ON (LOG) TWISTED CURVES

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Abstract. We describe an equivalence between the notion of balanced twisted curve introduced by Abramovich and Vistoli, and a new notion of log twisted curve, which is a nodal curve equipped with some logarithmic data in the sense of Fontaine and Illusie. As applications of this equivalence, we construct a universal balanced twisted curve, prove that a balanced twisted curve over a general base scheme admits étale locally on the base a finite flat cover by a scheme, and also give a new construction of the moduli space of stable maps into a Deligne–Mumford stack and a new proof that it is bounded.

1. Introduction

This work is motivated by the construction in ([2]) of a moduli space for stable maps into a Deligne–Mumford stack, and forthcoming work of Abramovich, Graber, and Vistoli on these moduli spaces.

1.1. Let S be a scheme and $\mathcal{C}/S$ a proper flat tame Deligne–Mumford stack $\mathcal{C} \to S$ whose fibers are purely 1–dimensional and geometrically connected with at most nodal singularities (recall that $\mathcal{C}$ is *tame* if for every algebraically closed field k and morphism $x : \mathrm{Spec}(k) \to \mathcal{C}$ the stabilizer group $\mathrm{Stab}_{\mathcal{C}}(x)(k)$ has order invertible in k). Let $\mathcal{C} \to C$ be the coarse moduli space of $\mathcal{C}$, and let $C^{\mathrm{sm}} \subset C$ be the open subset where $C \to S$ is smooth. Assume that the inverse image $\mathcal{C} \times_C C^{\mathrm{sm}} \subset \mathcal{C}$ is equal to the open substack of $\mathcal{C}$ where $\mathcal{C} \to S$ is smooth and that for every geometric point $\bar{s} \to S$ the map $\mathcal{C}_{\bar{s}} \to C_{\bar{s}}$ is an isomorphism over some dense open subset of $C_{\bar{s}}$. Then the coarse space C is a nodal curve over S, and as reviewed in (2.2) below, for any geometric point mapping to a node $\bar{s} \to C$ there exists an étale neighborhood $\mathrm{Spec}(A) \to C$ of $\bar{s}$ and an étale morphism

$$\mathrm{Spec}(A) \to \mathrm{Spec}_S(\mathcal{O}_S[x, y]/xy - t) \tag{1.1.1}$$

for some $t \in \mathcal{O}_S$, such that the pullback $\mathcal{C} \times_C \mathrm{Spec}(A)$ is isomorphic to

$$[\mathrm{Spec}(A[z, w]/zw = t', z^n = x, w^n = y)/\Gamma] \tag{1.1.2}$$

for some element $t' \in \mathcal{O}_S$, where Γ is a finite cyclic group of order n invertible in A such that if $\gamma \in \Gamma$ is a generator then $\gamma(z) = \zeta z$ and $\gamma(w) = \zeta' w$ for some primitive n–th roots of unity ζ and ζ'. The stack $\mathcal{C}$ is called *balanced* if étale locally there exists such a description with $\zeta' = \zeta^{-1}$.

Definition 1.2 ([2], 4.1.2)**.** A *twisted curve* is a stack $\mathcal{C} \to S$ as above such that the action at each nodal point $\bar{s} \to C$ is balanced. A twisted curve $\mathcal{C} \to S$ has *genus* g if the genus of $C_{\bar{s}}$ is g for every geometric point $\bar{s} \to S$. An *n–pointed twisted curve* is a twisted curve $\mathcal{C} \to S$ together with a collection of disjoint closed substacks $\{\Sigma_i\}_{i=1}^n$ of $\mathcal{C}$ such that:

(i) Each $\Sigma_i \subset \mathcal{C}$ is contained in the smooth locus of $\mathcal{C} \to S$;

(ii) The stacks Σ_i are étale gerbes over S.

(iii) If $\mathcal{C}_{\text{gen}}$ denotes the complement of the Σ_i in the smooth locus of $\mathcal{C} \to S$ then $\mathcal{C}_{\text{gen}}$ is a scheme.

Remark 1.3. This definition differs from that in ([2]) where the above notion of twisted curve is called a "balanced twisted curve". Since we will not use unbalanced twisted curves in this paper, we omit the adjective "balanced".

Remark 1.4. A priori, the collection of twisted curves over a scheme S form a 2–category. However, as explained in ([2], 4.4.2) this 2–category is equivalent to a 1–category. It therefore makes sense to speak of the category of twisted curves over S.

The main purpose of this note is to establish an equivalence between the above notion of twisted curve, and a new notion of "log twisted curve" defined using logarithmic structures in the sense of Fontaine and Illusie ([7]) (in what follows we will only consider fine log structures so we usually omit the adjective "fine").

Definition 1.5 ([10], 3.1)**.** Let X be a Deligne–Mumford stack.

(i) A log structure $\mathcal{M}$ on X is called *locally free* if for every geometric point $\bar{x} \to X$ the monoid $\overline{\mathcal{M}}_{\bar{x}} := \mathcal{M}_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}}$ is isomorphic to $\mathbb{N}^r$ for some r.

(ii) A morphism $\mathcal{M} \to \mathcal{N}$ of locally free log structures on X is called *simple* if for every geometric point $\bar{x} \to X$ the monoids $\overline{\mathcal{M}}_{\bar{x}}$ and $\overline{\mathcal{N}}_{\bar{x}}$ have the same rank, the morphism $\varphi : \overline{\mathcal{M}}_x \to \overline{\mathcal{N}}_{\bar{x}}$ is injective, and for every irreducible element $f \in \overline{\mathcal{N}}_{\bar{x}}$ there exists an irreducible element $g \in \overline{\mathcal{M}}_{\bar{x}}$ and a positive integer n such that $\varphi(g) = nf$.

Remark 1.6. Recall that if P is a sharp monoid (i.e. $P^* = \{0\}$), then an element $p \in P - \{0\}$ is *irreducible* if for any equality $p_1 + p_2 = p$ in P we have $p_1 = 0$ or $p_2 = 0$.

Let S be a scheme and $f : C \to S$ a nodal curve. As discussed in section 3, there exists canonical log structures $\mathcal{M}_C$ and $\mathcal{M}_S$ on C and S respectively, and an extension of f to a log smooth morphism $(C, \mathcal{M}_C) \to (S, \mathcal{M}_S)$.

Definition 1.7. A *n-pointed log twisted curve* over a scheme S is a collection of data

$$(C/S, \{\sigma_i, a_i\}_{i=1}^n, \ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S), \tag{1.7.1}$$

where C/S is a nodal curve, $\sigma_i : S \to C$ are sections, the a_i are positive integers invertible on S, and $\ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S$ is a simple morphism of log structures on S, where $\mathcal{M}_S$ denotes the canonical log structure on S mentioned above.

The main result of this paper is the following:

Theorem 1.8. *For any scheme S, there is a natural equivalence of groupoids between the groupoid of n–pointed twisted curves over S and the groupoid of log twisted n–pointed curves over S. Moreover, this equivalence is compatible with base change $S' \to S$.*

An important consequence of this is the following. Fix integers g and n, and let $\mathcal{S}_{g,n}$ denote the fibered category over $\mathbb{Z}$ which to any scheme S associates the groupoid of all (not necessarily stable) n–pointed genus g nodal curves C/S. The stack $\mathcal{S}_{g,n}$ is algebraic, and as explained in section 5 the substack $\mathcal{S}^0_{g,n} \subset \mathcal{S}_{g,n}$ classifying smooth curves defines a log structure $\mathcal{M}_{\mathcal{S}_{g,n}}$ on $\mathcal{S}_{g,n}$.

Theorem 1.9. *Let $\mathcal{M}^{\text{tw}}_{g,n}$ denote the fibered category over $\mathbb{Z}$ which to any scheme T associates the groupoid of n–marked genus g twisted curves $(\mathcal{C}, \{\Sigma_i\})$ over T. Then $\mathcal{M}^{\text{tw}}_{g,n}$ is a smooth Artin stack, and the natural map*

$$\pi : \mathcal{M}^{\text{tw}}_{g,n} \longrightarrow \mathcal{S}_{g,n} \tag{1.9.1}$$

sending $(\mathcal{C}, \{\Sigma_i\})$ to its coarse moduli space with the marked points induced by the Σ_i is representable by Deligne–Mumford stacks. Moreover, there is a natural locally free log structure $\mathcal{M}_{\mathcal{M}^{\text{tw}}_{g,n}}$ on $\mathcal{M}^{\text{tw}}_{g,n}$ and a log étale morphism

$$(\mathcal{M}^{\text{tw}}_{g,n}, \mathcal{M}_{\mathcal{M}^{\text{tw}}_{g,n}}) \longrightarrow (\mathcal{S}_{g,n}, \mathcal{M}_{\mathcal{S}_{g,n}}) \tag{1.9.2}$$

with underlying morphism of stacks (1.9.1).

Remark 1.10. Consider a field k and an object $(\mathcal{C}, \{\Sigma_i\}) \in \mathcal{M}^{\text{tw}}_{g,n}(k)$. Let $(C, \{\sigma_i\})$ be the coarse moduli space, and let R be a versal deformation space for the object $(C, \{\sigma_i\}) \in \mathcal{S}_{g,n}(k)$. Let $q_1, \ldots, q_m \in C$ be the nodes and let r_i be the order of the stabilizer group of a point of $\mathcal{C}$ lying above q_i. As in ([3], 1.5), there is a smooth divisor $D_i \subset \text{Spec}(R)$ classifying deformations where q_i remains a node. In other words, if $t_i \in R$ is an element defining D_i then in an étale neighborhood of q_i the versal deformation $\widetilde{C} \to \text{Spec}(R)$ of $(C, \{\sigma_i\})$ is isomorphic to

$$\text{Spec}(R[x, y]/xy - t_i). \tag{1.10.1}$$

It follows from the proof of (1.9) that a versal deformation space for the twisted curve $(\mathcal{C}, \{\Sigma_i\})$ is given by

$$R[z_1, \ldots, z_m]/(z_1^{r_1} - t_1, \ldots, z_m^{r_m} - t_m). \tag{1.10.2}$$

The proof of (1.9) also shows the following:

Corollary 1.11. *For any integer $N > 0$, let $\mathcal{M}^{\text{tw},\leq N}_{g,n}$ denote the substack of $\mathcal{M}^{\text{tw}}_{g,n}$ classifying n–pointed genus g twisted curves such that the order of the stabilizer group at every point is less than or equal to N. Then $\mathcal{M}^{\text{tw},\leq N}_{g,n}$ is an open substack of $\mathcal{M}^{\text{tw}}_{g,n}$ and the map $\mathcal{M}^{\text{tw},\leq N}_{g,n} \to \mathcal{S}_{g,n}$ is of finite type.*

Remark 1.12. For any fixed dual graph Γ, there is a locally closed substack $\mathcal{S}^{\Gamma}_{g,n} \subset \mathcal{S}_{g,n}$ (with the reduced structure) whose geometric points classify n–marked genus g nodal curves $(C, \{\sigma_i\})$ whose dual graph (see for example ([3], p. 86)) is isomorphic to Γ. This substack $\mathcal{S}^{\Gamma}_{g,n}$ is of finite type. Indeed for m sufficiently big, the substack $\mathcal{S}^{\Gamma}_{g,n}$ is in the image of the morphism

$$\overline{\mathcal{M}}_{g,m} \to \mathcal{S}_{g,n} \tag{1.12.1}$$

which associates to a stable m–marked curve $(C, \{\sigma_i\}_{i=1}^m)$ the object $(C, \{\sigma_i\}_{i=1}^n)$ of $\mathcal{S}_{g,n}$. For example, if r is the number of vertices of Γ, then we can take $m = 3r + n$ since any n–marked nodal curve $(C, \{\sigma_i\})$ defines an object of $\overline{\mathcal{M}}_{g,3r+n}$ after marking three points on each component.

Fix also an integer N, and write $\mathcal{M}^{\text{tw},\leq N,\Gamma}_{g,n}$ for the fiber product $\mathcal{M}^{\text{tw},\leq N}_{g,n} \times_{\mathcal{S}_{g,n}} \mathcal{S}^{\Gamma}_{g,n}$. The stack $\mathcal{M}^{\text{tw},\leq N,\Gamma}_{g,n}$ classifies N–marked twisted curves $(\mathcal{C}, \{\Sigma_i\})$ such that the coarse space C of $\mathcal{C}$ has dual graph Γ. Since the map $\mathcal{M}^{\text{tw},\leq N,\Gamma}_{g,n} \to \mathcal{S}^{\Gamma}_{g,n}$ is of finite type by (1.11), it follows that the stack $\mathcal{M}^{\text{tw},\leq N,\Gamma}_{g,n}$ is also of finite type.

The construction of the universal twisted curve also enables us to prove the following useful result (which holds automatically over a field by ([8], 2.1)).

Theorem 1.13. *Let $\mathcal{C}/S$ be a twisted curve. Then after replacing S by an étale cover there exists a finite flat morphism $Z \to \mathcal{C}$ with Z/S a projective scheme.*

As an application of these results, let us show how these Theorems can be combined with the general theory of Hom–stacks developed in ([13]) to construct the moduli space of stable maps into a Deligne–Mumford stack, and to prove that it is bounded.

1.14. Let S be a scheme and $\mathcal{X}/S$ a tame Deligne–Mumford stack of finite presentation with quasi–projective coarse moduli space. Let $\mathcal{K}_{g,n}(\mathcal{X}, d)$ be the fibered category over S which to any S–scheme T associates the groupoid of data as follows:

(1.14.1) A twisted n–pointed curve $(\mathcal{C}/T, \{\Sigma_i\})$ over T.

(1.14.2) A representable morphism $f : \mathcal{C} \to \mathcal{X}$ such that the induced morphism on coarse moduli spaces $C \to X$ is a stable n–pointed map of degree d.

Theorem 1.15 ([2], 1.4.1). *The fibered category $\mathcal{K}_{g,n}(\mathcal{X}, d)$ is a Deligne–Mumford stack of finite presentation over S.*

Remark 1.16. The most difficult part of this Theorem is the quasi–compactness of $\mathcal{K}_{g,n}(\mathcal{X}, d)$. Using the valuative criterion for properness as in ([2], 6.0.4) one can deduce from the above that $\mathcal{K}_{g,n}(\mathcal{X}, d)$ is a proper Deligne–Mumford stack over S.

Proof of (1.15). By a standard limit argument, we may assume that S is of finite type over $\mathbb{Z}$. Let X denote the coarse moduli space of $\mathcal{X}$. By ([1], 2.8), the stack $K_{g,n}(X, d)$ is representable and of finite presentation over S. Let N be the maximal order of any of the stabilizer groups of $\mathcal{X}$. Since the maps $f : \mathcal{C} \to \mathcal{X}$ are required to be representable, the natural map

(1.16.1) $$\mathcal{K}_{g,n}(\mathcal{X}, d) \longrightarrow \mathcal{M}^{\mathrm{tw}}_{g,n} \times_{\mathcal{S}_{g,n}} K_{g,n}(X, d)$$

has image in $\mathcal{M}^{\mathrm{tw},\leq N}_{g,n} \times_{\mathcal{S}_{g,n}} K_{g,n}(X, d)$. Hence by (1.11) it suffices to show that (1.16.1) is of finite type.

If $\mathcal{Z}$ and $\mathcal{W}$ are separated Deligne–Mumford stacks over S, and $\mathcal{Z}$ is proper and flat over S, let $\underline{\mathrm{Hom}}_S(\mathcal{Z}, \mathcal{W})$ be the stack over S which to any scheme $T \to S$ associates the groupoid of functors $\mathcal{Z} \times_S T \to \mathcal{W} \times_S T$. Assume that fppf–locally on S there exists a finite flat surjection $U \to \mathcal{Z}$ with U a scheme, and that the coarse moduli space Z of $\mathcal{Z}$ is projective over S. Then by ([13], 1.1), the stack $\underline{\mathrm{Hom}}_S(\mathcal{Z}, \mathcal{W})$ is a Deligne–Mumford stack locally of finite type over S, and the substack $\underline{\mathrm{Hom}}^{\mathrm{rep}}_S(\mathcal{Z}, \mathcal{W}) \subset \underline{\mathrm{Hom}}_S(\mathcal{Z}, \mathcal{W})$ classifying representable functors is an open substack. If furthermore the coarse space Z is also flat over S, then by ([13], 1.7) the natural map

(1.16.2) $$\underline{\mathrm{Hom}}_S(\mathcal{Z}, \mathcal{W}) \to \underline{\mathrm{Hom}}_S(Z, W)$$

is of finite type.

If $S \to \mathcal{M}^{\mathrm{tw},\leq N}_{g,n} \times_{\mathcal{S}_{g,n}} K_{g,n}(X, d)$ is a morphism corresponding to an n–pointed twisted curve $(\mathcal{C}/S, \{\Sigma_i\})$ of genus g with a stable map $f : C \to X$ of degree d, then the fiber product

(1.16.3) $$\mathcal{K}_{g,n}(\mathcal{X}, d) \times_{\mathcal{M}^{\mathrm{tw},\leq N}_{g,n} \times_{\mathcal{S}_{g,n}} K_{g,n}(X,d)} S$$

is isomorphic to the fiber product of

$$\begin{array}{ccc} & & S \\ & & \downarrow f \\ \underline{\mathrm{Hom}}^{\mathrm{rep}}_S(\mathcal{C}, \mathcal{X}) & \xrightarrow{\pi} & \underline{\mathrm{Hom}}_S(C, X). \end{array} \tag{1.16.4}$$

As stated above, the morphism labelled π is of finite type, and hence it follows that (1.16.3) is also of finite type. □

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2. Twisted curves

We recall some definitions and results from ([2]).

2.1. Let S be a scheme and $\mathcal{C}/S$ and $\pi : \mathcal{C} \to C$ be as in (1.1). By the same argument used in ([3], p. 81), the fact that $\mathcal{C}$ is flat over S and has nodal geometric fibers implies that $\mathcal{C}$ is étale locally on S and $\mathcal{C}$ isomorphic to the relative spectrum over S

$$\mathrm{Spec}_S(\mathcal{O}_S[x, y]/xy - t), \tag{2.1.1}$$

for some element $t \in \mathcal{O}_S$.

Recall that the smooth locus $\mathcal{C}^{\mathrm{sm}}_{\bar{s}}$ is assumed to be equal to the inverse image of the smooth locus $C^{\mathrm{sm}}_{\bar{s}} \subset C_{\bar{s}}$, and that over some dense open subset $U \subset C_{\bar{s}}$ the map $\mathcal{C}_{\bar{s}} \times_{C_{\bar{s}}} U \to U$ is an isomorphism. In what follows, we refer to the maximal open subspace of C over which the map π is an isomorphism as the *non–special locus.*

By ([2], 4.1.1), the coarse moduli space C is a nodal curve over S.

Proposition 2.2. *Let $\bar{s} \to C$ be a geometric point and set $\mathcal{C}^{\mathrm{sh}} := \mathcal{C} \times_C \mathrm{Spec}(\mathcal{O}_{C,\bar{s}})$.*

(i) *If $\bar{s}$ maps to a smooth point of C, then*

$$\mathcal{C}^{\mathrm{sh}} \simeq [\mathrm{Spec}\ \mathcal{O}_{C,\bar{s}}[z]/z^r = \pi/\Gamma], \tag{2.2.1}$$

where $\pi \in \mathfrak{m} \subset \mathcal{O}_{C,\bar{s}}$ is a local coordinate (i.e. defines an étale map $C \to \mathbb{A}^1_S$ in some étale neighborhood of $\bar{s}$), r is an integer invertible in $k(\bar{s})$, and Γ is a cyclic group of order r for which a generator $\gamma \in \Gamma$ acts by $z \mapsto \zeta z$ for some primitive r–th root ζ of 1.

(ii) *If $\bar{s}$ maps to a node of C, choose elements $t \in \mathcal{O}_{S,\bar{s}}$ and $x, y \in \mathcal{O}_{C,\bar{s}}$ such that $xy = t$ and $\mathcal{O}_{C,\bar{s}}$ is isomorphic to the strict henselization of $\mathcal{O}_{S,\bar{s}}[x, y]/(xy - t)$ at the point defined by $(\mathfrak{m}_{S,\bar{s}}, x, y)$. Then*

$$\mathcal{C}^{\mathrm{sh}} \simeq [\mathrm{Spec}(\mathcal{O}_{C,\bar{s}}[z, w]/zw = t',\ z^r = x,\ w^r = y)/\Gamma], \tag{2.2.2}$$

where r is an integer invertible in $k(\bar{s})$, $t' \in \mathcal{O}_{S,\bar{s}}$ is an element with $t'^r = t$, and Γ is a cyclic group of order r such that a generator $\gamma \in \Gamma$ acts by $z \mapsto \zeta_1 z$ and $w \mapsto \zeta_2 w$ for some primitive r–th roots ζ_1 and ζ_2 of 1.

Proof. By the proof of ([2], 2.2.3), the stack $\mathcal{C}^{\rm sh}$ is isomorphic to a quotient $[U/\Gamma]$, where U is a connected scheme finite over $\mathrm{Spec}(\mathcal{O}_{C,\bar{s}})$. Write $U = \mathrm{Spec}(A)$ with A a strictly henselian local ring, and let $\tilde{s} \in U$ be the closed point. In fact we can and will take $A = \mathcal{O}_{\mathcal{C},\tilde{s}}$.

If $\tilde{s}$ is a smooth point, choose an isomorphism between A and the strict henselization of $\mathrm{Spec}(\mathcal{O}_{S,\bar{s}}[z])$ at the point defined by $(\mathfrak{m}_{S,\bar{s}}, z)$. Let $\overline{A}$ denote the quotient $A/\mathfrak{m}_{S,\bar{s}}A$, and let $\bar{z}$ denote the image of z. The ring $\overline{A}$ is isomorphic to the strict henselization of $k(\bar{s})[\bar{z}]$ at the point $\{\bar{z} = 0\}$. Fix a generator $\gamma \in \Gamma$. Since γ fixes the point $\{\bar{z} = 0\}$, we must have $\gamma(\bar{z}) = u\bar{z}^i$ for some $u \in \overline{A}^*$. Since γ has finite order invertible in $k(\bar{s})$, we must have $i = 1$ and u a root of unity invertible in $k(\bar{s})$. Furthermore since the map $\mathcal{C}^{\rm sh}_{\bar{s}} \rightarrow \mathrm{Spec}(\mathcal{O}_{C_{\bar{s}},\bar{s}})$ is generically an isomorphism the action of Γ on the one–dimensional $k(\bar{s})$–space $(z)/(z^2) \otimes_{\mathcal{O}_{S,\bar{s}}} k(\bar{s})$ is faithful. It follows that Γ is a finite cyclic group of some order r invertible in $k(\bar{s})$ and that a generator $\gamma \in \Gamma$ acts by multiplying $\bar{z}$ by a primitive r–th root ζ of 1. Since the order of the group Γ is invertible in $k(\bar{s})$, there is a canonical decomposition $A = \oplus_i A_i$, where γ acts on A_i by multiplication by ζ^i. Let $z' \in A_1$ be a lifting of $\bar{z} \in A_1 \otimes_{\mathcal{O}_{S,\bar{s}}} k(\bar{s})$. The elements z and z' differ by an element of $\mathfrak{m}_{S,\bar{s}}A$, and in particular the map $\mathcal{O}_{S,\bar{s}}[t] \rightarrow A$ sending t to z' induces an isomorphism between A and the strict henselization of $\mathcal{O}_{S,\bar{s}}[t]$ at the point $(\mathfrak{m}_{S,\bar{s}}, t)$. Thus after replacing z by z' we obtain the description of $\mathcal{C}^{\rm sh}$ given in (i).

If $\tilde{s}$ is a node, there exists an element $t' \in \mathcal{O}_{S,\bar{s}}$ such that A is isomorphic to the strict henselization of $\mathrm{Spec}(\mathcal{O}_{S,\bar{s}}[z,w]/(zw - t'))$ at the point defined by $(z, w, \mathfrak{m}_{\bar{s}})$. Again since the group Γ is finite the action on the 2–dimensional $k(\bar{s})$–space $((z,w)/(z,w)^2) \otimes_{\mathcal{O}_{S,\bar{s}}} k(\bar{s})$ is faithful. Let $\bar{z}$ and $\bar{w}$ be the images of z and w in $\overline{A} := A \otimes_{\mathcal{O}_S} k(\bar{s})$. The action of Γ necessarily preserves the two components $\{\bar{z} = 0\}$ and $\{\bar{w} = 0\}$ of $\mathrm{Spec}(\overline{A})$ since $\mathrm{Spec}(\mathcal{O}_{C,\bar{s}} \otimes_{\mathcal{O}_S} k(\bar{s}))$ is reducible. It follows that an element $\gamma \in \Gamma$ acts by $\bar{z} \mapsto \zeta_1\bar{z}$ and $\bar{w} \mapsto \zeta_2\bar{w}$ for some roots of unity ζ_1 and ζ_2. Since the action of Γ on $\mathrm{Spec}(\overline{A})$ is generically free, it follows that Γ is a cyclic group of some order r invertible in $k(\bar{s})$. Fix a generator $\gamma \in \Gamma$. Then the elements $\gamma(z), \gamma(w) \in A$ have the property that $\gamma(z) \cdot \gamma(w) = t'$ and $(z, w, \mathfrak{m}_{S,\bar{s}}) = (\gamma(z), \gamma(w), \mathfrak{m}_{S,\bar{s}})$ (equality of ideals in A). By ([6], 2.1) this implies that $\gamma(z) = uz$ and $\gamma(w) = vw$ for some elements $u, v \in A^*$. Since γ is also of finite order we must have $u = \zeta_1$ and $v = \zeta_2$. This proves case (ii). □

Definition 2.3. Let $\mathcal{C}/S$ be as in (2.1), and let $\bar{s} \rightarrow C$ be a geometric point mapping to a node. We say that the stack $\mathcal{C}$ is *balanced at* $\bar{s}$ if we can describe the stack $\mathcal{C}^{\rm sh}$ as in (2.2 (ii)) such that $\zeta_1 = \zeta_2^{-1}$.

Remark 2.4. The notion of being balanced at $\bar{s}$ can be described more intrinsically as follows. Set $\mathcal{G} = \mathrm{Spec}(k(\bar{s})) \times_C \mathcal{C}$ and let $I \subset \mathcal{O}_{\mathcal{G}}$ be the nilradical. The sheaf I/I^2 defines a locally free sheaf on $\mathcal{G}_{\rm red}$ and the condition that the stack is balanced at $\bar{s}$ is equivalent to the condition that the invertible sheaf $\bigwedge^2(I/I^2)$ on $\mathcal{G}_{\rm red}$ is trivial. In (3.8), we give another interpretation of the notion of "balanced" in terms of the existence of certain log structures on the stack.

Remark 2.5. Since the group Γ has order invertible in the base, in (2.2 (ii)) either the stack is balanced at $\bar{s}$ or $t = 0$.

We define *twisted curves of genus* g and n*–pointed twisted curves* as in (1.2).

2.6. As discussed in ([2], 4.1.2), the image of the Σ_i in C is a collection of sections $\{\sigma_1, \ldots, \sigma_n\}$ of $C \rightarrow S$ and hence $(C, \{\sigma_i\})$ is an n–pointed nodal curve in the usual sense. Furthermore, if $C_{\rm gen}$ denotes the complement of the σ_i in the smooth locus of $C \to S$, then the map

$$\mathcal{C} \times_C C_{\rm gen} \longrightarrow C_{\rm gen} \tag{2.6.1}$$

is an isomorphism by assumption.

3. Log structures on twisted curves

3.1. Let us first make a motivational remark. Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne–Mumford compactification of the moduli space of n–pointed curves, and let $(\mathcal{U}, \{\sigma_i : \overline{\mathcal{M}}_{g,n} \to \mathcal{U}\}_{i=1}^n)$ be the universal n–pointed curve over $\overline{\mathcal{M}}_{g,n}$. The closed substack (with the reduced structure) $D \subset \overline{\mathcal{M}}_{g,n}$ classifying singular n–pointed curves is a divisor with normal crossings on $\overline{\mathcal{M}}_{g,n}$, and hence by ([7], 1.5) there is a natural log structure $\mathcal{M}_{\overline{\mathcal{M}}_{g,n}}$ on $\overline{\mathcal{M}}_{g,n}$. In addition, the inverse image of D in $\mathcal{U}$ together with the union of the sections σ_i is a divisor with normal crossings on $\mathcal{U}$ and hence there is also a natural log structure $\mathcal{M}_{\mathcal{U}}$ on the universal curve $\mathcal{U}$. There is also a natural morphism of log stacks

$$(\mathcal{U}, \mathcal{M}_{\mathcal{U}}) \longrightarrow (\overline{\mathcal{M}}_{g,n}, \mathcal{M}_{\overline{\mathcal{M}}_{g,n}}), \tag{3.1.1}$$

which is log smooth ([7], 3.7 (2)). This implies that if $(C, \{s_i\})$ is an n–pointed nodal curve over a scheme T arising from a morphism $T \to \overline{\mathcal{M}}_{g,n}$, then $C \to T$ can naturally be given the structure of a log smooth morphism $(C, \mathcal{M}_C) \to (T, \mathcal{M}_T)$ by pulling back the log structures on $\mathcal{U}$ and $\overline{\mathcal{M}}_{g,n}$. As we now explain, this structure can be defined intrinsically (i.e. without using $\overline{\mathcal{M}}_{g,n}$) and more generally for twisted curves.

3.2. Though the results of ([12]) are only stated there for schemes, they apply equally well to Deligne–Mumford stacks. We summarize here the results we need from (loc. cit.).

Definition 3.3. A log smooth morphism $f : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ is *essentially semi-stable* if for each geometric point $\bar{x} \to X$ the monoids $(f^{-1}\overline{\mathcal{M}}_S)_{\bar{x}}$ and $\overline{\mathcal{M}}_{X,\bar{x}}$ are free monoids, and if for suitable isomorphisms $(f^{-1}\overline{\mathcal{M}}_S)_{\bar{x}} \simeq \mathbb{N}^r$ and $\overline{\mathcal{M}}_{X,\bar{x}} \simeq \mathbb{N}^{r+s}$ the map

$$(f^{-1}\overline{\mathcal{M}}_S)_{\bar{x}} \to \overline{\mathcal{M}}_{X,\bar{x}} \tag{3.3.1}$$

is of the form

$$e_i \mapsto \begin{cases} e_i & \text{if } i \neq r \\ e_r + e_{r+1} + \cdots + e_{r+s} & \text{if } i = r, \end{cases} \tag{3.3.2}$$

where e_i denotes the i-th standard generator of $\mathbb{N}^r$.

Lemma 3.4. *If $f : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ is essentially semi-stable, then étale locally on X and S there exist charts $\mathbb{N}^r \to \mathcal{M}_S$, $\mathbb{N}^{r+s} \to \mathcal{M}_X$ such that the map $\mathbb{N}^r \to \mathbb{N}^{r+s}$ given by formula* (3.3.2) *is a chart for f, and such that the map*

$$\mathcal{O}_S \otimes_{\mathbb{Z}[\mathbb{N}^r]} \mathbb{Z}[\mathbb{N}^{r+s}] \to \mathcal{O}_X \tag{3.4.1}$$

is smooth.

Proof. Observe that if $s \in S$ is a point, then the stalk $\overline{\mathcal{M}}_{S,\bar{s}}$ is a free monoid and hence in some étale neighborhood of s there exists a chart $\mathbb{N}^r \to \mathcal{M}_S$ such that the induced map

$\mathbb{N}^r \to \overline{\mathcal{M}}_{S,\bar{s}}$ is bijective ([7], 2.10). If $x \in X$ is a point lying over s, then there exists in some étale neighborhood of x a chart

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{M}_X \\ \uparrow & & \uparrow \\ \mathbb{N}^r & \longrightarrow & \mathcal{M}_S \end{array} \tag{3.4.2}$$

such that the induced map

$$\mathcal{O}_S \otimes_{\mathbb{Z}[\mathbb{N}^r]} \mathbb{Z}[P] \to \mathcal{O}_X \tag{3.4.3}$$

is smooth and such that the map $P \to \overline{\mathcal{M}}_{X,\bar{x}}$ is bijective (this follows for example from the proof of ([7], 3.5)). From the bijectivity of $P \to \overline{\mathcal{M}}_{X,\bar{x}}$ we conclude that P is a free monoid, and that the map $\mathbb{N}^r \to P$ has the desired form (after perhaps applying an automorphism of $\mathbb{N}^r$). □

Let $S = \mathrm{Spec}(k)$, where k is a separably closed field, and let $f : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ be an essentially semi-stable morphism. Let $x \in X$ be a singular point. Then by (3.4) there exists a chart

$$\begin{array}{ccccc} \mathbb{N}^{r+s} & \longrightarrow & \mathcal{M}_X & \longrightarrow & \mathcal{O}_X \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{N}^r & \longrightarrow & \mathcal{M}_S & \xrightarrow{\alpha} & k \end{array} \tag{3.4.4}$$

in an étale neighborhood of x such that

$$k \otimes_{\mathbb{Z}[\mathbb{N}^r]} \mathbb{Z}[\mathbb{N}^{r+s}] \simeq k[x_r, \dots, x_{r+s}]/(x_r \cdots x_{r+s} - \alpha(e_r)) \to \mathcal{O}_X \tag{3.4.5}$$

is smooth. Since x is a singular point, it follows that $\alpha(e_r) = 0$ and hence the map

$$\overline{\mathcal{M}}_S \to \overline{\mathcal{M}}_{X,\bar{x}} \tag{3.4.6}$$

is of the form $\mathbb{N}^{r'} \to \mathbb{N}^{r'+s}$ as in (3.3) for some $r' \leq r$ and $s \geq 1$. It follows that if $\mathrm{Irr}(\overline{\mathcal{M}}_S)$ denotes the set of irreducible elements in $\overline{\mathcal{M}}_S$, then there is a unique element in $\mathrm{Irr}(\overline{\mathcal{M}}_S)$ whose image in $\overline{\mathcal{M}}_{X,\bar{x}}$ is not irreducible. This defines a canonical map

$$s_X : \{\text{singular points of } X\} \to \mathrm{Irr}(\overline{\mathcal{M}}_S). \tag{3.4.7}$$

Definition 3.5 ([12], 2.6)**.** An essentially semi-stable morphism of log Deligne–Mumford stacks $f : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ is *special at a geometric point* $\bar{s}$ if the map

$$s_{X_{\bar{s}}} : \{\text{singular points of } X_{\bar{s}}\} \to \mathrm{Irr}(\overline{\mathcal{M}}_{S,\bar{s}}) \tag{3.5.1}$$

induces a bijection between the set of connected components of the singular locus of $X_{\bar{s}}$ and $\mathrm{Irr}(\overline{\mathcal{M}}_{S,\bar{s}})$. If f is special at every geometric point $\bar{s} \to S$, then we call f a *special morphism.*

Theorem 3.6. *Let $f : \mathcal{C} \to S$ be a twisted curve. Then there exist log structures $\widetilde{\mathcal{M}}_{\mathcal{C}}$ and $\mathcal{M}_S$ on $\mathcal{C}$ and S respectively, and a special morphism*

$$(f, f^b) : (\mathcal{C}, \widetilde{\mathcal{M}}_{\mathcal{C}}) \longrightarrow (S, \mathcal{M}_S). \tag{3.6.1}$$

Moreover, the datum $(\widetilde{\mathcal{M}}_{\mathcal{C}}, \mathcal{M}_S, f^b)$ is unique up to unique isomorphism.

Proof. The uniqueness statement in the Theorem follows from the uniqueness part of ([12], 2.7). To prove (3.6) it therefore suffices to construct the data $(\widetilde{\mathcal{M}}_{\mathcal{C}}, \mathcal{M}_S, f^b)$. Furthermore, by the uniqueness it suffices to construct this data étale locally on S. By a limit argument using ([12], 2.17), it even suffices to consider the case when $S = \mathrm{Spec}(\mathcal{O}_S)$ is the spectrum of a strictly henselian local ring.

Let $p_1, \ldots, p_n$ be the nodes in the closed fiber of C, and choose for each $i \in \{1, n\}$ an open subset $U_i \subset C$ containing p_i and no other nodes. Let $\mathcal{U}_i \subset \mathcal{C}$ be the inverse image of U_i. Let $t_i \in \mathcal{O}_S$ be an element such that $\mathcal{U}_i$ is étale locally isomorphic to

(3.6.2) $$\mathrm{Spec}(\mathcal{O}_S[z, w]/(zw - t_i)),$$

and let $\mathcal{M}_S^i$ be the log structure on S defined by the map $\mathbb{N} \to \mathcal{O}_S$ sending 1 to t_i.

Following ([12], 3.1) define a semi-stable log structure on $\mathcal{U}_i$ to be a pair $(\mathcal{M}, f^b)$, where $\mathcal{M}$ is a log structure on $\mathcal{U}_i$ and $f^b : \mathcal{M}_S^i|_{\mathcal{U}_i} \to \mathcal{M}$ is a morphism of log structures such that the following hold:

(3.6.3) The induced morphism $(\mathcal{U}_i, \mathcal{M}) \to (S, \mathcal{M}_S^i)$ is log smooth.

(3.6.4) For every geometric point $\bar{u} \to \mathcal{U}_i$ the stalk $\overline{\mathcal{M}}_{\bar{u}}$ is a free monoid and the induced map

(3.6.5) $$\mathbb{N} \to \overline{\mathcal{M}}_S^i \to \overline{\mathcal{M}}_{\bar{u}}$$

is the diagonal map.

To prove the Theorem, it suffices to show that for each i there exists a semi–stable log structure on $\mathcal{U}_i$. To see this, assume given a semi–stable log structure $(\mathcal{M}^i, f^b)$ on $\mathcal{U}_i$ for each i. It follows for example from (3.4) that on the complement of the singular locus of $\mathcal{U}_i$ the map $f^b : \mathcal{M}_S^i|_{\mathcal{U}_i} \to \mathcal{M}^i$ is an isomorphism. In particular there is a global log structure $\mathcal{M}_{\mathcal{C}}^i$ with a map $f^*\mathcal{M}_S^i \to \mathcal{M}_{\mathcal{C}}^i$, whose restriction to $\mathcal{U}_i$ is equal to $(\mathcal{M}^i, f^b)$, and which equals the pullback of $\mathrm{id} : \mathcal{M}_S^i \to \mathcal{M}_S^i$ on any open substack of $\mathcal{C}$ which does not meet the singular locus of $\mathcal{U}_i$. Define $\mathcal{M}_S := \oplus_{\mathcal{O}_S^*}\mathcal{M}_S^i$ and $\mathcal{M}_{\mathcal{C}} := \oplus_{\mathcal{O}_{\mathcal{C}}^*}\mathcal{M}_{\mathcal{C}}^i$. Then $\mathcal{C} \to S$ extends naturally to a special morphism

(3.6.6) $$(\mathcal{C}, \mathcal{M}_{\mathcal{C}}) \to (S, \mathcal{M}_S).$$

To complete the proof of the Theorem, we therefore fix i and show that there exists a semi–stable log structure on $\mathcal{U}_i$. To ease the notation, we omit the index i and write simply $\mathcal{U}$ (t, etc.) for $\mathcal{U}_i$ (t_i, etc.).

As explained in ([12], 3.12) if $V \to \mathcal{U}$ is étale and

(3.6.7) $$\rho : V \to \mathrm{Spec}(\mathcal{O}_S[z, w]/(zw - t_i))$$

is an étale morphism, then the ideal $J \subset \mathcal{O}_V$ defined by (z, w) is independent of the choice of ρ, and hence there is a globally defined sheaf of ideals $J \subset \mathcal{O}_{\mathcal{U}}$. Let $\mathcal{D} \subset \mathcal{U}$ be the closed substack defined by this ideal. Also let $K_t \subset \mathcal{O}_S$ (resp. $K_t^{\mathcal{U}} \subset \mathcal{O}_{\mathcal{U}}$) be the kernel of multiplication by t. Since $\mathcal{U}$ is flat over S the sheaf $K_t^{\mathcal{U}}$ is equal to the pullback of K_t. Also, by the local description of the stack $\mathcal{C}$ in (2.2), the stack $\mathcal{D}$ is isomorphic to $B\Gamma \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathcal{O}_S/(t))$ for some finite cyclic group Γ of order invertible in $k(s)$.

Let $\mathcal{Z} \subset \mathcal{U}$ be the closed substack defined by $K_t^{\mathcal{U}} \cdot J$, and define

(3.6.8) $$G := \mathrm{Ker}(\mathcal{O}_{\mathcal{U}}^* \to \mathcal{O}_{\mathcal{Z}}^*).$$

Also let $G_2 \subset \mathcal{O}_{\mathcal{U}}^*$ be the subsheaf of units u such that $ut = t$. If $u \in G$ is a section, then $1 - u \in K_t^{\mathcal{U}}$ so $ut = t$. In particular, there is a natural inclusion $G \subset G_2$. By ([12], 3.18) there is a canonical obstruction $o \in H^1(\mathcal{U}, G_2/G)$ whose vanishing is necessary and sufficient for the existence of a semi–stable log structure on $\mathcal{U}$.

The obstruction o can be described as follows. Let $\widetilde{SS}_t$ be the stack over the étale site of $\mathcal{U}$ which to any étale $V \to \mathcal{U}$ associates the groupoid of semi–stable log structures on V. As explained in the proof of ([12], 3.18), the objects of $\widetilde{SS}_t$ admit no nontrivial automorphisms. It follows that the presheaf SS_t which to any V associates the isomorphism classes in $\widetilde{SS}_t$ is a sheaf on $\mathrm{Et}(\mathcal{U})$.

The sheaf G_2 is canonically identified with the sheaf of automorphisms of the pullback of $\mathcal{M}_S$ to $\mathcal{U}$ as follows. Let $e \in \mathcal{M}_S$ be the global section defined by the chart $\mathbb{N} \to \mathcal{M}_S$. Any automorphism α of $\mathcal{M}_S$ is by the universal property of the log structure associated to a pre–log structure given by $e \mapsto \lambda(u) + e$, where $\lambda : \mathcal{O}_{\mathcal{U}}^* \hookrightarrow \mathcal{M}_S|_{\mathcal{U}}$ is the natural inclusion and $u \in \mathcal{O}_{\mathcal{U}}^*$ is a unit such that $ut = t$. In other words, a section $u \in G_2$.

There is an action of G_2 on SS_t for which a unit $u \in G_2$ corresponding to an automorphism σ of $\mathcal{M}_S|_{\mathcal{U}}$ sends $(\mathcal{M}, f^b)$ to $(\mathcal{M}, f^b \circ \sigma)$. As explained in the proof of ([12], 3.18) this action descends to a torsorial action of G_2/G on SS_t. The obstruction $o \in H^1(\mathcal{U}, G_2/G)$ is the class of the torsor SS_t.

To complete the proof of (3.6), we show that the obstruction o is zero.

Lemma 3.7. *The map $H^1(\mathcal{U}, G_2/G) \to H^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}}^*)$ induced by the composite of the inclusion $G_2 \subset \mathcal{O}_{\mathcal{U}}^*$ with the projection $\mathcal{O}_{\mathcal{U}}^* \to \mathcal{O}_{\mathcal{D}}^*$ is injective.*

Proof. Let $\mathcal{F}$ denote the kernel of the map $\mathcal{O}_{\mathcal{D}}^* \to (\mathcal{O}_{\mathcal{D}}/K_t^{\mathcal{U}}\mathcal{O}_{\mathcal{D}})^*$. By ([12], 3.19) there is a natural exact sequence

(3.7.1) $$0 \to (K_t^{\mathcal{U}} \cap (t)) \otimes \mathcal{O}_{\mathcal{D}} \to G_2/G \to \mathcal{F} \to 0,$$

where $(t) \subset \mathcal{O}_{\mathcal{U}}$ denotes the ideal generated by t. Since $\mathcal{D}$ is a tame Deligne–Mumford stack with affine coarse moduli space and $(K_t^{\mathcal{U}} \cap (t)) \otimes \mathcal{O}_{\mathcal{D}}$ is a quasi–coherent sheaf, we have

(3.7.2) $$H^1(\mathcal{D}, (K_t^{\mathcal{U}} \cap (t)) \otimes \mathcal{O}_{\mathcal{D}}) = 0.$$

Looking at the long exact sequence of cohomology groups associated to (3.7.1), it follows that the natural map $H^1(\mathcal{U}, G_2/G) \to H^1(\mathcal{U}, \mathcal{F})$ is injective. Hence to prove the Lemma it suffices to show that the map $H^1(\mathcal{U}, \mathcal{F}) \to H^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}}^*)$ is injective. Consideration of the exact sequence

(3.7.3) $$0 \to \mathcal{F} \to \mathcal{O}_{\mathcal{D}}^* \to (\mathcal{O}_{\mathcal{D}}/K_t^{\mathcal{U}} \cdot \mathcal{O}_{\mathcal{D}})^* \to 0$$

shows that for this it suffices to show that the natural map

(3.7.4) $$H^0(\mathcal{D}, \mathcal{O}_{\mathcal{D}}^*) \to H^0(\mathcal{D}, (\mathcal{O}_{\mathcal{D}}/K_t^{\mathcal{U}} \cdot \mathcal{O}_{\mathcal{D}})^*)$$

is surjective. Since $\mathcal{O}_S$ is a local ring, this map is simply the natural surjection $(\mathcal{O}_S/(t))^* \to (\mathcal{O}_S/(K_t, t))^*$. □

By the Lemma, to prove (3.6), it suffices to show that the image of the obstruction o in $H^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}}^*)$ is zero. This class in $H^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}}^*)$ corresponds to an invertible sheaf $\mathcal{L}$ on $\mathcal{D}$ which we claim is trivial. For this note first of all that it suffices to show that if $\pi : \mathcal{U} \to U$ denotes the coarse moduli space, then the natural map $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is an isomorphism. For then the invertible sheaf $\mathcal{L}$ is obtained by pullback from an invertible sheaf on $\mathrm{Spec}(\mathcal{O}_S/(t))$ which is necessarily trivial since $\mathcal{O}_S$ is strictly henselian local.

Thus we may work étale locally on the coarse space U, and in particular may assume that we can choose an isomorphism

$$\mathcal{U} \simeq [\mathrm{Spec}(\mathcal{O}_S[z,w]/zw = t)/\Gamma], \tag{3.7.5}$$

where Γ is a finite cyclic group acting by multiplication by roots of unity on z and w. The stack $\mathcal{D}$ is the the closed substack

$$[\mathrm{Spec}(\mathcal{O}_S/(t))/\Gamma] \subset [\mathrm{Spec}(\mathcal{O}_S[z,w]/zw = t)/\Gamma], \tag{3.7.6}$$

and it is shown in ([12], proof of 3.16) that the invertible sheaf $\mathcal{L}$ is isomorphic to the dual of the rank 1 sheaf with basis $z \cdot w$. In other words, if $\gamma \in \Gamma$ is a generator acting by $\gamma(z) = \zeta_1 z$ and $\gamma(w) = \zeta_2 w$, then $\mathcal{L}$ is isomorphic to the rank 1 sheaf on $\mathcal{D}$ corresponding to the free $\mathcal{O}_S/(t)$–module on 1–generator e for which $\gamma(e) = (\zeta_1 \cdot \zeta_2)^{-1}e$. Since the stack $\mathcal{C}$ is balanced by assumption, it follows that $\mathcal{L}$ is trivial. □

Remark 3.8. From the proof it follows that the condition of the stack $\mathcal{C}$ being balanced is equivalent to the existence of log structures as in (3.6).

3.9. In fact the log structures $\widetilde{\mathcal{M}}_{\mathcal{C}}$ and $\mathcal{M}_S$ have a stronger universal property which will be needed below. Namely, consider two log structures $\mathcal{N}_{\mathcal{C}}$ and $\mathcal{N}_S$ on $\mathcal{C}$ and S respectively, and a morphism $g^b : f^*\mathcal{N}_S \to \mathcal{N}_{\mathcal{C}}$ such that the induced morphism of log spaces

$$(f, g^b) : (\mathcal{C}, \mathcal{N}_{\mathcal{C}}) \longrightarrow (S, \mathcal{N}_S) \tag{3.9.1}$$

is log smooth and integral in the sense of ([7], 4.3) and vertical (this last property means that the cokernel $\mathrm{Coker}(f^*\mathcal{N}_S \to \mathcal{N}_{\mathcal{C}})$ in the category of sheaves of monoids is a group). Then it is shown in ([12], 2.7) that there exist unique morphisms of log structures $h_S : \mathcal{M}_S \to \mathcal{N}_S$ and $h_{\mathcal{C}} : \widetilde{\mathcal{M}}_{\mathcal{C}} \to \mathcal{N}_{\mathcal{C}}$ such that the diagram

$$\begin{array}{ccc} f^*\mathcal{M}_S & \xrightarrow{f^*h_S} & f^*\mathcal{N}_S \\ {\scriptstyle f^b}\downarrow & & \downarrow{\scriptstyle g^b} \\ \widetilde{\mathcal{M}}_{\mathcal{C}} & \xrightarrow{h_{\mathcal{C}}} & \mathcal{N}_{\mathcal{C}} \end{array} \tag{3.9.2}$$

commutes and is cocartesian.

3.10. If $(\mathcal{C}, \Sigma_i)$ is an n–pointed twisted curve over some scheme S, we will also consider another log structure on $\mathcal{C}$. Let $\widetilde{\mathcal{M}}_{\mathcal{C}}$ be the log structure on $\mathcal{C}$ provided by (3.6). The ideal $\mathcal{J}_i$ defining Σ_i on $\mathcal{C}$ is an invertible sheaf equipped with a morphism $\mathcal{J}_i \to \mathcal{O}_{\mathcal{C}}$. As explained in ([7], Complement 1) it therefore corresponds to a log structure $\mathcal{N}_i$ on $\mathcal{C}$. This log structure can be described as follows. Étale locally we can choose a generator $f \in \mathcal{J}_i$ for the ideal and we define $\mathcal{N}_i$ to be the log structure associated to the prelog structure $\mathbb{N} \to \mathcal{O}_{\mathcal{C}}$ sending 1 to f. If f' is a second generator with corresponding log structure $\mathcal{N}_i'$, then there exists a unique unit $u \in \mathcal{O}_{\mathcal{C}}^*$ such that $uf' = f$. This unit defines an isomorphism $\mathcal{N}_i \to \mathcal{N}_i'$ by sending 1 to $\lambda(u) + 1$, where for $u \in \mathcal{O}_{\mathcal{C}}^*$ we write $\lambda(u)$ for the unique element of $\mathcal{N}_i'$ mapping to u.

It follows that the log structure constructed locally from generators of $\mathcal{J}_i$ glue to give the desired global log structure $\mathcal{N}_i$. We define $\mathcal{M}_{\mathcal{C}}$ to be the amalgemation

$$\mathcal{M}_{\mathcal{C}} := \widetilde{\mathcal{M}}_{\mathcal{C}} \oplus_{\mathcal{O}^*_{\mathcal{C}}} (\oplus_{i,\mathcal{O}^*_{\mathcal{C}}} \mathcal{N}_i). \tag{3.10.1}$$

The map $f^*\mathcal{M}_S \to \widetilde{\mathcal{M}}_{\mathcal{C}}$ induces a log smooth morphism of log stacks

$$(\mathcal{C}, \mathcal{M}_{\mathcal{C}}) \to (S, \mathcal{M}_S). \tag{3.10.2}$$

3.11. We define the notion of a n-pointed log twisted curve as in (1.7). The set of n–pointed twisted log curves over a scheme S form a groupoid as follows. Let $(C^1/S, \{\sigma^1_i, a^1_i\}^n_{i=1}, \ell_1 : \mathcal{M}^1_S \hookrightarrow \mathcal{M}^{1\prime}_S)$ and $(C^1/S, \{\sigma^2_i, a^2_i\}^n_{i=1}, \ell_2 : \mathcal{M}^2_S \hookrightarrow \mathcal{M}^{2\prime}_S)$ be two n–pointed twisted log curves over S. Then there are no morphisms between these two objects unless $a^1_i = a^2_i$ for every i in which case an isomorphism between consists of an isomorphism $\rho : (C^1/S, \{\sigma^1_i\}) \to (C^2/S, \{\sigma^2_i\})$ of pointed curves together with an isomorphism $\epsilon : \mathcal{M}^{1\prime}_S \to \mathcal{M}^{2\prime}_S$ such that the diagram

$$\begin{array}{ccc} \mathcal{M}^1_S & \xrightarrow{\ell_1} & \mathcal{M}^{1\prime}_S \\ {\scriptstyle\rho^*}\downarrow & & \downarrow{\scriptstyle\epsilon} \\ \mathcal{M}^2_S & \xrightarrow{\ell_2} & \mathcal{M}^{2\prime}_S \end{array} \tag{3.11.1}$$

commutes, where ρ^* denotes the isomorphism induced by the isomorphism ρ and the uniqueness statement in (3.6).

4. Proof of (1.8)

4.1. First consider a log twisted n–pointed curve $(C/S, \{\sigma_i, a_i\}, \ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S)$, and let

$$(C, \mathcal{M}_C) \longrightarrow (S, \mathcal{M}_S) \tag{4.1.1}$$

be the morphism of log schemes obtained from the pointed curve $(C/S, \{\sigma_i\})$ as in (3.10). We construct a twisted n–pointed curve $(\mathcal{C}, \{\Sigma_i\})/S$ from the log twisted curve $(C/S, \{\sigma_i, a_i\}, \ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S)$.

Define $\mathcal{C}$ to be the fibered category over S which to any $h : T \to S$ associated the groupoid of data consisting of a morphism $s : T \to C$ over h together with a commutative diagram of locally free log structures on T

$$\begin{array}{ccc} h^*\mathcal{M}_S & \xrightarrow{\ell} & h^*\mathcal{M}'_S \\ \downarrow & & \downarrow{\scriptstyle\tau} \\ s^*\mathcal{M}_C & \xrightarrow{k} & \mathcal{M}'_C, \end{array} \tag{4.1.2}$$

where:

(4.1 (i)) The map k is a simple, and for every geometric point $\bar{t} \to T$, the map $\overline{\mathcal{M}}'_{S,\bar{t}} \to \overline{\mathcal{M}}'_{C,\bar{t}}$ is either an isomorphism, or of the form $\mathbb{N}^r \to \mathbb{N}^{r+1}$ sending e_i to e_i for $i < r$ and e_r to either e_r or $e_r + e_{r+1}$.

(4.1 (ii)) For every i and geometric point $\bar{t} \to T$ with image under s in $\sigma_i(S) \subset C$, the group

$$\mathrm{Coker}(\overline{\mathcal{M}}'^{\mathrm{gp}}_{S,\bar{t}} \oplus \overline{\mathcal{M}}^{\mathrm{gp}}_{C,\bar{t}} \longrightarrow \overline{\mathcal{M}}'^{\mathrm{gp}}_{C,\bar{t}}) \tag{4.1.3}$$

is a cyclic group of order a_i.

For every i, define $\Sigma_i \subset \mathcal{C}$ to be the substack classifying morphisms $s : T \to C$ which factor through $\sigma_i(S) \subset C$ and diagrams (4.1.2) such that for every geometric point $\bar{t} \to T$ the image of

$$(\mathcal{M}'_{C,\bar{t}} - \tau(h^*\mathcal{M}'_{S,\bar{t}})) \to \mathcal{O}_{T,\bar{t}} \qquad (4.1.4)$$

is zero.

We claim that $\mathcal{C}$ together with the substacks Σ_i is a twisted n–pointed curve. Consider the tautological map $\mathcal{C} \to C$, and note that $\mathcal{C}$ is evidently a stack over C with respect to the étale topology. Thus to prove that $\mathcal{C}$ is an algebraic stack with the desired properties, we may work étale locally on C. We consider three cases.

4.2 (C is smooth and contains no σ_i). In this case, for every $s : T \to C$ as above, the map $h^*\mathcal{M}_S \to s^*\mathcal{M}_C$ is an isomorphism. It follows, that at every geometric point $\bar{t} \to T$ the rank of $\overline{\mathcal{M}}_S'^{\mathrm{gp}}$ is equal to the rank of $\overline{\mathcal{M}}_C'^{\mathrm{gp}}$. This implies that the map τ is an isomorphism, and hence $\mathcal{C} \simeq C$ in this case.

4.3 (C contains no σ_i but is not smooth). Let $p \to T$ be a geometric point mapping to the singular locus of C, and consider the diagram

$$\begin{array}{ccc} h^{-1}\overline{\mathcal{M}}_{S,p} & \xrightarrow{\ell} & h^{-1}\overline{\mathcal{M}}'_{S,p} \\ \downarrow & & \downarrow \tau \\ s^{-1}\overline{\mathcal{M}}_{C,p} & \xrightarrow{k} & \overline{\mathcal{M}}'_{C,p}. \end{array} \qquad (4.3.1)$$

After choosing suitable isomorphisms, this diagram is isomorphic to

$$\begin{array}{ccc} \mathbb{N}^r & \xrightarrow{(\alpha_1,\dots,\alpha_r)} & \mathbb{N}^r \\ \downarrow \kappa_1 & & \downarrow \kappa_2 \\ \mathbb{N}^{r+1} & \xrightarrow{(\beta_1,\dots,\beta_{r+1})} & \mathbb{N}^{r+1}, \end{array} \qquad (4.3.2)$$

where $\kappa_1(e_i) = e_i$ if $i < r$ and $\kappa_1(e_r) = e_r + e_{r+1}$. From the condition (4.1 (i)), it follows that $\alpha_i = \beta_i$ for all $i < r$, and that $\alpha_r = \beta_r = \beta_{r+1}$. From this and ([11], 5.20) it follows that if we choose an étale morphism

$$C \longrightarrow \mathrm{Spec}(\mathcal{O}_S[x, y]/xy - t) \qquad (4.3.3)$$

in some neighborhood of the image of p, where $t \in \mathcal{O}_S$, the stack $\mathcal{C}$ is isomorphic to

$$[\mathrm{Spec}(\mathcal{O}_C[z, w]/zw = t', z^{\alpha_r} = x, w^{\alpha_r} = y)/\mu_{\alpha_r}]. \qquad (4.3.4)$$

Here $t' \in \mathcal{O}_S$ is a section such that $t'^{\alpha_r} = t$, and a scheme–valued point $u \in \mu_{\alpha_r}$ acts by $z \mapsto uz$ and $w \mapsto u^{-1}w$. Such a function t' exists étale locally on S since we have the simple morphism ℓ.

4.4 (C is smooth and contains a single σ_i)**.** Let $p \to T$ be a geometric point mapping to the marked locus, and consider the diagram

$$\begin{array}{ccc} h^{-1}\overline{\mathcal{M}}_{S,p} & \xrightarrow{\ell} & h^{-1}\overline{\mathcal{M}}'_{S,p} \\ \downarrow & & \downarrow \tau \\ s^{-1}\overline{\mathcal{M}}_{C,p} & \xrightarrow{k} & \overline{\mathcal{M}}'_{C,p}. \end{array} \tag{4.4.1}$$

After choosing suitable isomorphisms, this diagram is isomorphic to

$$\begin{array}{ccc} \mathbb{N}^r & \xrightarrow{(\alpha_1,\ldots,\alpha_r)} & \mathbb{N}^r \\ \downarrow \kappa_1 & & \downarrow \kappa_2 \\ \mathbb{N}^{r+1} & \xrightarrow{(\beta_1,\ldots,\beta_{r+1})} & \mathbb{N}^{r+1}, \end{array} \tag{4.4.2}$$

where $\kappa_1(e_i) = e_i$ if $i \leq r$. From the condition (4.1 (i)), it follows that $\alpha_i = \beta_i$ for all $i \leq r$, and that the group (4.1.3) is isomorphic to $\mathbb{Z}/(\beta_{r+1})$. From (4.1 (ii)) it therefore follows that $\beta_{r+1} = a_i$. Using ([11], 5.20) we conclude that if $f \in \mathcal{O}_C$ denotes a local function defining σ_i, then the stack $\mathcal{C}$ is isomorphic to

$$[\mathrm{Spec}(\mathcal{O}_C[T]/T^{a_i} - f)/\mu_{a_i}], \tag{4.4.3}$$

where μ_{a_i} acts by multiplication on T. This also shows that the substack Σ_i is equal in this local situation to the closed substack defined by $T = 0$. In particular, Σ_i is an étale gerbe over S.

4.5 (Constructing log twisted curves from twisted curves)**.** Given a n–pointed twisted curve $(\mathcal{C}/S, \Sigma_i)$ let

$$f_{\mathcal{C}} : (\mathcal{C}, \mathcal{M}_{\mathcal{C}}) \longrightarrow (S, \mathcal{M}'_S) \tag{4.5.1}$$

be the morphism of log spaces constructed in (3.10) from the pointed stack $(\mathcal{C}, \{\Sigma_i\})$. Let $\pi : \mathcal{C} \to C$ be the coarse moduli space of $\mathcal{C}$, and let

$$f_C : (C, \mathcal{M}_C) \longrightarrow (S, \mathcal{M}_S) \tag{4.5.2}$$

denote the canonical morphism of log structures obtained from C. Let $\sigma_i : S \to C$ be the section induced by the gerbe Σ_i, and let a_i be the order of the stabilizer group of the gerbe Σ_i.

Lemma 4.6. (i) *Let $\pi_*\widetilde{\mathcal{M}}_{\mathcal{C}}$ denote the sheaf on C_{et} obtained by pushing forward the sheaf $\widetilde{\mathcal{M}}_{\mathcal{C}}$ and let $\beta : \pi_*\widetilde{\mathcal{M}}_{\mathcal{C}} \to \pi_*\mathcal{O}_{\mathcal{C}} \simeq \mathcal{O}_C$ (where the last isomorphism follows from* ([2], 2.2.1 (5))*) be the morphism of sheaves of monoids induced by the map $\widetilde{\mathcal{M}}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}$. Then $(\pi_*\widetilde{\mathcal{M}}_{\mathcal{C}}, \beta)$ is a fine log structure on C, and if $g^b : f_C^*\mathcal{M}'_S \to \pi_*\widetilde{\mathcal{M}}_{\mathcal{C}}$ denotes the morphism induced by the map $f_{\mathcal{C}}^b : f_{\mathcal{C}}^*\mathcal{M}'_S \to \widetilde{\mathcal{M}}_{\mathcal{C}}$, then the morphism of log spaces*

$$(f_C, g^b) : (C, \pi_*\widetilde{\mathcal{M}}_{\mathcal{C}}) \longrightarrow (S, \mathcal{M}'_S) \tag{4.6.1}$$

is log smooth, integral, and vertical (i.e. the conditions in (3.9) *hold).*

(ii) *For $i = 1, \ldots, r$, let $\mathcal{N}_{\mathcal{C},i}$ (resp. $\mathcal{N}_{C,i}$) denote the log structure on $\mathcal{C}$ (resp. C) defined by the gerbe Σ_i (resp. the section σ_i) as in* (3.10)*. Then there is a canonical isomorphism $\pi_*\mathcal{N}_{\mathcal{C},i} \simeq \mathcal{N}_{C,i}$ of log structures on C.*

Proof. For (i), we first compute the stalk of $\pi_*\widetilde{\mathcal{M}}_{\mathcal{C}}$ at a geometric point $\bar{s} \to C$. We consider the two cases (2.2 (i)) and (2.2 (ii)) separately.

In case (i), the log structure $\widetilde{\mathcal{M}}_{\mathcal{C}}$ is isomorphic to the pullback of $\mathcal{M}'_S$ and hence there exists an integer r and a morphism $\beta : \mathbb{N}^r \to \mathcal{O}_{C,\bar{s}}$ such that $\widetilde{\mathcal{M}}_{\mathcal{C}}$ is the log structure obtained from the induced morphism $\mathbb{N}^r \to \mathcal{O}_{C,\bar{s}}[z]/z^r = \pi$. It follows that $\pi_*\widetilde{\mathcal{M}}_{\mathcal{C},\bar{s}}$ is isomorphic to the Γ–invariants in $(\mathcal{O}_{C,\bar{s}}[z]/z^r = \pi)^* \oplus \mathbb{N}^r$ which is just the stalk $f_C^*\mathcal{M}'_{S,\bar{s}}$.

In case (ii), the stalk of $\widetilde{\mathcal{M}}_{\mathcal{C}}$ at the geometric point $\tilde{s}$ of $\mathcal{O}_{S,\bar{s}}[z,w]/zw = t'$ defined by $(\mathfrak{m}_{\bar{s}}, z, w)$ is isomorphic to $\mathcal{O}^*_{\mathcal{C},\tilde{s}} \oplus \mathbb{N}^{r-1} \oplus \mathbb{N}^2 \to \mathcal{O}_{\mathcal{C},\tilde{s}}$, where the map sends $\mathbb{N}^{r-1}$ to some elements of $\mathcal{O}_{S,\bar{s}}$ and the standard generators of $\mathbb{N}^2$ are sent to the images of z and w. The log structure $f_{\mathcal{C}}^*\mathcal{M}'_S$ is the log structure associated to the map $\mathbb{N}^r \to \mathbb{N}^{r-1} \oplus \mathbb{N}^2$ which is the identity on the first $r-1$ components and the diagonal on the last factor. Let $P \subset \mathbb{N}^{r-1} \oplus \mathbb{N}^2$ be the submonoid generated by $\mathbb{N}^{r-1}$, the diagonal of $\mathbb{N}^2$ and the elements $n \cdot e_i$, where e_i $(i = 1, 2)$ denote the generators of $\mathbb{N}^2$ and n is the order of the stabilizer group Γ of the closed point. Then it follows that the stalk of $\pi_*\widetilde{\mathcal{M}}_{\mathcal{C}}$ at $\bar{s}$ is equal to $\mathcal{O}^*_{C,\bar{s}} \oplus P$. It follows from these computations of stalks that étale locally on C the log structure $\pi_*\widetilde{\mathcal{M}}_{\mathcal{C}}$ is isomorphic to the pushout of the diagram

$$\begin{array}{ccc} f_C^*\mathcal{M}_S & \longrightarrow & f_C^*\mathcal{M}'_S \\ \downarrow & & \\ \mathcal{M}_C. & & \end{array} \tag{4.6.2}$$

From this part (i) follows since this shows that $\pi_*\widetilde{\mathcal{M}}_{\mathcal{C}}$ is a fine log structure, and all the other properties in (i) can be verified on stalks using the above local calculations.

For (ii), note first that there is a canonical map $\psi_i : \pi^*\mathcal{N}_{C,i} \to \mathcal{N}_{\mathcal{C},i}$ of log structures on $\mathcal{C}$. Locally this map is obtained as follows. The choice of a function $f \in \mathcal{O}_C$ defining the section σ_i induces a map $\mathbb{N} \to \mathcal{O}_C$ by sending 1 to f, and this map is a chart for $\mathcal{N}_{C,i}$. Similarly, étale locally on $\mathcal{C}$ we can choose a function $g \in \mathcal{O}_{\mathcal{C}}$ such that $\Sigma_i \subset \mathcal{C}$ is defined by the ideal (g) and $g^{a_i} = f$, and in this case the log structure $\mathcal{N}_{\mathcal{C},i}$ is the log structure associated to the chart $\mathbb{N} \to \mathcal{O}_{\mathcal{C}}$ sending 1 to g. The map $\psi_i : \pi^*\mathcal{N}_{C,i} \to \mathcal{N}_{\mathcal{C},i}$ is the map induced by sending $1 \in \mathbb{N}$ to $a_i \in \mathbb{N}$. If g' is a second generator of the ideal of Σ_i in $\mathcal{C}$ then $g' = ug$ for some $u \in \mathcal{O}^*_{\mathcal{C}}$. It follows that the induced map $\psi'_i : \pi^*\mathcal{N}_{C,i} \to \mathcal{N}_{\mathcal{C},i}$ sends $1 \in \mathbb{N}$ to $\psi_i(1)+\lambda(u^{a_i})$. In particular, if we further require that $g'^{a_i} = f$ then $\psi_i = \psi'_i$. It follows that the map ψ_i is independent of the choices and hence is defined globally. By adjunction, the map ψ_i induces a map $\mathcal{N}_{C,i} \to \pi_*\mathcal{N}_{\mathcal{C},i}$ of log structures on C which we claim is an isomorphism. For this it suffices to show that it becomes an isomorphism on stalks at every geometric point $\bar{s} \to C$. Again we consider the two cases (i) and (ii) in (2.2). In case (ii) the map is trivially an isomorphism since both log structures are isomorphic to the trivial log structures $\mathcal{O}^*_C$. In case (i), note that the stalk of $\mathcal{N}_{\mathcal{C},i}$ at the closed point of $\mathcal{O}_C[z]/z^{a_i} = \pi$ is equal to $(\mathcal{O}_C[z]/z^{a_i} = \pi)^* \oplus \mathbb{N}$ and the action of a generator of the cyclic group Γ sends $(0, 1)$ to $(\zeta, 1)$ for some a_i–th root of unity ζ. This implies (ii).

□

Corollary 4.7. *There exists canonical morphisms of log structures* $\pi^b : \pi^*\mathcal{M}_C \to \mathcal{M}_{\mathcal{C}}$ *and* $\ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S$ *such that the diagram of log spaces*

(4.7.1)
$$\begin{array}{ccc} (\mathcal{C}, \mathcal{M}_{\mathcal{C}}) & \xrightarrow{(\pi,\pi^b)} & (C, \mathcal{M}_C) \\ {\scriptstyle f_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle f_C} \\ (S, \mathcal{M}'_S) & \xrightarrow{(\mathrm{id},\ell)} & (S, \mathcal{M}_S) \end{array}$$

commutes. Moreover, $\ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S$ *is a simple extension.*

Proof. By (5.3 (i)) and the universal property in (3.9), there are canonical morphisms $\ell : \mathcal{M}_S \to \mathcal{M}'_S$ and $s : \widetilde{\mathcal{M}}_C \to \pi_*\widetilde{\mathcal{M}}_{\mathcal{C}}$ such that the diagram

(4.7.2)
$$\begin{array}{ccc} f_C^*\mathcal{M}_S & \xrightarrow{\ell} & f_C^*\mathcal{M}'_S \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{M}}_C & \xrightarrow{s} & \pi_*\widetilde{\mathcal{M}}_{\mathcal{C}} \end{array}$$

commutes. Let $t : \mathcal{M}_C \to \pi_*\mathcal{M}_{\mathcal{C}}$ be the composite

(4.7.3)
$$\mathcal{M}_C \simeq \widetilde{\mathcal{M}}_C \oplus_{\mathcal{O}_C^*} (\oplus_{\mathcal{O}_C^*}\mathcal{N}_{C,i}) \xrightarrow{s\times(4.6\ (\mathrm{ii}))} \pi_*\widetilde{\mathcal{M}}_{\mathcal{C}} \oplus_{\mathcal{O}_C^*} (\oplus_{\mathcal{O}_C^*}\pi_*\mathcal{N}_{\mathcal{C},i}) \xrightarrow{\mathrm{can}} \pi_*\mathcal{M}_{\mathcal{C}},$$

where can denotes the map induced by the isomorphism $\mathcal{M}_{\mathcal{C}} \simeq \widetilde{\mathcal{M}}_{\mathcal{C}} \oplus_{\mathcal{O}_{\mathcal{C}}^*} (\oplus_{\mathcal{O}_{\mathcal{C}}^*}\mathcal{N}_{\mathcal{C},i})$. Define π^b to be the map induced by t by adjunction. The commutativity of (4.7.2) implies that (4.7.1) commutes. That ℓ is a simple extension follows from the construction of ℓ in ([12], proof of 2.7). □

4.8. We therefore obtain an n–pointed log twisted curve $(C/S, \{\sigma_i, a_i\}, \ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S)$. Let $\mathcal{C}' \to C$ be the pointed twisted curve obtained from this data and the construction (4.1)–(4.4). The stack $\mathcal{C}'$ is defined as the classifying stack of diagrams of log structures as in (4.1.2). The morphisms of log structures π^b and ℓ in (4.7) and the commutativity of (4.7.1) therefore defines a canonical map $\mathcal{C} \to \mathcal{C}'$ over S. That this map is an isomorphism follows from the local description of the stack $\mathcal{C}'$ in (4.2)–(4.4).

Finally if we start with an n–pointed log twisted curve $(C/S, \{\sigma_i, a_i\}, \ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S)$ and let $(\mathcal{C}, \{\Sigma_i\})$ denote the twisted pointed curve obtained from (4.1)–(4.4), then the log structures $\mathcal{M}_{S'}$ and $\mathcal{M}_{\mathcal{C}}$ are simply the tautological data defined by the modular definition of $\mathcal{C}$. In particular, the simple extension $\ell : \mathcal{M}_S \to \mathcal{M}'_S$ is canonically isomorphic to that obtained from (4.7). We conclude that the preceding two constructions define quasi–inverse functors between the two categories in (1.8).

This completes the proof of (1.8). □

5. Proof of (1.9)

Fix integers g and n and let $\mathcal{S}_{g,n}$ denote the fibered category over $\mathbb{Z}$ which to any scheme S associates the groupoid of all (not necessarily stable) n–pointed nodal curves C/T.

Lemma 5.1. *The fibered category $\mathcal{S}_{g,n}$ is an algebraic stack locally of finite type over $\mathbb{Z}$. There is a natural open substack $\mathcal{S}^0_{g,n} \subset \mathcal{S}_{g,n}$ classifying smooth n–marked curves, and the complement $\mathcal{S}_{g,n} - \mathcal{S}^0_{g,n}$ is a divisor with normal crossings.*

Proof. Note first that the forgetful map $\mathcal{S}_{g,n} \to \mathcal{S}_g$ identifies $\mathcal{S}_{g,n}$ with an open subset of the smooth locus of the n–fold fiber product of the universal curve over $\mathcal{S}_g := \mathcal{S}_{g,0}$. Hence it suffices to prove the theorem for $\mathcal{S}_g$ (observe that $\mathcal{S}^0_{g,n} = \mathcal{S}_{g,n} \times_{\mathcal{S}_g} \mathcal{S}^0_g$, and that $\mathcal{S}_{g,n} \to \mathcal{S}_g$ is smooth).

Note also, that for each integer $m \geq 0$, there is a natural map

(5.1.1) $$\overline{\mathcal{M}}_{g,m} \longrightarrow \mathcal{S}_g,$$

where $\overline{\mathcal{M}}_{g,m}$ denotes the Deligne–Mumford–Knudsen compactification of the moduli space of m–pointed smooth curves. This map is relatively representable and of finite type: For any $T \to \mathcal{S}_g$ corresponding to a nodal curve $C \to T$, the fiber product $\overline{\mathcal{M}}_{g,m} \times_{\mathcal{S}_g} T$ is isomorphic to an open subscheme of the smooth locus of the m–fold fiber product $C \times_T \cdots \times_T C$. Furthermore, the disjoint union

(5.1.2) $$\coprod_m \overline{\mathcal{M}}_{g,m} \longrightarrow \mathcal{S}_g$$

is smooth and surjective and

(5.1.3) $$\mathcal{S}^0_g \times_{\mathcal{S}_g} \coprod_m \overline{\mathcal{M}}_{g,m} \simeq \coprod_m \mathcal{M}_{g,m},$$

where $\mathcal{M}_{g,m} \subset \overline{\mathcal{M}}_{g,m}$ denotes the open substack classifying smooth m–pointed curves. Hence to prove the lemma it remains only to see that if C_1 and C_2 are objects in $\mathcal{S}(T)$ for some scheme T, then the functor I over T which to any T'/T associates the set of isomorphisms $C_{1,T'} \to C_{2,T'}$ is representable by an algebraic space. For this note that such an isomorphism is determined by a closed subscheme $\Gamma \subset C_{1,T'} \times_{T'} C_{2,T'}$ such that the projections $\Gamma \to C_{i,T'}$ are isomorphisms. It follows that I is isomorphic to an open subfunctor of the Hilbert functor $\underline{\mathrm{Hilb}}(C_1 \times_T C_2)$ of the projective scheme $C_1 \times_T C_2$. By ([4], IV.3.2) it follows that I is representable by a scheme and so the lemma follows. □

In particular, by ([7], 3.7 (1)) there is a natural log structure $\mathcal{M}_{\mathcal{S}_{g,n}}$ on $\mathcal{S}_{g,n}$ defined by $\mathcal{S}_{g,n} - \mathcal{S}^0_{g,n}$ (see ([11], 5.1) for a discussion of log structures on stacks).

5.2. Let S be a scheme, $r > 0$ an integer, and $\mathcal{M}_S$ the log structure on S induced by a map $c : \mathbb{N}^r \to \mathcal{O}_S$. Let $\mathcal{F} \to S$ be the stack which to any $t : T \to S$ associates the groupoid of simple extensions $t^*\mathcal{M}_S \hookrightarrow \mathcal{M}'_S$. For any collection of integers $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$, let $\mathcal{F}(\underline{\alpha})$ denote the substack of $\mathcal{F}$ consisting of simple extensions $t^*\mathcal{M}_S \hookrightarrow \mathcal{M}'_S$ such that for every geometric point $\bar{t} \to T$ there exists a morphism $\beta : \mathbb{N}^r \to \overline{\mathcal{M}}'_{S,\bar{t}}$ such that the diagram

(5.2.1) $$\begin{array}{ccc} \mathbb{N}^r & \xrightarrow{(\alpha_1,\ldots,\alpha_r)} & \mathbb{N}^r \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \beta} \\ \overline{\mathcal{M}}_{S,\bar{t}} & \longrightarrow & \overline{\mathcal{M}}'_{S,\bar{t}} \end{array}$$

commutes.

Lemma 5.3. (i) *The inclusion* $\mathcal{F}(\underline{\alpha}) \subset \mathcal{F}$ *is representable by open immersions.*

(ii) *The stack* $\mathcal{F}(\underline{\alpha})$ *is isomorphic to*

$$[\mathrm{Spec}_S(\mathcal{O}_S[t_1, \dots, t_r]/(t_i^{\alpha_i} = c(e_i)))/\mu], \tag{5.3.1}$$

where e_i *denote the standard generators of* $\mathbb{N}^r$ *and* μ *denotes the group scheme* $\mu_{\alpha_1} \times \cdots \mu_{\alpha_r}$ *acting by* $(\zeta_1, \dots, \zeta_r) \cdot t_i = \zeta_i t_i$. *In particular,* $\mathcal{F}(\underline{\alpha})$ *and hence also* $\mathcal{F}$ *is algebraic.*

Proof. To see (i), note that by ([11], 3.5 (ii)) the sheaves $\overline{\mathcal{M}}_S$ and $\overline{\mathcal{M}}'_S$ are constructible, so it suffices to show that if ζ is a generization of a point $t \in T$ for which there exists a commutative diagram as in (5.2.1), then there also exists such a commutative diagram for $\bar{\zeta}$. But this is clear because we have a commutative diagram of cospecialization maps

$$\begin{array}{ccc} \overline{\mathcal{M}}_{S,\bar{t}} & \xrightarrow{\text{cosp}} & \overline{\mathcal{M}}_{S,\bar{\zeta}} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}'_{S,\bar{t}} & \xrightarrow{\text{cosp}} & \overline{\mathcal{M}}'_{S,\bar{\zeta}}. \end{array} \tag{5.3.2}$$

As for (ii), note that if $\mathcal{M}_S \hookrightarrow \mathcal{M}'_S$ is an object in $\mathcal{F}(\underline{\alpha})(T)$, then there is a unique morphism $\beta : \mathbb{N}^r \to \overline{\mathcal{M}}'_S$ such that the diagram

$$\begin{array}{ccc} \mathbb{N}^r & \xrightarrow{(\alpha_1, \dots, \alpha_r)} & \mathbb{N}^r \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \beta} \\ \overline{\mathcal{M}}_S & \longrightarrow & \overline{\mathcal{M}}'_S \end{array} \tag{5.3.3}$$

commutes. By a standard limit argument the map β exists locally, and since it is unique it exists globally. Part (ii) then follows from ([11], 5.20). □

5.4. Turning to the proof of (1.9), we can by (1.8) view $\mathcal{M}^{\text{tw}}_{g,n}$ as the stack over $\mathbb{Z}$ which to any scheme S assocaites the groupoid of n–marked genus g log twisted curves $(\mathcal{C}/S, \{\sigma_i, a_i\}, \ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S)$. For a sequence of natural numbers $\underline{b} = \{b_1, \dots, b_n\}$, let $\mathcal{M}^{\text{tw}}_{g,n}(\underline{b})$ be the substack of $\mathcal{M}^{\text{tw}}_{g,n}$ classifying log twisted curves $(\mathcal{C}/S, \{\sigma_i, a_i\}, \ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S)$ with $a_i = b_i$ for all i. There is a natural decomposition

$$\mathcal{M}^{\text{tw}}_{g,n} = \coprod_{\underline{b}} \mathcal{M}^{\text{tw}}_{g,n}(\underline{b}), \tag{5.4.1}$$

so it suffices to prove the theorem for each of the $\mathcal{M}^{\text{tw}}_{g,n}(\underline{b})$.

Now the stack $\mathcal{M}^{\text{tw}}_{g,n}(\underline{b})$ is simply the stack over $\mathcal{S}_{g,n}$ which to any $s : S \to \mathcal{S}_{g,n}$ associates the groupoid of simple extensions $\ell : s^*\mathcal{M}_{\mathcal{S}_{g,n}} \hookrightarrow \mathcal{M}'_S$ such that for every geometric point $\bar{s} \to S$ the order of the group

$$\mathrm{Coker}(\overline{\mathcal{M}}^{\text{gp}}_{\mathcal{S}_{g,n},\bar{s}} \longrightarrow \overline{\mathcal{M}}'^{\text{gp}}_{S,\bar{s}}) \tag{5.4.2}$$

is invertible in $k(\bar{s})$. Since the log structure $\mathcal{M}_{\mathcal{S}_{g,n}}$ is locally in the smooth topology obtained from a smooth map to $\mathrm{Spec}(\mathbb{Z}[\mathbb{N}^r])$ for some r, Theorem (1.9) follows from (5.3 (ii)). The log structure on $\mathcal{M}^{\text{tw}}_{g,n}(\underline{b})$ is the tautological log structure provided by the interpretation of $\mathcal{M}^{\text{tw}}_{g,n}(\underline{b})$ as a moduli space for log structures above. The statement that (1.9.2) is log étale follows from ([7], 3.4). □

6. Proof of (1.13)

By standard limit considerations, we may assume that S is the spectrum of the strict henselization at some point of a finite type $\mathbb{Z}$–algebra. Let C/S be the coarse moduli space of $\mathcal{C}$. After marking some more points (with multiplicity 1) in the non–special locus of C, we may assume that $(C, \{\sigma_i\})$ is obtained from a morphism $S \to \overline{\mathcal{M}}_{g,n}$. Let $\mathcal{X} \to \overline{\mathcal{M}}_{g,n}$ be the stack classifying simple morphisms of log structures $\ell : \mathcal{M}_{\overline{\mathcal{M}}_{g,n}} \hookrightarrow \mathcal{M}$. To prove the theorem, it suffices by the proof of (1.9) to consider the universal twisted curve over an étale cover of $\mathcal{X}$, and hence we are reduced to the case when S is a smooth quasi–projective scheme over $\mathrm{Spec}(\mathbb{Z})$, and the log structure $\mathcal{M}'_S$ is defined by a divisor with normal crossings on S.

Observe that in this case the composite $\mathcal{C} \to S \to \mathrm{Spec}(\mathbb{Z})$ is smooth. Indeed since $(\mathcal{C}, \mathcal{M}_{\mathcal{C}}) \to (S, \mathcal{M}'_S)$ is essentially semi–stable in the sense of (3.3) and $\mathcal{M}'_S$ is defined by a divisor with normal crossings relative to $\mathbb{Z}$, there exists by (3.4) étale locally on S and $\mathcal{C}$ a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{a} & \mathrm{Spec}(\mathbb{Z}[t,z,w]/(zw-t)) \\ \downarrow & & \downarrow \\ S & \xrightarrow{b} & \mathrm{Spec}(\mathbb{Z}[t]), \end{array} \tag{6.0.3}$$

where the morphisms b is smooth and also the induced map

$$a' : \mathcal{C} \to S \times_{\mathrm{Spec}(\mathbb{Z}[t])} \mathrm{Spec}(\mathbb{Z}[t,z,w]/(zw-t)) \tag{6.0.4}$$

is smooth.

Let $s \in S$ be a point and let $\zeta \in \mathrm{Spec}(\mathbb{Z})$ be the image of s. Let S_ζ, $\mathcal{C}_\zeta$ etc. denote the fibers over ζ. The approach is using ([8], proof of 2.1). Consider the bundle $\Omega^1_{\mathcal{C}/\mathrm{Spec}(\mathbb{Z})}$, and set

$$A := \mathbb{V}(\Omega^1_{\mathcal{C}/\mathrm{Spec}(\mathbb{Z})}) \to \mathcal{C}. \tag{6.0.5}$$

We claim that there exists a dense open subset $U \subset A$ which is representable by an algebraic space such that the map $U \to \mathcal{C}$ is surjective. To see this, we may work étale locally on the coarse moduli space C of $\mathcal{C}$ and consider each of the two cases in (2.2) separately.

In an étale neighborhood of a marked point, the stack $\mathcal{C}$ is by (2.2 (i)) isomorphic to

$$[\mathrm{Spec}(\mathbb{Z}[1/n][t, t_1, \ldots, t_r, z])/\Gamma], \tag{6.0.6}$$

where r is an integer and Γ is a cyclic group of some order n acting by multiplication on z. In this case the module $\Omega^1_{\mathcal{C}/\mathbb{Z}}$ is the free module on generators dt, $\{dt_i\}$, and dz. The subspace $U \subset A$ where Γ acts non–trivially is the open substack which to any ring R associates the set of maps $\Omega^1_{\mathcal{C}/\mathbb{Z}} \to R$ sending dz to an element of R^*. In particular, the map $U \to \mathcal{C}$ is surjective.

In an étale neighborhood of a node, the stack $\mathcal{C}$ is by (2.2 (ii)) étale locally isomorphic to the stack

$$[\mathrm{Spec}(\mathbb{Z}[t,z,w]/(t-zw))/\Gamma] \times_{\mathrm{Spec}(\mathbb{Z}[t]),a} S, \tag{6.0.7}$$

where $a : S \to \mathrm{Spec}(\mathbb{Z}[t])$ is a smooth map defined by a smooth divisor, and Γ is a finite cyclic group acting as in (2.2). In this local situation, the module $\Omega^1_{\mathcal{C}/\mathrm{Spec}(\mathbb{Z})}$ corresponds to the locally free module over

$$\text{Spec}(\mathbb{Z}[t,z,w]/(t-zw))\times_{\text{Spec}(\mathbb{Z}[t])} S \tag{6.0.8}$$

obtained from the direct sum of $\Omega^1_{S/\mathbb{Z}[t]}$ and the free module with basis dz and dw. The action of $\gamma\in\Gamma$ is trivial on $\Omega^1_{S/\mathbb{Z}[t]}$ and $\gamma(dz)=d\gamma(z)$ and $\gamma(dw)=d\gamma(w)$. It follows that if $\rho:\Omega^1_{\mathcal{C}/\text{Spec}(\mathbb{Z})}\to R$ is a map corresponding to an R–valued point of A, then ρ is Γ–invariant if and only if $\rho(dz)=\rho(dw)=0$. This proves in particular the existence of the dense open subset $U\subset A$.

In fact, the space U is a quasi–projective scheme since it is a subspace of the coarse moduli space of A which is projective over S. Let $P\to\mathcal{C}$ denote $\mathbb{P}(\Omega^1_{\mathcal{C}/\text{Spec}(\mathbb{Z})}\oplus\mathcal{O}_{\mathcal{C}})$, and view U as an open substack of P. Let $S=P-U$ and for $t>0$ set $P^t:=P\times_{\mathcal{C}}\times\cdots\times_{\mathcal{C}}P$ and $S^t=S\times_{\mathcal{C}}\times\cdots\times_{\mathcal{C}}S$ (t copies). Choose t_0 sufficiently big so that the dimension of $S^{t_0}_\zeta$ is less than the all the fiber dimensions of the morphism $P^{t_0}_\zeta\to\mathcal{C}_\zeta$, and set $P=P^{t_0}$, $Q=P-S^{t_0}$. Furthermore, choose an embedding $Q\hookrightarrow\mathbb{P}^N_{\mathbb{Z}}$ for some N.

The proof of ([8], 2.1) now shows that after making an étale base change $\mathbb{Z}\to\mathcal{O}$, there exists hypersurfaces $H^1,\dots,H^r\subset\mathbb{P}^N_{\mathcal{O}}$ such that if $Z=Q\cap H^1\cap\cdots\cap H^r$ then the map $Z_\zeta\to P_\zeta$ is a closed immersion, $Z_\zeta\to\zeta$ is smooth and $Z_\zeta\to\mathcal{C}_\zeta$ is finite and flat. We can also without loss of generality assume that Z is smooth over $\text{Spec}(\mathcal{O})$. It follows from the fiber–by–fiber criterion for flatness ([5], IV.5.9), that the map $Z\to\mathcal{C}$ is flat at every point lying over $s\in S$. Furthermore, since Z and $\mathcal{C}$ are proper, we can after replacing S by a Zariski neighborhood of s assume that $Z\to\mathcal{C}$ is flat everywhere. Since this map is also proper, it follows that if S is connected then all the fibers have dimension 0 and $Z\to\mathcal{C}$ is therefore the desired finite flat cover.

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