TANGENT SPACES AND OBSTRUCTION THEORIES

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These are notes from my series of 8 lectures on tangent spaces and obstruction theories, which were part of a MSRI summer workshop on Deformation Theory and Moduli in Algebraic Geometry (July 23 – August 3, 2007). Nothing in these notes is original. For a list of references where this material (and much more!) can be found see the end of the notes.

These are the (slightly cleaned up) notes prepared in advance of the lectures. The actual content of the lectures may differ slightly (and in particular several of the examples included here were left out of the lectures due to time constraints).

Lecture 1. The ring of dual numbers.

1.1. **Motivation.** Let X be a scheme over a field k, and let $x \in X(k)$ be a point. The *tangent space* $T_X(x)$ of X at x is the dual of the k-vector space $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m} \subset \mathcal{O}_{X,x}$ is the maximal ideal. If X represents some functor

$$F : (\text{schemes})^{\text{op}} \to \text{Set},$$

then the point $x \in X(k)$ corresponds to an element of $F(\text{Spec}(k))$, and elements of the tangent space should correspond to infinitesimal deformations of x. At first approximation, the purpose of my lecture series is to understand from a functorial point of view this tangent space as well as the obstruction spaces that arise if X is singular at x.

1.2. **Dual numbers.** For a ring R, and an R-module I we define the *ring of dual numbers* $R[I]$ as follows. This ring is an R-algebra whose underlying R-module is $R \oplus I$. The algebra structure is given by the following rule:

$$(r, i) \cdot (r', i') = (rr', ri' + r'i), \quad r, r' \in R, \quad i, i' \in I.$$

There is a natural projection $\pi : R[I] \to R$ sending R to itself by the identity map, and I to zero. We therefore have a commutative diagram

$$\begin{array}{ccc} R[I] & \xrightarrow{\pi} & R \\ & \nwarrow & \uparrow \text{id} \\ & & R. \end{array}$$

Note that $R[I]$ is functorial in the pair (R, I). Namely, if $R \to R'$ is a morphism of rings, and $I \to I'$ is a morphism (of R-modules) from I to an R'-module I', then there is an induced morphism

$$R[I] \to R'[I'].$$

Remark 1.3. We will often consider the special case when $I = R$. In this case $R[I]$ will often be denoted $R[\epsilon]$ (it really should be $R[\epsilon]/(\epsilon^2)]$, but the ideal (ϵ^2) is usually omitted from the notation).

Remark 1.4. Note that the preceding discussion also makes sense for sheaves on a topological space. If X is a topological space, $\mathcal{O}$ a sheaf of rings on X, and I a sheaf of $\mathcal{O}$-modules on X then we obtain a sheaf of $\mathcal{O}$-algebras $\mathcal{O}[I]$ together with a surjection $\mathcal{O}[I] \rightarrow \mathcal{O}$.

In particular, if $X = (|X|, \mathcal{O}_X)$ is a scheme and I is a quasi-coherent $\mathcal{O}_X$-module, then we can consider the ringed space $X[I] := (|X|, \mathcal{O}_X[I])$.

Exercise 1.5. Show that $X[I]$ is a scheme.

Relationship with derivations.

Let $A \rightarrow R$ be a ring homomorphism, and let M be an R-module. Recall that an *A-derivation from R to M* is a homomorphism of A-modules

$$\partial : R \rightarrow M$$

such that for any $x, y \in R$ we have

$$\partial(xy) = x\partial(y) + y\partial(x).$$

Let $\mathrm{Der}_A(R, M)$ denote the R-module of A-derivations from R to M.

Let $A - \mathrm{Alg}/R$ denote the category of pairs (C, f), where C is an A-algebra and $f : C \rightarrow R$ is a morphism of A-algebras. A morphism $(C, f) \rightarrow (C', f')$ in $A - \mathrm{Alg}/R$ is an A-algebra homomorphism $g : C \rightarrow C'$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & C' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & R & \end{array}$$

commutes. Note in particular that for an R-module I, the projection $\pi : R[I] \rightarrow R$ defines an object $(R[I], \pi) \in A - \mathrm{Alg}/R$.

Remark 1.6. If $A = \mathbb{Z}$ we sometimes write simply Alg/R for $\mathbb{Z} - \mathrm{Alg}/R$.

Proposition 1.7. *For any A-derivation $\partial : R \rightarrow I$ the induced map*

$$R \rightarrow R[I], \quad x \mapsto x + \partial(x)$$

is a morphism in $A - \mathrm{Alg}/R$, and the induced map

$$\mathrm{Der}_A(R, I) \rightarrow \mathrm{Hom}_{A-\mathrm{Alg}/R}(R, R[I])$$

is an R-module isomorphism.

Note that in general, if $f : C \rightarrow R$ is an object of $A - \mathrm{Alg}/R$, and the kernel I of f is a square-zero ideal in C, then any section $s : R \rightarrow C$ of f identifies $f : C \rightarrow R$ with

$\pi : R[I] \to R$. Indeed, given the section s, define a map $\sigma : R[I] \to C$ by $\sigma(x+i) = s(x)+i$ ($x \in R$ and $i \in I$). The commutativity of the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & R[I] & \xrightarrow{\pi} & R & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \sigma} & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & C & \xrightarrow{f} & R & \longrightarrow & 0
\end{array}$$

shows that σ in fact is an isomorphism.

An important special case is when $C = R \otimes_A R/J^2$, where $J \subset R \otimes_A R$ is the kernel of the diagonal $R \otimes_A R \to R$. Let $I \subset C$ denote the ideal J/J^2. Then I is equal to the R-module of Kahler differentials $\Omega^1_{R/A}$. Let $s : R \to C$ be the section defined by the map $R \to R \otimes_A R$ sending $x \in R$ to $x \otimes 1$. This section gives an identification

$$R \otimes_A R/J^2 \simeq R[\Omega^1_{R/A}]$$

and therefore by 1.7 also identifies sections of the diagonal map $R \otimes_A R/J^2 \to R$ with $\mathrm{Der}_A(R, \Omega^1_{R/A})$.

Exercise 1.8. Show that under this identification, the universal derivation $d : R \to \Omega^1_{R/A}$ with the section of $R \otimes R/J^2 \to R$ given by sending x to $1 \otimes x$.

1.9. **The tangent space of a functor.** Let Mod_R denote the category of finitely generated R-modules. If

$$H : \mathrm{Mod}_R \to \mathrm{Set}$$

is a functor which commutes with finite products, then there is a canonical factorization

$$\begin{array}{ccccc}
\mathrm{Mod}_R & & \xrightarrow{\quad H \quad} & & \mathrm{Set} \\
 & {\scriptstyle \overline{H}}\searrow & & \nearrow{\scriptstyle \text{forget}} & \\
 & & \mathrm{Mod}_R & &
\end{array}$$

of H through a functor $\overline{H} : \mathrm{Mod}_R \to \mathrm{Mod}_R$. Indeed, if $I \in \mathrm{Mod}_R$ then the additive structure on $H(I)$ is given by the composite map

$$H(I) \times H(I) \simeq H(I \oplus I) \xrightarrow{+} H(I).$$

For $f \in R$ multiplication by f in $H(I)$ is given by the map

$$H(I) \xrightarrow{H(\times f)} H(I).$$

Exercise 1.10. Show that these maps define an R-module structure on $H(I)$.

Let $A \to R$ be a ring homomorphism. The category $A - \mathrm{Alg}/R$ has finite products. If $f : C \to R$ and $f' : C' \to R$ are two objects, then the product in $A - \mathrm{Alg}/R$ is given by the fiber product $C \times_R C'$ with the natural projection to R.

Lemma 1.11. *The functor*

$$\mathrm{Mod}_R \to A - \mathrm{Alg}/R, \quad I \mapsto R[I]$$

commutes with finite products.

Proof. This amounts to the claim that for $I, J \in \mathrm{Mod}_R$ the natural map

$$R[I \oplus J] \to R[I] \times_R R[J]$$

is an isomorphism, which is immediate. □

Corollary 1.12. *Suppose $F : A - \mathrm{Alg}/R \to \mathrm{Set}$ is a functor such that for $I, J \in \mathrm{Mod}_R$ the canonical map*

$$F(R[I] \times_R R[J]) \to F(R[I]) \times F(R[J])$$

is an isomorphism. Then for any $I \in \mathrm{Mod}_R$, the set $F(R[I])$ has a canonical R-module structure.

Definition 1.13. Let $F : A - \mathrm{Alg}/R \to \mathrm{Set}$ be a functor such that for $I, J \in \mathrm{Mod}_R$ the canonical map

$$F(R[I] \times_R R[J]) \to F(R[I]) \times F(R[J])$$

is an isomorphism. The *tangent space* of F, denoted T_F, is the R-module $F(R[\epsilon])$.

Remark 1.14. In the above we do not need that F is defined on the full category $A - \mathrm{Alg}/R$. If $\mathscr{C} \subset A - \mathrm{Alg}/R$ is a full subcategory closed under products containing the objects $R[I]$ for R-modules I, and if $F : \mathscr{C} \to \mathrm{Set}$ is a functor such that for all I, J the map

$$F(R[I] \times_R R[J]) \to F(R[I]) \times F(R[J])$$

is a bijection, then we can talk about the tangent space of F.

This will often be applied when R is a field k, and A is a complete noetherian local ring with residue field k. In this case we will frequently consider the category $\mathscr{C}_A$ of artinian local A-algebras with residue field k.

Exercise 1.15. Let k be a field, X/k a scheme, and $x \in X$ a point. Let $k \to R$ be the composite homomorphism $k \to \mathcal{O}_{X,x} \to k(x)$. Define

$$F : k - \mathrm{Alg}/k(x) \to \mathrm{Set}$$

to the functor sending a diagram $k \to C \to k(x)$ to the set of dotted arrows over $\mathrm{Spec}(k)$ filling in the following diagram:

$$\begin{array}{ccc} \mathrm{Spec}(k(x)) & \longrightarrow & \mathrm{Spec}(C) \\ \downarrow{\scriptstyle x} & \swarrow & \\ X. & & \end{array}$$

Show that the hypotheses of 1.13 are satisfied, and that the resulting k-vector space T_F is canonically isomorphic to the tangent space of X at x.

Lecture 2. Computation of tangent spaces, examples

2.1. **Deformations of smooth schemes.** Let R be a ring, and let $g : X \to \mathrm{Spec}(R)$ be a smooth separated morphism of schemes (the separatedness assumption is not necessary but included for expository reasons).

Let

$$\mathrm{Def}_X : \mathrm{Alg}/R \to \mathrm{Set}$$

be the functor which to any $(C, f : C \to R) \in \mathrm{Alg}/R$ associates the set of cartesian diagrams

(2.1.1)
$$\begin{array}{ccc} X & \longrightarrow & X' \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle g'} \\ \mathrm{Spec}(R) & \xrightarrow{f} & \mathrm{Spec}(C), \end{array}$$

where g' is smooth. We consider a second diagram

(2.1.2)
$$\begin{array}{ccc} X & \longrightarrow & X'' \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle g''} \\ \mathrm{Spec}(R) & \xrightarrow{f} & \mathrm{Spec}(C), \end{array}$$

equal to 2.1.1 if there exists a dotted arrow h filling in the following diagram

(2.1.3)
$$\begin{tikzcd} & & X'' \arrow[dl, dashed, "h"'] \arrow[ddl, "g''"] \\ X \arrow[rru] \arrow[r] \arrow[d, "g"'] & X' \arrow[d, "g'"'] & \\ \mathrm{Spec}(R) \arrow[r, "f"] & \mathrm{Spec}(C). & \end{tikzcd}$$

Proposition 2.2. *The functor* Def_X *satisfies the assumptions in 1.13.*

Proof. Done in Brian Osserman's lecture. □

We can compute the tangent space T_{Def_X} as follows.

In the case when X is affine, then we have the following facts. Let $f : C \to R$ be a surjection with nilpotent kernel J.

(i) $\mathrm{Def}_X(C, f)$ consists of one element.

(ii) Given two diagrams 2.1.1 and 2.1.2 the set of arrows h filling in 2.1.3 is a torsor under $H^0(X, T_X \otimes J)$, where T_X denotes the relative tangent bundle of $X \to \mathrm{Spec}(R)$.

The following exercise should also be noted somewhere:

Exercise 2.3. Let $j : X_0 \hookrightarrow X$ be a closed immersion of schemes defined by a nilpotent ideal. If X_0 is affine, then X is also affine.

We can use the affine case to study the general case as follows. Let I be an R-module, and let I_X denote $I \otimes_R \mathscr{O}_X$.

Lemma 2.4. *The morphism* $f : X[I_X] \to \mathrm{Spec}(R[I])$ *is smooth.*

Proof. This is clear as it is obtained by base change from X along the projection $\mathrm{Spec}(R[I]) \to \mathrm{Spec}(R)$. □

Now suppose given a diagram 2.1.1 with $C = R[I]$. Note that the topological spaces $|X|$ and $|X'|$ are equal, so if $\mathscr{U} = \{U_i\}$ is a covering of X by affine open subsets, then we also obtain a covering $X' = \cup_i U_i'$ of X' by affine open subsets. Note also that $U_i' \cap U_j'$ is the unique lifting to an open subset of X' of $U_i \cap U_j$.

By the affine case, there exists for each i an isomorphism

$$\sigma_i : U_i' \to U_i[I_{U_i}]$$

reducing to the identity modulo I. Fix a collection $\{\sigma_i\}$ of such isomorphisms. On the intersections $U_i \cap U_j$ we therefore get two maps

$$\sigma_i|_{U_{ij}}, \sigma_j|_{U_{ij}} : U_{ij}' \to U_{ij}[I_{U_{ij}}].$$

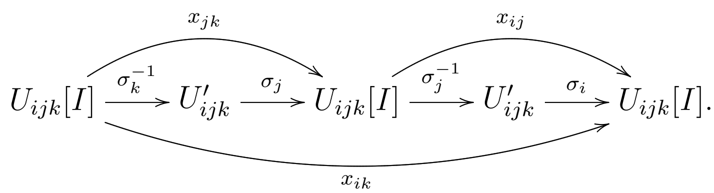
The difference $\sigma_i|_{U_{ij}} - \sigma_j|_{U_{ij}}$ of these two maps is by (ii) above given by an element $x_{ij} \in H^0(U_{ij}, T_X \otimes I)$.

Lemma 2.5. *For $i, j, k \in I$, we have*

$$x_{ik} = x_{ij} + x_{jk}$$

in $H^0(U_{ijk}, T_X \otimes I)$.

Proof. Consider the diagram



□

It follows that $\{x_{ij}\}$ define a Cech cocycle in $Z^1(\mathscr{U}, T_X \otimes I)$. Let $c(X') \in H^1(X, T_X \otimes I)$ denote the corresponding cohomology class.

Proposition 2.6. *Assume that X is separated so that the intersections U_{ij} are affine. Then the association*

$$X' \mapsto c(X') \in H^1(X, T_X \otimes I)$$

defines a bijection

$$\mathrm{Def}_X(R[I]) \to H^1(X, T_X \otimes I).$$

Exercise 2.7. Show that 2.6 still holds without the assumption that X is separated.

Exercise 2.8. Prove that the R-module structure on T_{Def_X} defined in 1.13 agrees with the standard R-module structure on $H^1(X, T_X)$ under the identification in 2.6.

2.9. **Deformations of nodes.** Let k be a field and let $A = k[[x, y]]/(xy)$ be the completion of the ring $k[x, y]/(xy)$ at the maximal ideal (x, y). Let

$$F : k - \mathrm{Alg}/k \to \mathrm{Set}$$

be the functor which associates to $k \to C \to k$ the set of isomorphism classes of pairs (A_C, ι), where A_C is a flat C-algebra and $\iota : A_C \otimes_C k \to A$ is an isomorphism of rings.

Then as above the conditions in 1.13 are met, so F has a tangent space T_F. We can calculate T_F as follows.

First note that there are some obvious deformations of A to $k[\epsilon]$. Namely, for any element $a \in k$ we can consider the deformation

$$A_a := k[\epsilon][[x,y]]/(xy - a\epsilon).$$

Lemma 2.10. *Let*

$$\begin{array}{ccc} k[\epsilon] & \longrightarrow & A' \\ \downarrow & & \downarrow \iota \\ k & \longrightarrow & A \end{array}$$

be an element of $F(k[\epsilon])$. Then there exists an element $a \in k$ such that $A' \simeq A_a$ (as a deformation of A). Moreover, the element a is unique.

Proof. Note that since A' is flat over $k[\epsilon]$, the kernel of ι is canonically isomorphic to $A \cdot \epsilon$.

Let $\tilde{x}, \tilde{y} \in A'$ be liftings of x and y respectively. Then we have

$$\tilde{x} \cdot \tilde{y} = (a + \sum_{i \geq 1} \alpha_i x^i + \sum_{j \geq 1} \beta_j y^j) \cdot \epsilon.$$

Let $\tilde{y}'$ denote $\tilde{y} + (\sum_{i \geq 1} \alpha_i x^{i-1})\epsilon$ and $\tilde{x}'$ denote $\tilde{x} + (\sum_{j \geq 1} \beta_j y^{j-1})\epsilon$. Then

$$\tilde{y}' \cdot \tilde{x}' = a\epsilon.$$

We therefore obtain a map

$$A_a \to A'$$

over A, which must be an isomorphism since both are flat over $k[\epsilon]$.

To see that a is unique, suppose $f : A_a \to A_{a'}$ is an isomorphism over A. Write

$$f(x) = x + \alpha\epsilon, \quad f(y) = y + \beta\epsilon, \quad \alpha, \beta \in A.$$

Then we find that

$$a\epsilon = f(x) \cdot f(y) = (x + \alpha\epsilon) \cdot (y + \beta\epsilon) = a' + (x\beta + y\alpha) \cdot \epsilon,$$

which implies that $a = a'$. $\square$

We therefore have a bijection

$$k \to T_F, \quad a \mapsto A_a.$$

Let us verify that this is compatible with the vector space structure.

For the additive structure, note that the sum of A_a and A_b in T_F is given by the image of (A_a, A_b) under the map

$$F(k[\epsilon]) \times F(k[\epsilon]) \xrightarrow{\simeq} F(k[\epsilon_1, \epsilon_2]) \xrightarrow{\epsilon_i \mapsto \epsilon} F(k[\epsilon]).$$

This image is given by the algebra

$$(A_a + A_b) := (A_a \times_A A_b) \otimes_{k[\epsilon] \times_k k[\epsilon], \Delta} k[\epsilon].$$

The two elements $(x,x), (y,y) \in A_a \times_A A_b$ satisfy

$$(x,x) \cdot (y,y) = (a\epsilon, b\epsilon),$$

and therefore define a map $A_{a+b} \to (A_a + A_b)$ over A. Again by flatness this map must be an isomorphism.

The statement about the module structure follows from noting that if $f \in k$ then the map

$$F(\times f) : F(k[\epsilon]) \to F(k[\epsilon])$$

sends A_a to

$$A_a \otimes_{k[\epsilon], \epsilon \mapsto f\epsilon} k[\epsilon] \simeq A_{af}.$$

Remark 2.11. This example is generalized in one of the exercises where it is shown that the line $k[[a]]$ defines a hull for the functor of deformations of the node.

Lecture 3. Obstruction theories

Before stating the general setup, let us start with an example.

Let $\pi : A' \to A$ be a surjection of rings with kernel J a square-zero ideal. Let $f : X \to \mathrm{Spec}(A)$ be a smooth separated morphism.

Problem 3.1. *Find a smooth lifting $f' : X' \to \mathrm{Spec}(A')$ of f.*

Fix a covering $\mathscr{U} = \{U_i\}_{i \in I}$ of X by affines. For each $i \in I$ choose a smooth lifting $U_i' \to \mathrm{Spec}(A')$ of U_i. Choose for each $i, j \in I$ an isomorphism

$$\varphi_{ji} : U_i'|_{U_{ij}} \to U_j'|_{U_{ij}}.$$

We would like these to give gluing data for a lifting X' of X. For this we need the morphisms on the overlaps

$$\varphi_{ki}, \varphi_{jk} \circ \varphi_{ji} : U_i'|_{U_{ijk}} \to U_k'|_{U_{ijk}}$$

to be equal. Let ∂_{ijk} denote the automorphism of $U_i'|_{U_{ijk}}$ given by

$$\varphi_{ki}^{-1} \circ (\varphi_{jk} \circ \varphi_{ji}).$$

This automorphism reduces to the identity on U_{ijk} and therefore corresponds to an element of $T_{X/A} \otimes J$, which we again denote by ∂_{ijk}.

Lemma 3.2. *(i) $\{\partial_{ijk}\} \in Z^2(\mathscr{U}, T_{X/A} \otimes J)$.*

(ii) If φ_{ji}' is a second choice of isomorphisms with corresponding $\{\partial_{ijk}'\} \in Z^2(\mathscr{U}, T_{X/A} \otimes J)$ then $\{\partial_{ijk}\} - \{\partial_{ijk}'\} \in B^2(\mathscr{U}, T_{X/A} \otimes J)$.

Proof. Exercise. □

Let $o(f) \in H^2(X, T_{X/A} \otimes J)$ denote the corresponding cohomology class.

Proposition 3.3. *There exists a lifting $X' \to \mathrm{Spec}(A')$ of X if and only if $o(f) = 0$.*

Proof. The class $o(f) = 0$ if and only if there exists infinitesimal automorphisms σ_{ij} of $U_i'|_{U_{ij}}$ such that if we replace φ_{ji} by $\varphi_{ji} \circ \sigma_{ij}$ then $\partial_{ijk} = 0$. □

Exercise 3.4. Show that $o(f) \in H^2(X, T_{X/A} \otimes J)$ is independent of the choice of $\mathscr{U}$.

Summary 3.5. *(i) There is a canonical obstruction $o(f) \in H^2(X, T_{X/A} \otimes J)$ such that $o(f) = 0$ if and only if there exists a lifting of X to A'.*

(ii) If $o(f) = 0$, then the set of isomorphism classes of liftings of X to A' is a torsor under $H^1(X, T_{X/A} \otimes J)$.

(iii) For any lifting X'/A' of X to A, the group of automorphisms of X' reducing to the identity over A is canonically isomorphic to $H^0(X, T_{X/A} \otimes J)$.

This is the general pattern, as we will see when we discuss the cotangent complex.

Let Λ be a ring, and consider a functor

$$F : \Lambda - \mathrm{Alg} \longrightarrow \mathrm{Set}.$$

We will often consider the following data which we will refer to as a *deformation situation*:

(i) A diagram in $\Lambda - \mathrm{Alg}$

$$A' \xrightarrow{p} A \xrightarrow{q} A_0,$$

where A_0 is reduced, p and q are surjections with nilpotent kernels, and the kernel J of p is annihilated by $\mathrm{Ker}(A' \to A_0$ (so in particular $J^2 = 0$ and J can be viewed as an A_0-module). We will also assume that J is a finite type A_0-module.

(ii) An element $a \in F(A)$. We denote the image of a in $F(A_0)$ by a_0.

Definition 3.6. An *obstruction theory* for F consists of the following data:

(i) For every morphism $A \to A_0$ of Λ-algebras with kernel a nilpotent ideal and A_0 reduced, and element $a \in F(A)$ a functor

$$\mathcal{O}_a : (\text{finite type } A_0\text{-modules}) \to (\text{finite type } A_0\text{-modules}).$$

(ii) For every deformation situation $A' \to A \to A_0$ and $a \in F(A)$ an element $o(a) \in \mathcal{O}_a(\mathrm{Ker}(A' \to A))$ which is zero if and only if a lifts to $F(A')$.

This data is further required to be functorial in the following sense. For a commutative diagram of rings

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ A_0 & \xrightarrow{f_0} & B_0, \end{array}$$

where A_0 and B_0 are reduced, p and q are surjective, and $\mathrm{Ker}(p)$ and $\mathrm{Ker}(q)$ are nilpotent, and $a \in F(A)$ a morphism of functors from the category of finite type A_0-modules to finite type B_0-modules

$$\alpha : (- \otimes_{A_0} B_0) \circ \mathcal{O}_a(-) \to \mathcal{O}_{f_*(a)}(-) \circ (- \otimes_{A_0} B_0).$$

This morphism of functors is required to be compatible with composition and for a morphism of deformation situations

$$\begin{array}{ccccc} A' & \xrightarrow{\sigma} & A & \longrightarrow & A_0 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} \\ B' & \xrightarrow{\tau} & B & \longrightarrow & B_0 \end{array}$$

with $J = \mathrm{Ker}(\sigma)$ and $I = \mathrm{Ker}(\tau)$ square-zero ideals, and for $a \in F(A)$ we have

$$\alpha(o_a(A')) = o_{f_*(a)}(B').$$

Remark 3.7. In most examples the functor $\mathcal{O}_a(-)$ depends only on a_0 and not the particular lifting a.

Remark 3.8. Often one considers certain subcategories of $\Lambda-\mathrm{Alg}$ (for example the category of finite type Λ-algebras). One sometimes may consider obstruction theories defined on such smaller categories.

Example 3.9. Let $X \hookrightarrow X'$ be a closed immersion defined by a square-zero ideal J. Let L be a line bundle on X. We then wish to understand the deformations of L to X'.

For this consider the exponential sequence

$$0 \to 1 + J \to \mathscr{O}_{X'}^* \to \mathscr{O}_X^* \to 0,$$

and the associated long exact sequence of cohomology groups

$$0 \to H^0(J) \to H^0(\mathscr{O}_{X'}^*) \to H^0(\mathscr{O}_X^*) \to H^1(J) \to \mathrm{Pic}(X') \to \mathrm{Pic}(X) \to H^2(J).$$

Proposition 3.10. *Assume that the map $H^0(X', \mathscr{O}_{X'}) \to H^0(X, \mathscr{O}_X)$ is surjective. Then the following hold:*

(i) There is a canonical obstruction $o(L) \in H^2(X, J)$ whose vanishing is necessary and sufficient for there to exists a lifting of L to X'.

(ii) If $o(L) = 0$, then the set of isomorphism classes of liftings of L is a torsor under $H^1(X, J)$.

(iii) The group of automorphisms of any lifting L' is canonically isomorphic to $H^0(X, J)$.

Example 3.11. Let $S = \mathrm{Spec}(k)$ be the spectrum of a field, and let $(A, e)/k$ be an abelian variety. Consider the functor

$$(\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$$

assigning to any scheme $T \to S$ the set of isomorphism classes of pairs (L, ι), where L is a line bundle on A_T and $\iota : e^*L \to \mathscr{O}_T$ is an isomorphism of line bundles. This functor factors through the category of abelian groups, with the group structure given by tensor product. One can show that this functor is representable by an abelian variety A^t/S called the *dual abelian variety.* The above discussion shows that there is a canonical isomorphism

$$H^1(A, \mathscr{O}_A) \simeq \mathfrak{t}^t,$$

where $\mathfrak{t}^t$ denotes the tangent space of A^t at the origin.

Example 3.12. The above discussion of line bundles ($= \mathbb{G}_m$-torsors) can be generalized as follows. Let $X \hookrightarrow X'$ be as above, and let $G' \to X'$ be a smooth group scheme with reduction $G \to X$. Let $P \to X$ be a G-torsor. We then wish to understand the deformations of P to X'.

For this let $\mathscr{G}_P$ denote the group scheme of automorphisms of P, and let $\mathrm{Lie}(\mathscr{G}_P)$ denote the Lie algebra of $\mathscr{G}_P$. For example, if $G = GL_n$ then P corresponds to a vector bundle E of rank n and $\mathrm{Lie}(\mathscr{G}_P) = \mathscr{E}nd(E)$.

Étale locally on X there exists a lifting of P to X' and any two such liftings are locally isomorphic. We can then apply the same argument we used for deformations of smooth schemes, and one finds the following:

Proposition 3.13. *(i) There is a canonical obstruction $o(P) \in H^2(X, \mathrm{Lie}(\mathscr{G}_P) \otimes J)$ whose vanishing is necessary and sufficient for there to exists a lifting of P to X'.*

(ii) If $o(P) = 0$, then the set of isomorphism classes of liftings of P is a torsor under $H^1(X, \mathrm{Lie}(\mathscr{G}_P) \otimes J)$.

(iii) The group of automorphisms of any lifting P' is canonically isomorphic to $H^0(X, \mathrm{Lie}(\mathscr{G}_P) \otimes J)$.

Lecture 4. More examples

Let $A' \to A$ be a surjection of rings with kernel J a square-zero ideal. Let $P' \to \mathrm{Spec}(A')$ be a smooth morphism with reduction $P \to \mathrm{Spec}(A)$, and let $j : X \hookrightarrow P$ be a closed immersion with $X \to \mathrm{Spec}(A)$ smooth.

Problem 4.1. *Find $j' : X' \hookrightarrow P'$ lifting j with $X' \to \mathrm{Spec}(A')$ smooth.*

To solve this problem, let $\mathcal{L}$ denote the presheaf of sets on the topological space $|X|$ which to any open subset $U \subset X$ associates the set of diagrams

(4.1.1)
$$\begin{array}{ccc} U' & \hookleftarrow & U \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\ P' & \hookleftarrow & P \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \hookleftarrow & \mathrm{Spec}(A) \end{array}$$

with j' an immersion and $U' \to \mathrm{Spec}(A')$ smooth.

Lemma 4.2. *$\mathcal{L}$ is a sheaf.*

Let $\mathcal{N}$ denote the normal bundle of X in P. By definition $\mathcal{N} = I_X/I_X^2$, where I_X denotes the ideal sheaf of X in P. From the exact sequence

$$0 \longrightarrow I_X/I_X^2 \xrightarrow{d} j^*\Omega^1_{P/A} \longrightarrow \Omega^1_{X/A} \longrightarrow 0,$$

we get an exact sequence

$$0 \to T_{X/A} \to j^*T_{P/A} \to \mathcal{N} \to 0.$$

There is an action of $j^*T_{P/A} \otimes J$ on $\mathcal{L}$ defined as follows. Given a commutative diagram 4.1.1 and a section $\partial \in j^*T_{P/A} \otimes J(U)$ we obtain a new element of $\mathcal{L}(U)$

$$\begin{array}{ccc} U' & \hookleftarrow & U \\ \downarrow{\scriptstyle \partial * j'} & & \downarrow{\scriptstyle j} \\ P' & \hookleftarrow & P \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \hookleftarrow & \mathrm{Spec}(A) \end{array}$$

where $\partial * j'$ denotes the map obtained from the fact that the set of dotted arrows filling in the diagram

$$\begin{array}{ccc} U' & \hookleftarrow & U \\ \vdots & & \downarrow{\scriptstyle j} \\ P' & \hookleftarrow & P \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \hookleftarrow & \mathrm{Spec}(A) \end{array}$$

are a torsor under $j^*T_{P/A} \otimes J(U)$.

Lemma 4.3. *The action of $j^*T_{P/A} \otimes J$ on $\mathcal{L}$ descends to a torsorial action of $\mathcal{N} \otimes J$ on $\mathcal{L}$.*

Proof. It suffices to consider the case when U is affine in which case U' is unique and the only issue is the choice of j'. □

Summary 4.4. *(i) There exists a canonical obstruction $o(j) \in H^1(X, \mathcal{N} \otimes_A J)$ with $o(j) = 0$ if and only if there exists a lifting j' of j.*

(ii) If $o(j) = 0$ then the set of liftings of j form a torsor under $H^0(X, \mathcal{N} \otimes_A J)$.

Remark 4.5. The sequence

$$0 \to T_{X/A} \to j^*T_{P/A} \to \mathcal{N} \to 0$$

induces a long exact sequence

$$\begin{array}{l} H^0(X, \mathcal{N} \otimes J) \longrightarrow H^1(X, T_{X/A} \otimes J) \longrightarrow H^1(X, j^*T_{P/A} \otimes J) \\ \longrightarrow H^1(X, \mathcal{N} \otimes J) \xrightarrow{\delta} H^2(X, T_{X/A} \otimes J). \end{array}$$

Exercise 4.6. Show that $\delta(o(j))$ is equal to the obstruction to finding a smooth lifting of X, and of $o(j) = 0$ then the map

$$H^0(X, \mathcal{N} \otimes J) \to H^1(X, T_{X/A} \otimes J)$$

is identified with the map

$$[j' : X' \hookrightarrow P'] \mapsto [X'].$$

Example 4.7. Let P be a surface over a field k, and let $X \subset P$ be a smooth rational curve with $X.X = -1$. Then by [Hartshorne, V.1.4.1] we have $\deg \mathcal{N} = -1$. Therefore

$$H^1(X, \mathcal{N} \otimes J) = 0, \quad \text{and} \quad H^0(X, \mathcal{N} \otimes J) = 0.$$

It follows that X can be uniquely deformed.

Example 4.8. Let $(A' \to A, J)$ be as above, and assume in addition that A is artinian local with residue field k, and that J is a k-vector space. Let $(X, e)/\mathrm{Spec}(A)$ be an abelian scheme with reduction (X_0, e) to k.. Assume further that $2 \in A^*$ (this assumption is not necessary as you show in an exercise). I claim that there exists a lifting (X', e') of (X, e) to an abelian scheme over A'.

For this note that this is equivalent to the existence of a lifting of X to $\mathrm{Spec}(A')$, and therefore we need to show that the obstruction $o(X) \in H^2(X_0, T_{X_0} \otimes J)$ is zero.

Let $\iota : X \to X$ be the map $x \mapsto -x$ (scheme-valued points). Since the formation of the obstruction $o(X)$ is invariant under automorphisms of X we have that

$$\iota^* o(X) = o(X).$$

On the other hand, the following lemma shows that $\iota^* o(X) = -o(X)$, and therefore $o(X) = 0$ (since $2 \in A^*$).

Proposition 4.9. *The map* $\iota^* : H^2(X_0, T_{X_0} \otimes J) \to H^2(X_0, T_{X_0} \otimes J)$ *is equal to multiplication by* -1.

Proof. This follows from the following observations:

(a) Since X_0 is a group scheme, there is a canonical isomorphism $\mathscr{O}_{X_0} \otimes \mathfrak{t}$, where $\mathfrak{t}$ denotes the tangent space of X_0 at the origin.

(b) There is a canonical isomorphism $\mathfrak{t}^t \simeq H^1(X_0, \mathscr{O}_{X_0})$, where $\mathfrak{t}^t$ denotes the tangent space at the origin of the dual abelian variety X_0^t. This follows from our earlier discussion of deformations of line bundles.

(c) The canonical map $H^1(X_0, \mathscr{O}_{X_0}) \wedge H^1(X_0, \mathscr{O}_{X_0}) \to H^2(X_0, \mathscr{O}_{X_0})$ is an isomorphism. This is a fact from the theory of abelian varieties.

Therefore

$$H^2(X_0, T_{X_0} \otimes J) \simeq (\mathfrak{t}^t \wedge \mathfrak{t}^t) \otimes \mathfrak{t} \otimes J.$$

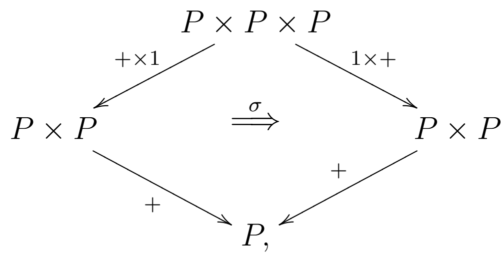
From the above observations one finds that ι^* acts by multiplication by -1 on the first three terms in the above expression. □

Lecture 5. Picard stacks

A *Picard category* is a groupoid P with the following extra structure:

(a) A functor $+ : P \times P \to P$.

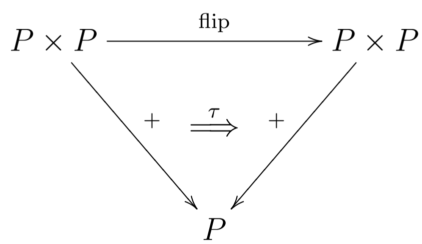
(b) A natural transformation of functors σ



which we write as

$$\sigma_{x,y,z} : (x + y) + z \to x + (y + z).$$

(c) A natural transformation of functors τ



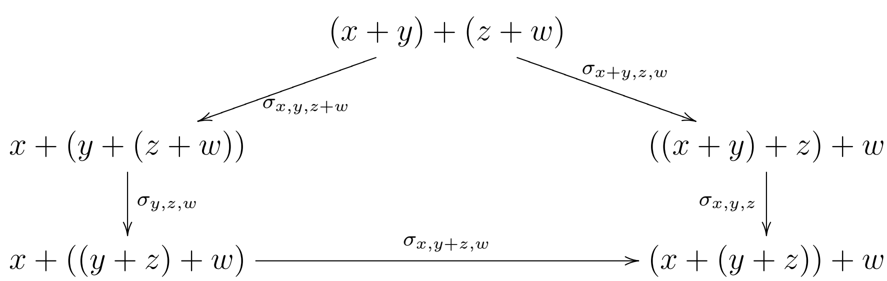
which we write as

$$\tau_{x,y} : x + y \to y + x.$$

This data is required to satisfy the following:

(0) For every $x \in P$ the functor $P \to P$ sending y to $x + y$ is an equivalence.

(i) (Pentagon Axiom) For objects $x, y, z, w \in P$ the diagram



commutes.

(ii) $\tau_{x,x} = \mathrm{id}$ for every $x \in P$.

(iii) For $x, y \in P$ we have $\tau_{x,y} \circ \tau_{y,x} = \mathrm{id}_{y+x}$.

(iv) (Hexagon Axiom) The diagram

$$\begin{array}{ccc} x+(y+z) & \xrightarrow{\tau} & x+(y+z) \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ (x+y)+z & & (x+z)+y \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ z+(x+y) & \xrightarrow{\sigma} & (z+x)+y. \end{array}$$

Example 5.1. Let X be a scheme, and let $\mathscr{P}ic(X)$ denote the category of line bundles on X. Then $\mathscr{P}ic(X)$ is a Picard category with $+$ given by $\otimes$.

Example 5.2. Let $f : X \to Y$ be a morphism of schemes, and let I be a quasi-coherent $\mathcal{O}_X$-module. An *I-extension of X over Y* is a commutative diagram

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & X' \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle f'} & \\ Y, & & \end{array}$$

with j a square-zero closed immersion, together with an isomorphism

$$\sigma : I \to \mathrm{Ker}(\mathcal{O}_{X'} \to \mathcal{O}_X).$$

Let $\underline{\mathrm{Exal}}_Y(X, I)$ denote the category of I-extensions of X over Y.

It is sometimes useful to view the category $\underline{\mathrm{Exal}}_Y(X, I)$ as the category of diagrams of sheaves of rings on $|X|$

$$\begin{array}{ccc} \mathcal{O}_{X'} & \xrightarrow{\pi} & \mathcal{O}_X \\ \uparrow & \nearrow & \\ \mathcal{O}_Y, & & \end{array}$$

where π is a surjection morphism with square zero-kernel, together with an isomorphism $J \simeq \mathrm{Ker}(\pi)$. Note that this description in particular implies that if $U \subset X$ is an open subset then there is a restriction functor

$$\underline{\mathrm{Exal}}_Y(X, I) \to \underline{\mathrm{Exal}}_Y(U, I).$$

Let $u : I \to J$ be a morphism of $\mathcal{O}_X$-modules, and let

$$\begin{array}{ccc} \mathcal{O}_{X'} & \xrightarrow{\pi} & \mathcal{O}_X \\ \uparrow & \nearrow & \\ \mathcal{O}_Y, & & \end{array}$$

be an object of $\underline{\mathrm{Exal}}_Y(X, I)$. Then

$$\mathcal{O}_{X'_u} := \mathcal{O}_{X'} \oplus_I J = (\mathcal{O}_{X'} \oplus J)/\{(i, -i) | i \in I\}$$

is an $\mathcal{O}_{X'}$-algebra giving

$$\begin{array}{ccc} & & X'_u \\ & \overset{J}{\nearrow} & \downarrow \\ X & \xrightarrow{I} & X. \end{array}$$

In this way we get a functor

$$u_* : \underline{\mathrm{Exal}}_Y(X, I) \to \underline{\mathrm{Exal}}_Y(X, J).$$

Lemma 5.3. *If I and J are two quasi-coherent $\mathcal{O}_X$-modules, then*

$$(\mathrm{pr}_{1*}, \mathrm{pr}_{2*}) : \underline{\mathrm{Exal}}_Y(X, I \oplus J) \to \underline{\mathrm{Exal}}_Y(X, I) \times \underline{\mathrm{Exal}}_Y(X, J)$$

is an equivalence of categories.

Proof. Exercise. □

Let $\Sigma : I \oplus I \to I$ be the summation map. The composite

$$\begin{array}{c} \underline{\mathrm{Exal}}_Y(X, I) \times \underline{\mathrm{Exal}}_Y(X, I) \\ \downarrow \simeq \\ \underline{\mathrm{Exal}}_Y(X, I) \\ \downarrow \Sigma_* \\ \underline{\mathrm{Exal}}_Y(X, I) \end{array}$$

then gives $\underline{\mathrm{Exal}}_Y(X, I)$ the structure of a Picard stack.

Example 5.4. Let $f : A \to B$ be a homomorphism of abelian groups. Define P_f to be the category whose objects are elements of B, and for which a morphism $x \to y$ is given by a section $h \in A$ such that $f(h) = y - x$. Then P_f is a Picard category.

Let T be a topological space (or site). A *Picard (pre)-stack over T* is a (pre)-stack in groupoids $\mathscr{P}$ with a functor

$$+ : \mathscr{P} \times \mathscr{P} \to \mathscr{P}$$

and isomorphisms of functors σ and τ such that for every $U \subset T$ the category $\mathscr{P}(U)$ with the restrictions of the functors is a Picard category.

Remark 5.5. Examples 5.1 and 5.2 give Picard stacks, and 5.4 gives a Picard prestack.

Let T be a topological space. For Picard stacks $\mathscr{P}_1, \mathscr{P}_2$ over T a *morphism of Picard stacks* $\mathscr{P}_1 \to \mathscr{P}_2$ is a morphism of stacks $F : \mathscr{P}_1 \to \mathscr{P}_2$ together with an isomorphism of functors

$$\iota : F(x + y) \to F(x) + F(y)$$

such that the following two diagrams commute:

$$\begin{array}{ccc} F(x+y) & \xrightarrow{\iota} & F(x) + F(y) \\ \downarrow F(\tau) & & \\ F(y+x) & \xrightarrow{\iota} & F(y) + F(x), \end{array}$$

and

$$\begin{array}{ccccc} F((x+y)+z) & \xrightarrow{\iota} & F(x+y)+F(z) & \xrightarrow{\iota} & (F(x)+F(y))+F(z) \\ \downarrow{\scriptstyle F(\sigma)} & & & & \downarrow{\scriptstyle \sigma} \\ F(x+(y+z)) & \xrightarrow{\iota} & F(x)+F(y+z) & \xrightarrow{\iota} & F(x)+(F(y)+F(z)). \end{array}$$

If $F_1, F_2 : \mathscr{P}_1 \to \mathscr{P}_2$ are two morphisms of Picard stacks then a morphism of functors $u : F_1 \to F_2$ is a natural transformation of functors (necessarily an isomorphism) such that the diagram

$$\begin{array}{ccc} F_1(x+y) & \xrightarrow{u} & F_2(x+y) \\ \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} \\ F_1(x)+F_1(y) & \xrightarrow{u_x+u_y} & F_2(x)+F_2(y). \end{array}$$

In this way the collection of Picard stacks over T form a 2-category (though the details of the theory of 2-categories will not be worked out here). The main point is that we have a category

$$\mathrm{HOM}(\mathscr{P}_1, \mathscr{P}_2)$$

instead of a set of morphisms.

Exercise 5.6. Let $\underline{\mathrm{HOM}}(\mathscr{P}_1, \mathscr{P}_2)$ denote the stack over T defined as follows: For $U \subset T$ the fiber $\underline{\mathrm{HOM}}(\mathscr{P}_1, \mathscr{P}_2)(U)$ is the groupoid of morphisms of Picard stacks $\mathscr{P}_{1,U} \to \mathscr{P}_{2,U}$ (where $\mathscr{P}_{i,U}$ denotes the restriction of $\mathscr{P}_i$ to U). Define a sum functor

$$+ : \underline{\mathrm{HOM}}(\mathscr{P}_1, \mathscr{P}_2) \times \underline{\mathrm{HOM}}(\mathscr{P}_1, \mathscr{P}_2) \to \underline{\mathrm{HOM}}(\mathscr{P}_1, \mathscr{P}_2)$$

by the formula

$$(F_1 + F_2)(x) = F_1(x) + F_2(x).$$

The structural isomorphism $\iota : (F_1 + F_2)(x) \to F_1(x) + F_2(x)$ is defined to be the unique isomorphism making the following diagram commute:

$$\begin{array}{ccc} (F_1+F_2)(x+y) & \xrightarrow{\qquad\qquad \iota \qquad\qquad} & (F_1+F_2)(x)+(F_1+F_2)(y) \\ \| & & \| \\ F_1(x+y)+F_2(x+y) & & (F_1(x)+F_2(x))+(F_1(y)+F_2(y)) \\ & \searrow^{\iota_1+\iota_2} \quad \swarrow_{\tau} & \\ & (F_1(x)+F_1(y))+(F_2(x)+F_2(y)). & \end{array}$$

Define appropriate isomorphisms of functors σ and τ for $\underline{\mathrm{HOM}}(\mathscr{P}_1, \mathscr{P}_2)$ such that $\underline{\mathrm{HOM}}(\mathscr{P}_1, \mathscr{P}_2)$ becomes a Picard stack.

Exercise 5.7. (Sheafification). Let $\mathscr{P}$ be a Picard prestack over a topological space T. Then there exists a morphism $\pi : \mathscr{P} \to \mathscr{P}^a$ from $\mathscr{P}$ to a Picard stack $\mathscr{P}^a$ which is universal in the sense that for any Picard stack $\mathscr{Q}$ over T the canonical functor (composition with π)

$$\underline{\mathrm{HOM}}(\mathscr{P}^a, \mathscr{Q}) \to \underline{\mathrm{HOM}}(\mathscr{P}, \mathscr{Q})$$

is an equivalence of categories.

Exercise 5.8. (Identity element) Let $\mathscr{P}$ be a Picard stack over a topological space T. An *identity element* for $\mathscr{P}$ is a pair (e, φ), where $e \in \mathscr{P}(T)$ and $\varphi : e + e \to e$ is an isomorphism in $\mathscr{P}(T)$.

(a) Show that an identity element (e, φ) exists and is unique up to unique isomorphism.

(b) If (e, φ) is an identity element there exists a unique isomorphism of functors α from

$$e + (-) : \mathscr{P} \to \mathscr{P}$$

to the identity functor, such that the diagram of functors

$$\begin{array}{ccc} e + (e + (-)) & \xrightarrow{\sigma} & (e + e) + (-) \\ & {\scriptstyle \mathrm{id}+\alpha} \searrow \quad \swarrow {\scriptstyle \varphi} & \\ & e + (-) & \end{array}$$

commutes.

Exercise 5.9. (Kernels) Let $F : \mathscr{P}_1 \to \mathscr{P}_2$ be a morphism of Picard stacks over a topological space T. Define the *kernel* of F, denoted $\mathrm{Ker}(F)$, to be the stack over T which to any $U \subset T$ associated the groupoid of pairs (p, ι), where $p \in \mathscr{P}_1(U)$ and $\iota : 0_2 \to F(p)$ is an isomorphism in $\mathscr{P}_2(U)$. There is an additive structure on $\mathrm{Ker}(F)$ for which the sum of (p, ι) and (p', ι') is given by $p + p'$ with the isomorphism

$$0_2 \xrightarrow{\iota+\iota'} F(p) + F(p') \xrightarrow{\simeq} F(p + p').$$

Show that with these definitions the stack $\mathrm{Ker}(F)$ is a Picard stack.

Lecture 6. Structure theorem for Picard stacks

For a two-term complex of abelian groups $K^{\cdot} \in C^{[-1,0]}(T)$ let $\mathrm{pch}(K^{\cdot})$ denote the Picard prestack defined in the last lecture, and let $\mathrm{ch}(K^{\cdot})$ denote the associated Picard stack 5.7.

Note that if $f : K_1^{\cdot} \to K_2^{\cdot}$ is a morphism of complexes then f inducese a morphism of Picard prestacks $F : \mathrm{pch}(K_1) \to \mathrm{pch}(K_2)$, and hence also a morphism of Picard stacks $F : \mathrm{ch}(K_1) \to \mathrm{ch}(K_2)$. Moreover, if $f_1, f_2 : K_1^{\cdot} \to K_2^{\cdot}$ is a morphism of complexes with associated morphisms of stacks F_1 and F_2, then a homotopy $h : f_1 \to f_2$ then h induces an isomorphism of morphisms $\mathrm{ch}(h) : F_1 \to F_2$.

Lemma 6.1. *If K^{-1} is flasque, then $\mathrm{pch}(K^{\cdot})$ is a Picard stack.*

Proof. It suffices to show that the projection $\pi : \mathrm{pch}(K^{\cdot}) \to \mathrm{ch}(K^{\cdot})$ is an equivalence.

Let $U \subset T$ be an open subset and let $x \in \mathrm{ch}(K^{\cdot})$ be an object. Consider the sheaf $\mathcal{L}$ on U which to any open $V \subset U$ associates the set of pairs (l, z), where $l \in K^0(V)$ and $z : \pi(l) \to x|_V$ is an isomorphism in $\mathrm{ch}(K^{\cdot})$. Then $\mathcal{L}$ is a torsor under $K^{-1}|_U$, and since $H^1(U, K^{-1}) = 0$ since K^{-1} is flasque it follows that the functor $\mathrm{pch}(K^{\cdot}) \to \mathrm{ch}(K^{\cdot})$ is essentially surjective and therefore an equivalence. □

For a complex $K^{\cdot} \in C^{[-1,0]}(T)$, we have the following:

(a) The sheaf associated to the presheaf sending $U \subset T$ to the set of isomorphism classes of objects in $\mathrm{ch}(K)(U)$ is canonically isomorphic to $\mathscr{H}^0(K)$.

(b) For any object of $\mathrm{ch}(K)(U)$ the sheaf of automorphisms is canonically isomorphic to $\mathscr{H}^{-1}(K)|_U$.

Corollary 6.2. *A morphism of complexes $f : K_1 \to K_2$ induces an equivalence $\mathrm{ch}(K_1) \to \mathrm{ch}(K_2)$ if and only if f is a quasi-isomorphism.*

Let $\widetilde{C}^{[-1,0]}(T) \subset C^{[-1,0]}(T)$ denote the full subcategory of two-term complexes $K^{\cdot}$ with K^{-1} injective.

Theorem 6.3. *The 2-functor* ch *induces an equivalence of 2-categories*

$$\mathrm{ch} : \widetilde{C}^{[-1,0]}(T) \to (\textit{Picard stacks over } T).$$

In particular, ch *induces an (ordinary) equivalence of categories between $D^{[-1,0]}(T)$ and the category whose objects are Picard stacks over T and whose morphisms are isomorphism classes of morphisms of Picard stacks.*

Lemma 6.4. *Let $\mathscr{P}$ be a Picard stack over T, and let $\{U_i\}_{i\in I}$ be a collection of open subsets of T. Suppose given for each $i \in I$ an object $k_i \in \mathscr{P}(U_i)$ and set $K := \oplus_{i\in I}\mathbb{Z}_{U_i}$. Then there exists a morphism of Picard stacks*

$$F : \mathrm{ch}(0 \to K) \to \mathscr{P}$$

and isomorphisms $\sigma_i : F(1 \in \mathbb{Z}_{U_i}(U_i)) \simeq k_i$. Moreover, the data (F, σ_i) is unique up to unique isomorphism.

Proof. Left as exercise. □

Lemma 6.5. *Let $\mathscr{P}$ be a Picard stack over T. Then there exists a complex $K^{\cdot} \in C^{[-1,0]}(T)$ and an isomorphism $\mathrm{ch}(K^{\cdot}) \to \mathscr{P}$.*

Proof. Choose data

(a) A collection of open subsets $\{U_i \subset T\}_{i\in I}$.

(b) For each $i \in I$ an object $k_i \in \mathscr{P}(U_i)$.

in such a way that for every open $V \subset T$ and object $k \in \mathscr{P}(V)$ there exists a covering $V = \cup_j V_j$ such that for every j there exists an $i \in I$ with $V_j \subset U_i$ and $k|_{V_j}$ isomorphic to $k_i|_{V_j}$.

Define $K^0 := \oplus_{i\in I}\mathbb{Z}_{U_i}$. By 6.4 there exists a unique morphism of Picard stacks

$$F : \mathrm{pch}(0 \to K^0) = \mathrm{ch}(0 \to K^0) \to \mathscr{P}.$$

Now define K^{-1} to be the sheaf which to any open subset $V \subset T$ associates the set of pairs (x, l) where $x \in K^0(V)$ and $l : F(0) \to F(x)$ is an isomorphism in $\mathscr{P}(V)$, and let $K^{-1} \to K^0$ be the map sending (x, l) to x.

We define an abelian group structure on K^{-1} as follows. Given $(x, l), (x', l') \in K^{-1}$ define their sum to be the element $x + x' \in K^0$ together with the isomorphism

$$F(0) \xrightarrow{\simeq} F(0) + F(0) \xrightarrow{l+l'} x + x'.$$

One verifies then that this defines an abelian group structure on K^{-1} and there is a natural morphism

$$\mathrm{pch}(K^{\cdot}) \to \mathscr{P}$$

and therefore we get a morphism of Picard stacks

$$\mathrm{ch}(K^{\cdot}) \to \mathscr{P}.$$

We claim that this map is an equivalence. This is clear, because the map $\mathrm{pch}(K^{\cdot}) \to \mathscr{P}$ is clearly fully faithful and every object is locally in the image. □

Lemma 6.6. *Let $K, L \in C^{[-1,0]}(T)$ be two complexes and let $F : \mathrm{ch}(K) \to \mathrm{ch}(L)$ be a morphism of Picard stacks. Then there exists a quasi-isomorphism $k : K' \to K$ and a morphism $l : K' \to L$ such that F is isomorphic $\mathrm{ch}(l)\mathrm{ch}(k)^{-1}$. In particular, if $K \in \widetilde{C}^{[-1,0]}(T)$ then any morphism of Picard stacks $F : \mathrm{ch}(K) \to \mathrm{ch}(L)$ is isomorphic to a morphism of the form $\mathrm{ch}(f)$ for a morphism of complexes $f : K \to L$.*

Proof. Choose a collection of data $\{(U_i, k_i, l_i, \sigma_i)\}_{i \in I}$ such that:

(a) $U_i \subset T$ is an open subset;

(b) $k_i \in K^0(U_i)$ and $l_i \in \mathbb{L}^0(U_i)$ and $\sigma : F(k_i) \to l_i$ is an isomorphism in $\mathrm{ch}(L)(U_i)$;

(c) The map $K'^0 := \oplus_{i \in I} \mathbb{Z}_{U_i} \to K^0$ is surjective.

Now define K'^{-1} to be the fiber product $K^{-1} \times_{K^0} K'^0$, and let

$$k : K' \to K$$

be the natural quasi-isomorphism. We also have a map

$$l : K' \to L$$

to be the map which in degree 0 sends $(U_i, k_i, l_i, \sigma_i)$ to l_i and which sends $(v, (U_i, k_i, l_i, \sigma_i)) \in K^{-1}$ to the unique element $t \in L^{-1}$ such that the diagram

$$\begin{array}{ccc} F(0) & \xrightarrow{F(v)} & F(k_i) \\ \downarrow{\simeq} & & \downarrow{\sigma_i} \\ 0 & \xrightarrow{t} & l_i \end{array}$$

commutes. The maps σ_i define an isomorphism of functors $F \simeq \mathrm{ch}(l)\mathrm{ch}(k)^{-1}$.

The last statement follows from noting that if $K \in \widetilde{C}^{[-1,0]}(T)$ then there exists a morphism $s : K \to K'$ such that the composite $K \to K' \to K$ is homotopic to the identity. □

Lemma 6.7. *Let $K_1, K_2 \in \widetilde{C}^{[-1,0]}(T)$. For two morphisms of complexes $f_1, f_2 : K_1 \to K_2$ with associated morphisms of Picard stacks $F_1, F_2 : \mathrm{ch}(K_1) \to \mathrm{ch}(K_2)$, and any isomorphism $H : F_1 \to F_2$ there exists a unique homotopy $h : K_1^0 \to K_2^{-1}$ such that $u = \mathrm{ch}(h)$.*

Proof. The morphism h is defined by sending a local section $k \in K_1^0$ to the unique section $v \in K_2^{-1}$ such that $dv = f_1(k) - f_2(k)$, and v is the morphism corresponding to H. □

Lecture 7. The truncated cotangent complex

Definition 7.1. Let $f : X \to S$ be a morphism of schemes. Define the *truncated tangent complex* $\tau_{\leq 1}\mathbb{T}_{X/S} \in D^{[0,1]}(X)$ to be the complex defined by

$$\operatorname{ch}(\tau_{\leq 1}\mathbb{T}_{X/S}[1]) = \underline{\operatorname{Exal}}_S(X, \mathscr{O}_X).$$

This is only a preliminary definition for two reasons:

(a) This is only the truncated complex. The full cotangent complex will be discussed in the next lecture.

(b) This definition does not include the $\mathscr{O}_X$-module structure on $\underline{\operatorname{Exal}}_S(X, \mathscr{O}_X)$. It would be very interesting to have a notion of $\mathscr{O}_X$-linear Picard stack and generalize the structure theorem of last lecture to get an equivalence of categories between $D^{[-1,0]}(\mathscr{O}_X)$ and a suitable category of $\mathscr{O}_X$-linear Picard stacks.

Both problems will be taken care of by Illusie's cotangent complex. However, in the meantime we can understand $\tau_{\leq 1}\mathbb{T}_{X/S}$ in some special cases.

Proposition 7.2. *Let $j : X \to S$ be a closed immersion. Then $\tau_{\leq 1}\mathbb{T}_{X/S}$ is isomorphic to $\mathscr{N}_{X/S}[-1]$, where $\mathscr{N}_{X/S} := \mathscr{H}om(j^*I, \mathscr{O}_X)$, where I is the ideal of X in S.*

Proof. The category $\underline{\operatorname{Exal}}_S(X, \mathscr{O}_X)$ classifies diagrams

$$\begin{array}{ccccc} \mathscr{O}_X & \overset{\pi}{\twoheadleftarrow} & \mathscr{O}_{X'} & \hookleftarrow & \mathscr{O}_X \cdot \epsilon \\ & \nwarrow & \uparrow & & \uparrow \partial \\ & & j^{-1}\mathscr{O}_S/I^2 & \hookleftarrow & j^*I. \end{array}$$

This implies that $\mathscr{O}_{X'} = j^{-1}\mathscr{O}_S/I^2 \oplus_{j^*I,\partial} \mathscr{O}_X$. □

Proposition 7.3. *Let $f : X \to S$ be a smooth morphism. Then $\tau_{\leq 1}\mathbb{T}_{X/S} \simeq T_{X/S}$.*

Proof. Let $K \in \widetilde{C}^{[-1,0]}(|X|)$ be the complex with $\operatorname{ch}(K) \simeq \underline{\operatorname{Exal}}_S(X, \mathscr{O}_X)$. Since any two objects are locally isomorphic, we have $\mathscr{H}^1(K) = 0$ so K is quasi-isomorphic to $H^0(K)$. But $H^0(K)$ is equal to the sheaf of automorphisms of $X[\epsilon]$ over S reducing to the identity on X. This is we have already encountered as $T_{X/S}$. □

Proposition 7.4. *Suppose given a diagram*

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & P \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & \\ S, & & \end{array}$$

where g is smooth and j is an immersion. Then there is a canonical isomorphism

$$\tau_{\leq 1}\mathbb{T}_{X/S} \simeq (j^*T_{P/S} \to \mathscr{N}_{X/S}),$$

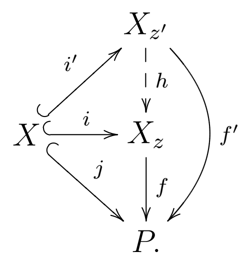
where $j^*T_{X/S} \to \mathscr{N}_{X/S}$ is the dual of the map $d : j^*I \to j^*\Omega^1_{P/S}$.

Proof. Given a section $z : j^*I \to \mathscr{O}_X$ of $\mathscr{N}_{X/S}$, we obtain an object of $\underline{\mathrm{Exal}}_S(X, \mathscr{O}_X)$ by setting $\mathscr{O}_{X'}$ equal to the pushout of the diagram

$$\begin{array}{ccc} j^*I & \xrightarrow{\ z\ } & \mathscr{O}_X \cdot \epsilon \\ \big\downarrow & & \\ j^{-1}\mathscr{O}_P/I^2 & & \end{array}$$

with the natural map to $\mathscr{O}_X$. Let $X \hookrightarrow X_z$ denote this $\mathscr{O}_X$-extension of X over S.

Consider two sections $z, z' \in \mathscr{N}_{X/S}$ defining a diagram of solid arrows over S



Since f is a closed immersion, the morphism h is determined by the composite $f \circ h$, and this composite is in turn specified by a section $\partial \in j^*T_{P/S}$. One verifies that the condition that $f \circ h$ factors through X_z is precisely the condition that the image of ∂ in $\mathscr{N}_{X/S}$ is equal to $z - z'$.

We therefore obtain a morphism of Picard prestacks

$$\mathrm{pch}(j^*T_{P/S} \to \mathscr{N}_{X/S}) \to \underline{\mathrm{Exal}}_S(X, \mathscr{O}_X),$$

and hence a morphism of Picard stacks

$$\mathrm{ch}(j^*T_{P/S} \to \mathscr{N}_{X/S}) \to \underline{\mathrm{Exal}}_S(X, \mathscr{O}_X).$$

This morphism of Picard stacks is an equivalence as it is fully faithful and locally essentially surjective. $\square$

Definition 7.5. The *truncated cotangent complex* $\tau_{\geq -1}L_{X/S}$ is the complex $j^*I \to j^*\Omega^1_{P/S}$ so we have

$$\underline{\mathrm{Exal}}_S(X, \mathscr{O}_X) \simeq \mathrm{ch}(\tau_{\leq 1}\mathscr{R}\mathscr{H}om(\tau_{\geq -1}L_{X/S}, \mathscr{O}_X)[1]).$$

In fact the above arguments show more. For any $\mathscr{O}_X$-module I we obtain a canonical isomorphism of Picard stacks

$$\mathrm{ch}(\tau_{\leq 1}\mathscr{R}\mathscr{H}om(\tau_{\geq -1}L_{X/S}, I)[1]) \to \underline{\mathrm{Exal}}_S(X, I)$$

functorial in I. This can even be extended to 2-term complexes as follows. For a morphism of $\mathscr{O}_X$-modules $F \to J$ with F injective we get a complex $C^\cdot$

$$\mathscr{H}om(j^*\Omega^1_{P/S}, F) \to \mathscr{H}om(j^*\Omega^1_{P/S}, J) \times_{\mathscr{H}om(j^*I, J)} \mathscr{H}om(j^*I, F).$$

The complex $C^\cdot$ represents

$$\tau_{\leq 1}\mathscr{R}\mathscr{H}om(j^*I \to j^*\Omega^1_{P/S}, F \to J)[1].$$

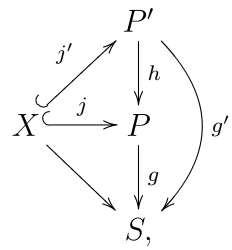
Let $\underline{\mathrm{Exal}}_S(X, F \to J)$ denote the kernel of the natural map

$$\underline{\mathrm{Exal}}_S(X, F) \to \underline{\mathrm{Exal}}_S(X, J).$$

Then we obtain from the above and your exercise that

$$\underline{\mathrm{Exal}}_S(X, F \to J) \simeq \mathrm{ch}(\tau_{\leq 1}\mathscr{R}\mathscr{H}om(j^*I \to j^*\Omega^1_{P/S}, F \to J)[1]).$$

The complex $j^*I \to j^*\Omega^1_{P/S}$ is independent of the choice of $j : X \hookrightarrow P$ in the following sense. Suppose given a commutative diagram



where g and g' are smooth. Then there is an induced morphism of complexes

$$\varphi : (j^*I \to j^*\Omega^1_{P/S}) \to (j'^*I' \to j^*\Omega^1_{P'/S})$$

which we claim is a quasi-isomorphism. Let $C^\cdot$ (resp. $C'^\cdot$) denote the complex $(j^*I \to j^*\Omega^1_{P/S})$ (resp. $(j'^*I' \to j^*\Omega^1_{P'/S})$). We then obtain a natural transformation of functors $\varphi : h'_C \to h_C$, where h_C is the composite

$$D^{[-1,0]}(\mathscr{O}_X) \xrightarrow{\tau_{\leq 1}\mathscr{R}\mathscr{H}om(C,-)[1]} D^{[-1,0]}(\mathscr{O}_X) \xrightarrow{H^1} (\text{Groups})$$

and $h_{C'}$ is defined similarly. By the above this map φ is an isomorphism of functors, and therefore by Yoneda's lemma the map $C^\cdot \to C'^\cdot$ is a quasi-isomorphism.

Of course for a general morphism $f : X \to S$ there need not exist a factorization through a smooth morphism. This can be remedied as follows. For a sheaf of sets F on $|X|$, let $\mathscr{O}_X\{F\}$ denote the free algebra on the sheaf F. The functor $F \mapsto \mathscr{O}_X\{F\}$ is left adjoint to the forgetful functor

$$(\mathscr{O}_X - \text{algebras}) \to (\text{sheaves of sets}).$$

As a substitute for the existence of an embedding into a smooth scheme, we can choose any map from a sheaf of sets F to $\mathscr{O}_X$ such that the resulting map

$$f^{-1}\mathscr{O}_S\{F\} \to \mathscr{O}_X$$

is surjective (for example we can take $F = \mathscr{O}_X$). We can then define a complex

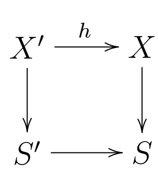
(7.5.1) $$I/I^2 \to \Omega^1_{f^{-1}\mathscr{O}_S\{F\}/f^{-1}\mathscr{O}_S},$$

where I is the kernel of $f^{-1}\mathscr{O}_S \to \mathscr{O}_X$.

Then one shows as above that this is independent of the choices.

Definition 7.6 (Good definition)**.** The *truncated cotangent complex* of $f : X \to S$, denoted $\tau_{\geq -1}L_{X/S}$, is the complex 7.5.1 obtained by taking $F = \mathscr{O}_X$.

Remark 7.7. Note that for a commutative square



there is a natural map $h^*\tau_{\geq -1}L_{X/S} \to \tau_{\geq -1}L_{X'/S'}$.

Remark 7.8. Note that the cohomology sheaves of $\tau_{\geq -1}L_{X/S}$ are quasi-coherent and coherent if f is of finite type and S is locally noetherian.

Exercise 7.9. Let S be a scheme and $f : X \to Y$ an étale morphism of S-schemes. Show that $f^*\tau_{\geq -1}L_{Y/S} \to \tau_{\geq -1}L_{X/S}$ is a quasi-isomorphism.

Lecture 8. The cotangent complex, an overview

In this lecture I summarize some of the basic features of Illusie's cotangent complex and its applications to deformation theory. It should be emphasized, however, that the theory of cotangent complex is not just important in the study of deformation theory but is a basic structure associated to any morphism of schemes.

Illusie's construction gives the following:

(i) For every morphism of schemes $f : X \to Y$ a complex $L_{X/Y} \in C^{\leq 0}(\mathscr{O}_X)$ of flat $\mathscr{O}_X$-modules with quasi-coherent cohomology sheaves. If Y is locally noetherian and f is locally of finite type then the cohomology sheaves of $L_{X/Y}$ are coherent. Note that $L_{X/Y}$ is a complex and not just an object of the derived category.

(ii) For a commutative diagram

(8.0.1)
$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{v} & Y \end{array}$$

there is a base change morphism

$$u^*L_{X/Y} \to L_{X'/Y'}.$$

If 8.0.1 is cartesian and either f or v is flat (or more generally the square is tor-independent) then the base change morphism is a quasi-isomorphism. Moreover, in this case the sum map

$$f'^*L_{Y'/Y} \oplus u^*L_{X/Y} \to L_{X'/Y}$$

is a quasi-isomorphism.

(iii) For a composite

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

there is a distinguished triangle in $D^-_{\text{qcoh}}(\mathscr{O}_X)$

$$f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to f^*L_{Y/Z}[1],$$

which is functorial in the natural sense.

(iv) The truncation $\tau_{\geq 1}L_{X/Y}$ is equal to our earlier defined truncated cotangent complex. In particular $\mathscr{H}_0(L_{X/Y}) = \Omega^1_{X/Y}$.

Remark 8.1. In a few special cases one can compute the cotangent complex:

(a) There is always a canonical isomorphism $H_0(L_{X/Y}) \simeq \Omega^1_{X/Y}$.

(b) If $X \to Y$ is smooth then $L_{X/Y} = \Omega^1_{X/Y}$.

(c) If $X \hookrightarrow Y$ is a closed immersion which is a local complete intersection then $L_{X/Y} = I/I^2[1]$, where I is the ideal of X in Y.

Theorem 8.2 (Illusie)**.** *For any quasi-coherent sheaf I on X there is a canonical isomorphism of Picard stacks*

$$\mathrm{ch}(\tau_{\geq -1}(\mathscr{R}\mathscr{H}om(L_{X/Y}, I)[1])) \simeq \underline{\mathrm{Exal}}_Y(X, I).$$

Combining this with the above properties we can use the cotangent complex to study almost every deformation theory problem.

Problem 8.3. *Suppose given the diagram of schemes indicated by the solid arrows*

$$\begin{array}{ccc} X_0 & \overset{i}{\dashrightarrow} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ Y_0 & \overset{j}{\hookrightarrow} & Y \\ \downarrow & \swarrow & \\ S, & & \end{array}$$

where j is a closed immersion defined by a square zero ideal J. Fill in the diagram as indicated with i a square-zero closed immersion such that the induced map

$$f_0^*J \to \mathrm{Ker}(\mathscr{O}_X \to \mathscr{O}_{X_0})$$

is an isomorphism.

We need to find an element of $\mathrm{Exal}_Y(X, f_0^*I)$ whose image in $\mathrm{Hom}_{\mathscr{O}_{X_0}}(f_0^*J, f_0^*J)$ is the identity. Consider the distinguished triangle

$$f_0^*L_{Y_0/Y} \to L_{X_0/Y} \to L_{X_0/Y_0} \to f_0^*L_{Y_0/Y}[1].$$

From this we get a long exact sequence

$$0 \longrightarrow \mathrm{Ext}^0(L_{X_0/Y_0}, f_0^*J) \longrightarrow \mathrm{Ext}^0(L_{X_0/Y}, f_0^*J) \longrightarrow \mathrm{Ext}^0(f_0^*L_{Y_0/Y}, f_0^*J) \longrightarrow \mathrm{Ext}^1(L_{X_0/Y_0}, f_0^*J)$$
$$\longrightarrow \mathrm{Ext}^1(L_{X_0/Y}, f_0^*J) \longrightarrow \mathrm{Ext}^1(f_0^*L_{Y_0/Y}, f_0^*J) \xrightarrow{\partial} \mathrm{Ext}^2(L_{X_0/Y_0}, f_0^*J)$$

which can also be written as

$$0 \longrightarrow \mathrm{Ext}^0(L_{X_0/Y_0}, f_0^*J) \longrightarrow \mathrm{Ext}^0(L_{X_0/Y}, f_0^*J) \longrightarrow 0 \longrightarrow \mathrm{Ext}^1(L_{X_0/Y_0}, f_0^*J)$$
$$\longrightarrow \mathrm{Exal}_Y(X_0, f_0^*J) \longrightarrow \mathrm{Hom}(f_0^*J, f_0^*J) \xrightarrow{\partial} \mathrm{Ext}^2(L_{X_0/Y_0}, f_0^*J).$$

We conclude

Theorem 8.4. *(i) There exists an obstruction*

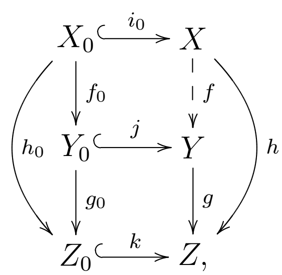
$$o(f_0) := \partial(\mathrm{id}) \in \mathrm{Ext}^2(L_{X_0/Y_0}, f_0^*J)$$

whose vanishing is necessary and sufficient for a solution to the problem.

(ii) If $o(f_0) = 0$, then the set of isomorphism classes of solutions is a torsor under $\mathrm{Ext}^1(L_{X_0/Y_0}, f_0^* J)$.

(iii) For any solution, the group of automorphisms is canonically isomorphic to $\mathrm{Ext}^0(L_{X_0/Y_0}, f_0^* J)$.

Problem 8.5. *Suppose given a commutative diagram of solid arrows*



where i (resp. j, k) is a closed immersion defined by a square-zero ideal I (resp. J, K).

Find an arrow f filling in the diagram.

Theorem 8.6. *There is a canonical class $o \in \mathrm{Ext}^1(f_0^* L_{Y_0/Z_0}, I)$ whose vanishing is necessary and sufficient for the existence of a morphism $f : X \to Y$ filling in the diagram. If $o = 0$ then the set of such maps f is a torsor under the group $\mathrm{Ext}^0(f_0^* L_{Y_0/Z_0}, I)$.*

Proof. Consider the classes

$$e(X) \in \mathrm{Ext}^1_{X_0}(L_{X_0/Z}, I), \quad e(Y) \in \mathrm{Ext}^1_{Y_0}(L_{Y_0/Z}, J)$$

defined by the extensions. Consider the images z_X and z_Y of these classes under the maps

$$\mathrm{Ext}^1_{X_0}(L_{X_0/Z}, I) \to \mathrm{Ext}^1_{X_0}(f_0^* L_{Y_0/Z}, I),$$

and

$$\mathrm{Ext}^1_{Y_0}(L_{Y_0/Z}, J) \to \mathrm{Ext}^1_{X_0}(f_0^* L_{Y_0/Z}, J) \to \mathrm{Ext}^1_{X_0}(f_0^* L_{Y_0/Z}, I).$$

Then one sees that there exists a morphism if and only if $z_X = z_Y$.

Now consider the distinguished triangle

$$h_0^* L_{Z_0/Z} \to f_0^* L_{Y_0/Z} \to f_0^* L_{Y_0/Z_0} \to h_0^* L_{Z_0/Z}[1].$$

This induces an exact sequence

$$\begin{array}{ccccccc}
\mathrm{Ext}^0(h_0^* L_{Z_0/Z}, I) & \longrightarrow & \mathrm{Ext}^1(f_0^* L_{Y_0/Z_0}, I) & \longrightarrow & \mathrm{Ext}^1(f_0^* L_{Y_0/Z}, I) & \longrightarrow & \mathrm{Ext}^1(h_0^* L_{Z_0/Z}, I) \\
\| & & & & & & \| \\
0 & & & & & & \mathrm{Hom}(h_0^* K, I).
\end{array}$$

It follows from the construction that $z_X - z_Y$ maps to zero in $\mathrm{Hom}(h_0^* K, I)$, and therefore the class $z_X - z_Y$ gives an element $o \in \mathrm{Ext}^1(f_0^* L_{Y_0/Z_0}, I)$.

The second statement follows from the isomorphism $H_0(L_{Y_0/Z_0}) \simeq \Omega^1_{Y_0/Z_0}$. □

Remark 8.7. The above results can often be used to show that the tangent space of moduli functors is finite-dimensional.

Remark 8.8. There is also a theory of cotangent complexes for morphisms of algebraic stacks. Using this and appropriate generalizations of the above results (in particular 8.4) one sees that Artin's theorem is in fact an "if and only if".

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