\alpha_0), \alpha_0, \dots, \alpha_n). \tag{C.11.11}$$

The map g_n defines a morphism of topoi

$$g_n : \mathscr{X}_{\mathrm{fpqc}}|_{P_n} \to \mathscr{X}_{\mathrm{fpqc}}|_{P_{n+1}}, \tag{C.11.12}$$

and therefore also a map

$$g_n^* : B_{n+1*}\widetilde{J}_{n+1} \to B_{n*}\widetilde{J}_n. \tag{C.11.13}$$

Exactly as in the case of faithfully flat descent [Mi, I, proof of 2.18] these maps give a homotopy between the identity map on $C^\cdot$ and the zero map. From this it follows that

$$\widetilde{J}_{-1} \to C^\cdot \tag{C.11.14}$$

is a quasi-isomorphism. □

Lemma C.12. *Let F be a quasi-coherent sheaf on $\mathscr{X}$. Then the adjunction map*

$$F \to RA_*A^*F \tag{C.12.1}$$

is an isomorphism.

Proof. Again by [De4, 1.4.5] there is a spectral sequence

$$E_1^{pq} = R^qA_{p*}A_p^*F \implies R^{p+q}A_*A^*F. \tag{C.12.2}$$

By C.10 (ii) we have $R^qA_{p*}A_p^*F = 0$ for $q > 0$, which implies that RA_*A^*F is represented by the complex

$$A_{0*}A_0^*F \to A_{1*}A_1^*F \to A_{2*}A_2^*F \to \cdots. \tag{C.12.3}$$

By classical fpqc descent [Mi, I.2.18] the adjunction map $F \to A_{0*}A_0^*F$ induces a quasi-isomorphism between F and C.12.3. □

Lemma C.13. *Let F be a quasi-coherent sheaf on $\mathscr{X}$. Then the adjunction map*

$$jRu(j(F)) \to j(F) \tag{C.13.1}$$

is an isomorphism.

Proof. Choose an injective resolution $F \to I^\cdot$ in the category $\mathrm{Mod}(\mathscr{X})$. Let $\widetilde{I}^\cdot_\cdot$ be the complex on $\mathscr{X}_{\mathrm{fpqc}}|_{U_\cdot}$ whose j-th term is $\widetilde{I}^j_\cdot$. For any $n \geq 0$, the complex

$$\Gamma(\mathscr{X}_{\mathrm{fpqc}}|_{U_n}, I^0) \to \Gamma(\mathscr{X}_{\mathrm{fpqc}}|_{U_n}, I^1) \to \Gamma(\mathscr{X}_{\mathrm{fpqc}}|_{U_n}, I^2) \to \cdots \tag{C.13.2}$$

computes $R\Gamma(\mathscr{X}_{\mathrm{fpqc}}|_{U_n}, F)$. By C.10 (i) it follows that the natural map

$$A_{n*}A_n^*F \to A_{n*}\widetilde{I}^\cdot_n \tag{C.13.3}$$

is a quasi-isomorphism. By C.12 we therefore obtain an isomorphism in the derived category

$$F \simeq RA_*A^*F \simeq \mathrm{Tot}((p,q) \mapsto A_{p*}\widetilde{I}^q_p), \tag{C.13.4}$$

where the right side denotes the total complex of the indicated complex. On the other hand, by C.11 the natural map

$$ju(I^{\cdot}) = (A_* \widetilde{I}^0_{\cdot} \to A_* \widetilde{I}^1_{\cdot} \to \cdots) \to \mathrm{Tot}((p,q) \mapsto A_{p*}\widetilde{I}^q_p) \tag{C.13.5}$$

is a quasi-isomorphism. Since $jRu(F) = ju(I^{\cdot})$ we conclude that the adjunction map

$$jRu(j(F)) \to j(F) \tag{C.13.6}$$

is an isomorphism. □

We can now complete the proof of C.9. We need to show that the adjunction maps

$$jRu \to \mathrm{id}, \quad \text{and} \quad \mathrm{id} \to (Ru) \circ j \tag{C.13.7}$$

are isomorphisms.

Note first that if $\varphi : F \to G$ is a morphism in $D^+(\mathrm{Qcoh}(\mathscr{X}))$, then φ is an isomorphism if and only if $j(\varphi) : j(F) \to j(G)$ is an isomorphism in $D^+(\mathrm{Mod}(\mathscr{X}))$. Therefore to verify that the adjunction map

$$\mathrm{id} \to (Ru) \circ j \tag{C.13.8}$$

is an isomorphism it suffices to show that the adjunction map

$$j \to j \circ (Ru) \circ j \tag{C.13.9}$$

is an isomorphism. For this in turn it suffices to show that the adjunction map $jRu \to \mathrm{id}$ is an isomorphism.

For this note that if $F \in D^+_{\mathrm{qcoh}}(\mathrm{Mod}(\mathscr{X}))$ then there is a spectral sequence (the spectral sequence of a filtered complex [De3, 1.4.6])

$$E_1^{pq} = jR^q u(\mathscr{H}^p(F)) \implies jR^{p+q}u(F). \tag{C.13.10}$$

Since each $\mathscr{H}^p(F)$ is quasi-coherent this implies that the natural map

$$jR^n u(F) \to ju\mathscr{H}^n(F) = \mathscr{H}^n(F) \tag{C.13.11}$$

is an isomorphism. □

Remark C.14. In the case of an algebraic stack in the usual sense [LM-B], one could replace the small fpqc topology in the above with the lisse-étale topology [LM-B, §12].

Appendix D. $\widetilde{B}_{\mathrm{cris}}(V)$-admissible implies crystalline.

D.1. Let V be a complete discrete valuation ring of mixed characteristic $(0,p)$, field of fractions K, and perfect residue field k. Let W be the ring of Witt vectors of k, and let $K_0 \subset K$ be the field of fractions of W. Let $K \hookrightarrow \overline{K}$ be an algebraic closure, and let $A_{\mathrm{cris}}(V)$, $B_{\mathrm{cris}}(V)$, and $\widetilde{B}_{\mathrm{cris}}(V)$ be the rings defined in 6.2.

Let us recall the construction of these rings. Let $\overline{V} \subset \overline{K}$ denote the integral closure of V, and let S_V denote the ring of sequences $(a_n)_{n\geq 0}$, where $a_n \in \overline{V}/p\overline{V}$ and $a_{n+1}^p = a_n$ for all $n \geq 0$. Then S_V is a perfect ring and we can form the ring of Witt vectors $W(S_V)$. As in 6.2.2 there is a surjection

$$\theta : W(S_V) \to \overline{V}^{\wedge}, \tag{D.1.1}$$

where $\overline{V}^{\wedge}$ denotes the p-adic completion of $\overline{V}$. If J denotes the kernel of θ then $A_{\mathrm{cris}}(V)$ is defined to be the p-adic completion of the divided power envelope $D_J(W(S_V))$.

Fix elements $\epsilon_m \in \overline{V}$ with $\epsilon_0 = 1$, $\epsilon_{m+1}^p = \epsilon_m$, and $\epsilon_1 \neq 1$. Let $\epsilon \in S_V$ be the element defined by the reductions of the ϵ_m, and let $[\epsilon] \in W(S_V)$ be its Teichmuller lifting. Define $\pi_\epsilon := [\epsilon] - 1 \in W(S_V)$. Then one verifies (see for example 6.2.5) that the series

$$\sum_{m\geq 1}(-1)^{m-1}(m-1)!\pi_\epsilon^{[m]} \tag{D.1.2}$$

converges to an element $t \in A_{\mathrm{cris}}(V)$. We define $B_{\mathrm{cris}}(V)$ to be the localization $A_{\mathrm{cris}}(V)[1/t]$.

Fix a sequence of elements τ_m $(m \geq 0)$ with $\tau_0 = p$ and $\tau_{m+1}^p = \tau_m$. As in 6.8 let λ_{1/p^n} denote the element given by the sequence $(a_m)_{m\geq 0}$ with $a_m = \tau_{m+n}$, and let $\delta_{1/p^n} := [\lambda_{1/p^n}]$. We define $\widetilde{B}_{\mathrm{cris}}(V)$ to be the ring obtained from $B_{\mathrm{cris}}(V)$ by inverting the elements δ_{1/p^n} $(n \geq 0)$. Note that $\delta_{1/p^n}^{p^n} = \delta_1$ so we also have $\widetilde{B}_{\mathrm{cris}}(V) = B_{\mathrm{cris}}(V)[1/\delta_1]$.

The action of G_K on $\overline{V}$ induces an action of G_K on S_V, $W(S_V)$, and $A_{\mathrm{cris}}(V)$ by functoriality. Let $\chi : G_K \to \mathbb{Z}_p^*$ denote the cyclotomic character. Then it follows from the construction that G_K acts on t by

$$g * t = \chi(g)t. \tag{D.1.3}$$

In particular, the action of G_K on $A_{\mathrm{cris}}(V)$ induces an action on $B_{\mathrm{cris}}(V)$. Also the choice of the elements τ_m defines a homomorphism

$$\rho : G_K \to \mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}. \tag{D.1.4}$$

If $g \in G_K$ then the image $\rho(g) = (\zeta_n)_{n\geq 0}$ in $\mathbb{Z}_p(1)$ is characterized by the equality

$$\rho(\tau_n) = \zeta_n \tau_n. \tag{D.1.5}$$

There is also a map

$$\alpha : \mathbb{Z}_p(1) \to A_{\mathrm{cris}}(V)^* \tag{D.1.6}$$

sending a sequence $(\zeta_n)_{n\geq 0}$ to the Teichmuller lifting $[\zeta]$ of the element $\zeta \in S_V$ defined by the reductions of the ζ_n. One verifies immediately from the construction that for $g \in G_K$ we have

$$g * \delta_1 = \alpha(\rho(g)) \cdot \delta_1. \tag{D.1.7}$$

In particular, the G_K-action on $B_{\mathrm{cris}}(V)$ induces an action of G_K on $\widetilde{B}_{\mathrm{cris}}(V)$.

Definition D.2. Let $A/\mathbb{Q}_p$ be a (possibly infinite dimensional) vector space with action of the group G_K (not necessarily continuous). We say that A is $\widetilde{B}_{\mathrm{cris}}(V)$-*admissible* if there exists a K_0-vector space M_0 and an isomorphism

$$A \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \simeq M_0 \otimes_{K_0} \widetilde{B}_{\mathrm{cris}}(V) \tag{D.2.1}$$

compatible with the Galois actions, where G_K acts on the left through the action on each factor and on the right with trivial action on M_0 and the natural action on $\widetilde{B}_{\mathrm{cris}}(V)$.

We say that A is *crystalline* if $A = \cup_i A_i$, where each $A_i \subset A$ is a finite dimensional subrepresentation, the action on each A_i is continuous (where A_i is given the usual p-adic topology), and A_i is crystalline (in the usual sense [Fo1, 5.1.4]).

The main result of this appendix is the following:

Theorem D.3. *Let $A/\mathbb{Q}_p$ be a $\widetilde{B}_{\mathrm{cris}}(V)$-admissible representation. Then A is crystalline.*

D.4. We begin the proof by recalling some facts about the ring B^+_{dR} and its field of fractions B_{dR}. Let $(B_n, J_n, [\,])$ be the divided power ring which is the reduction of $A_{\mathrm{cris}}(V)$ modulo p^n. Then we set

$$B^+_{\mathrm{dR}} := \varprojlim_r(\mathbb{Q} \otimes \varprojlim_n B_n/J_n^{[r]}). \tag{D.4.1}$$

There is a natural map $A_{\mathrm{cris}}(V) \otimes \mathbb{Q} \to B^+_{\mathrm{dR}}$, and one can show (see for example [Fo4, 1.5.2]) that B^+_{dR} is a complete discrete valuation ring and that the image of $t \in A_{\mathrm{cris}}(V)$ in B^+_{dR} is a uniformizer. The field B_{dR} is defined to be the field of fractions of B^+_{dR}. There is a natural inclusion $B_{\mathrm{cris}}(V) \hookrightarrow B_{\mathrm{dR}}$. Note that the action of G_K on $A_{\mathrm{cris}}(V)$ induces an action of G_K on B^+_{dR} and B_{dR}.

Proposition D.5. *Let d be an integer, and let $A \subset (B_{\mathrm{dR}})^d$ be a finite dimensional $\mathbb{Q}_p$-subspace that is stable under G_K. Then the restriction of the G_K-action to A defines a continuous action on A (where A is given the usual p-adic topology).*

Proof. First note that the action of G_K on A is continuous if and only if for some i the action of G_K on $A(i)$ is continuous, since $A \simeq A(i) \otimes \mathbb{Q}_p(-i)$ and the tensor product of two continuous representations is again continuous. After possible replacing A by $t^i \cdot A \simeq A(i)$ we may therefore assume that $A \subset (B^+_{\mathrm{dR}})^d$.

Since A is finite dimensional we get in this case an injection

$$A \hookrightarrow (\mathbb{Q} \otimes \varprojlim_n B_n/J_n^{[r]})^d \tag{D.5.1}$$

for some r. By [Ts1, A2.10] each $B_n/J_n^{[r]}$ is flat over $\mathbb{Z}/(p^n)$ and the natural map

$$B_{n+1}/J_{n+1}^{[r]} \otimes \mathbb{Z}/(p^n) \to B_n/J_n^{[r]} \tag{D.5.2}$$

is an isomorphism for all n. This also implies that $\varprojlim_n B_n/J_n^{[r]}$ is flat over $\mathbb{Z}_p$.

To ease notation we write C_n (resp. C) for the ring $B_n/J_n^{[r]}$ (resp. $\varprojlim B_n/J_n^{[r]}$) in what follows. We view C as a topological ring with the p-adic topology (so each C_n is given the discrete topology). The topology on C also defines a topology on $\mathbb{Q} \otimes C$.

Lemma D.6. *The map $\rho : G_K \times A \to A$ is continuous if and only for every $v \in A$ the map*

$$\rho_v : G_K \to A, \quad g \mapsto g(v) \tag{D.6.1}$$

is continuous.

Proof. The 'only if' direction is immediate as for every $v \in A$ there is a commutative diagram

$$\begin{tikzcd} G_K \times \{v\} \arrow[d, hook] \arrow[dr, "\rho_v"] & \\ G_K \times A \arrow[r, "\rho"] & A, \end{tikzcd} \tag{D.6.2}$$

and the inclusion $\{v\} \hookrightarrow A$ is the inclusion of a closed point.

For the 'if' direction choose a lattice

$$\Lambda = \mathbb{Z}_p e_1 + \cdots \mathbb{Z}_p e_r \subset A, \tag{D.6.3}$$

and for $i = 1, \ldots, r$ define

$$U_i := \rho_{e_i}^{-1}(\Lambda) \subset G_K. \tag{D.6.4}$$

By assumption each U_i is open in G_K, and therefore

$$U := U_1 \cap \cdots \cap U_r = \{g \in G_K | g(\Lambda) \subset \Lambda\} \tag{D.6.5}$$

is an open subgroup of G_K. Since G_K is compact it follows that there exists $g_1, \ldots, g_s \in G_K$ such that

$$G_K = \cup_{i=1}^s U \cdot g_i. \tag{D.6.6}$$

Let t be an integer such that

$$g_i(\Lambda) \subset \frac{1}{p^t}\Lambda \tag{D.6.7}$$

for $i = 1, \ldots, s$. Then we find that $g(p^t\Lambda) \subset \Lambda$ for all $g \in G_K$.

Now consider an open subset $v + p^k\Lambda \subset A$ for some $v \in \Lambda$, and let $(g, x) \in \rho^{-1}(v + p^k\Lambda)$. Denote by H the intersection $U \cap \rho_v^{-1}(v + p^k\Lambda)$ which is an open subset of G_K. Then for any $y \in p^{k+t}\Lambda$ and $u \in H$ we have

$$\rho_u\rho_g(x + y) = \rho_u(v + p^k\lambda) \in v + p^k\Lambda, \tag{D.6.8}$$

where $\lambda \in \Lambda$. We conclude that

$$H \cdot g \times \{x + p^{k+t}\Lambda\} \subset \rho^{-1}(v + p^k\Lambda). \tag{D.6.9}$$

□

Lemma D.7. *The p-adic topology on A agrees with the topology induced by the topology on $\mathbb{Q} \otimes C$.*

Proof. Let $\Lambda \subset A$ be a lattice whose image in $\mathbb{Q} \otimes C$ is contained in C. For every n let $\Psi_n \subset C_n$ be the image of Λ. Let K_n denote the kernel of the projection $\Lambda/p^n\Lambda \to \Psi_n$ so we have an exact sequence

$$0 \to K_n \to \Lambda/p^n\Lambda \to \Psi_n \to 0. \tag{D.7.1}$$

Passing to the limit (and using the fact that the kernels $\{K_n\}$ satisfy the Mittag-Leffler condition since $\Lambda/p^n\Lambda$ is an artinian module) we get an exact sequence

$$0 \to \varprojlim K_n \to \Lambda \to \varprojlim \Psi_n \to 0. \tag{D.7.2}$$

Moreover the composite map

$$\Lambda \to \varprojlim \Psi_n \to C = \varprojlim C_n \tag{D.7.3}$$

is injective, which implies that $\varprojlim K_n = 0$ and that $\Lambda \simeq \varprojlim \Psi_n$.

For every n let $U_n \subset \Lambda$ denote the kernel of the map $\Lambda \to \Psi_n$. Then the $\{U_n\}$ form a basis of open subsets around $0 \in A$ for the induced topology. Note also that we have $p^n\Lambda \subset U_n$.

To prove that the p-adic topology agrees with the induced topology, it therefore suffices to show that for any integer n there exists an integer k such that $U_{n+k} \subset p^n\Lambda$. For this note that since $\varprojlim K_n = 0$ there exists an integer k such that the image of K_{n+k} in K_n is zero. This implies that there exists a dotted arrow filling in the following diagram

(D.7.4)
$$\begin{tikzcd} \Lambda \arrow[d] \arrow[dr] & \\ \Lambda/p^{n+k}\Lambda \arrow[r, two heads] \arrow[d] & \Psi_{n+k} \arrow[dl, dashed] \\ \Lambda/p^n\Lambda, & \end{tikzcd}$$

and hence $U_{n+k} \subset p^n\Lambda$. □

Since A has the induced topology, to prove that for any $v \in A$ the map

(D.7.5)
$$\rho_v : G_K \to A, \quad g \mapsto g \cdot v$$

is continuous, it suffices to show that the composite map

(D.7.6)
$$G_K \xrightarrow{\rho_v} A \hookrightarrow \mathbb{Q} \otimes C$$

is continuous. In particular, this will follow if we show that for every $x \in \mathbb{Q} \otimes C$ the map

(D.7.7)
$$\rho_x : G_K \to \mathbb{Q} \otimes C, \quad g \mapsto g \cdot x$$

is continuous. For this in turn it suffices to show that for every $x \in C$ and integer n the subgroup

(D.7.8)
$$G_x(k) := \{g \in G_K | gx - x \in p^n C\} \subset G_K$$

is open. For this in turn it suffices to show that for any integer n and $x \in B_n/J_n^{[r]}$ the subgroup

(D.7.9)
$$H_x := \{g \in G_K | g(x) = x\}$$

is open. Let $f : W_n(S_V) \to B_n/J_n^{[r]}$ denote the natural map, and let $J_n \subset W_n(S_V)$ denote the image of $\mathrm{Ker}(\theta)$. Then any element $x \in B_n/J_n^{[r]}$ can be written as a finite sum of terms of the form $f(y)$ ($y \in W_n(S_V)$) and $f(y)^{[i]}$ ($y \in J_n$, $0 \leq i < r$). It therefore suffices to show that for $x = f(y)$ the subgroup $H_x \subset G_K$ is open. This can be seen as follows. Write

(D.7.10)
$$y = (a_0, \ldots, a_{n-1}) \in W_n(S_V)$$

with $a_i = (a_{im}) \in S_V$ (so we have $a_{im} \in \overline{V}/p\overline{V}$). Define $a_i^{1/p^i} \in S_V$ to be the element $(b_m) \in S_V$ with

(D.7.11)
$$b_m = a_{i,m+i}.$$

We have a commutative diagram

(D.7.12)
$$\begin{tikzcd} & B_n \arrow[d, "g"] \\ W_n(S_V) \arrow[ur, "f"] \arrow[r, "\theta_n"] & \overline{V}/p^n\overline{V}, \end{tikzcd}$$

where θ_n denotes the reduction of the map θ. Choose for $i = 0, \dots, n-1$ a lifting $\tilde{y}_i$ of $\theta_n([a_i^{1/p^i}])$. Then one shows as in [Ts1, A1.5] that we have

$$f(y) = \sum_{i=0}^{n-1} p^i \tilde{y}_i^{p^{n-i}}. \tag{D.7.13}$$

It therefore suffices to show that for any $e = (e_m) \in S_V$ and $u := \theta_n([e]) \in \overline{V}/p^n\overline{V}$ the subgroup

$$K_u := \{g \in G_K | g(u) = u\} \tag{D.7.14}$$

is open in G_K. For this note that by definition of the map θ we have

$$\theta_n([e]) = \tilde{e}_n^{p^n} \tag{D.7.15}$$

for any lifting $\tilde{e}_n \in \overline{V}/p^n\overline{V}$ of e_n. This therefore reduces the proof to showing that for any element $z \in \overline{V}/p^n\overline{V}$ the subgroup

$$\{g \in G_K | g(z) = z\} \subset G_K \tag{D.7.16}$$

is open which is immediate. This completes the proof of D.5. $\square$

Returning to the proof of D.3, let A be a $\widetilde{B}_{\mathrm{cris}}(V)$-admissible representation, and fix a K_0-space M_0 with a G_K-equivariant isomorphism

$$A \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \simeq M_0 \otimes_{K_0} \widetilde{B}_{\mathrm{cris}}(V). \tag{D.7.17}$$

Write $M_0 = \cup_i N_i$ where $N_i \subset M_0$ is a finite dimensional subspace, and set (intersection inside $A \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V)$)

$$A_i := A \cap (N_i \otimes_{K_0} \widetilde{B}_{\mathrm{cris}}(V)). \tag{D.7.18}$$

Then $A_i \subset A$ is a subrepresentation of G_K. We have a commutative diagram

$$\begin{tikzcd} A_i \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \arrow[r] \arrow[rr, bend left, "a"] & N_i \otimes_{K_0} \widetilde{B}_{\mathrm{cris}}(V) \arrow[r, hook] & A \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V), \end{tikzcd} \tag{D.7.19}$$

where the map a is an inclusion. It follows that the natural map

$$A_i \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \to N_i \otimes_{K_0} \widetilde{B}_{\mathrm{cris}}(V) \tag{D.7.20}$$

is an inclusion. Since N_i is finite dimensional, this implies that A_i is also finite dimensional. We conclude that $A = \cup_i A_i$, where $A_i \subset A$ is a finite dimensional subrepresentation. The composite map

$$A_i \hookrightarrow N_i \otimes_{K_0} \widetilde{B}_{\mathrm{cris}}(V) \hookrightarrow N_i \otimes_{K_0} B_{\mathrm{dR}} \simeq B_{\mathrm{dR}}^{\dim(N_i)} \tag{D.7.21}$$

identifies A_i with a finite dimensional $\mathbb{Q}_p$-subspace of $B_{\mathrm{dR}}^{\dim(N_i)}$ which is G_K-stable. It follows from this and D.5 that the action of G_K on A_i is continuous. Since $A = \cup_i A_i$ we have proven the following:

Corollary D.8. *Let A be a $\widetilde{B}_{\mathrm{cris}}(V)$-admissible G_K-representation. Then $A = \cup_i A_i$, where each $A_i \subset A$ is a continuous subrepresentation of finite dimension.*

For a $\mathbb{Q}_p$-representation A (possibly infinite dimensional) define

$$D_{\widetilde{\mathrm{cris}}}(A) := (A \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V))^{G_K}. \tag{D.8.1}$$

Then $D_{\widetilde{\mathrm{cris}}(A)}$ is a K_0-vector space.

Recall from [Ol3, 15.4 and 15.5] that the functor $D_{\widetilde{\mathrm{cris}}}$ has the following properties:

(i) For any finite dimensional continuous G_K-representation A the natural map

$$D_{\widetilde{\mathrm{cris}}}(A) \otimes_{K_0} \widetilde{B}_{\mathrm{cris}}(V) \to A \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \tag{D.8.2}$$

is injective.

(ii) For any finite dimensional continuous G_K-representation A the K_0-space $D_{\widetilde{\mathrm{cris}}}(A)$ is finite dimensional and

$$\dim_{\mathbb{Q}_p} A \geq \dim_{K_0} D_{\widetilde{\mathrm{cris}}}(A). \tag{D.8.3}$$

If A is an infinite dimensional G_K-representation which can be written as a union $A = \cup_i A_i$ of finite dimensional continuous subrepresentations, then it follows from (i) and the isomorphism

$$D_{\widetilde{\mathrm{cris}}}(A) \simeq \varinjlim_i D_{\widetilde{\mathrm{cris}}}(A_i) \tag{D.8.4}$$

that the natural map

$$D_{\widetilde{\mathrm{cris}}}(A) \otimes_{K_0} \widetilde{B}_{\mathrm{cris}}(V) \to A \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \tag{D.8.5}$$

is injective.

Lemma D.9. *Let A be a $\widetilde{B}_{\mathrm{cris}}(V)$-admissible G_K-representation. Then any subrepresentation and quotient representation of A is also $\widetilde{B}_{\mathrm{cris}}(V)$-admissible.*

Proof. Consider an exact sequence of G_K-representations

$$0 \to A' \to A \to A'' \to 0 \tag{D.9.1}$$

with A a $\widetilde{B}_{\mathrm{cris}}(V)$-admissible representation. It follows from D.8 that both A' and A'' are equal to the unions of their finite dimensional continuous subrepresentations. This implies that we have a commutative diagram

(D.9.2)

$$\begin{tikzcd} 0 \arrow[r] & D_{\widetilde{\mathrm{cris}}}(A') \otimes_{K_0} \widetilde{B}_{\mathrm{cris}}(V) \arrow[r] \arrow[d, hook] & D_{\widetilde{\mathrm{cris}}}(A) \otimes_{K_0} \widetilde{B}_{\mathrm{cris}}(V) \arrow[r] \arrow[d, "\simeq"] & D_{\widetilde{\mathrm{cris}}}(A'') \otimes_{K_0} \widetilde{B}_{\mathrm{cris}}(V) \arrow[d, hook] & \\ 0 \arrow[r] & A' \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \arrow[r] & A \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \arrow[r] & A'' \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \arrow[r] & 0, \end{tikzcd}$$

where all the vertical arrows are inclusions and the middle arrow is an isomorphism. A diagram chase then shows that all the vertical arrows in fact are isomorphisms. □

This now completes the proof of D.3. For if A is a $\widetilde{B}_{\mathrm{cris}}(V)$-admissible representation, we can by D.8 and D.9 write $A = \cup_i A_i$ where each $A_i \subset A$ is a continuous finite dimensional subrepresentation which is also $\widetilde{B}_{\mathrm{cris}}(V)$-admissible. The theorem now follows from [Ol3, 15.5] which shows that any finite dimensional continuous $\widetilde{B}_{\mathrm{cris}}(V)$-admissible representation is crystalline. □

Remark D.10. If A is a (possibly infinite dimensional) crystalline representation of G_K, and of M_0 is a K_0-vector space with a G_K-equivariant isomorphism

$$A \otimes_{\mathbb{Q}_p} \widetilde{B}_{\mathrm{cris}}(V) \simeq M_0 \otimes_{K_0} \widetilde{B}_{\mathrm{cris}}(V), \tag{D.10.1}$$

then by [Ol3, 15.3 and 15.7] we have $D_{\widetilde{\mathrm{cris}}}(A) = M_0$, the natural map $D_{\mathrm{cris}}(A) \to D_{\widetilde{\mathrm{cris}}}(A)$ is an isomorphism, and the isomorphism D.10.1 is induced by the isomorphism

$$A \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \simeq D_{\mathrm{cris}}(A) \otimes_{K_0} B_{\mathrm{cris}}(V) \simeq M_0 \otimes_{K_0} B_{\mathrm{cris}}(V). \tag{D.10.2}$$

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