SHEAVES ON CURVES AND THEIR LOCAL TERMS

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Contents

1. Statement of the main result

One of the many fundamental results proven in [9] is that if V is an irreducible lisse sheaf on a curve over a finite field then V is of "geometric origin". The purpose of this note is to elucidate this result, and to discuss some applications to local terms continuing our study in [12, 13, 14]. All the key ideas in this paper can be found in [9].

1.1. Let $\mathbb{F}_q$ be a finite field of characteristic p and let ℓ be a prime different from p. Fix an algebraic closure $\mathbb{F}_q \hookrightarrow k$ and let ℓ be a prime different from p.

1.2. If $\mathscr{V}$ is a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on a finite type separated Deligne-Mumford stack Y/k, we denote by $\mathscr{V}^{ss}$ the *semisimplification of* $\mathscr{V}$ in the Tannakian category of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on Y. More concretely, this semisimplification can be described as follows in the case when Y is connected. Let $\bar{y} \to Y$ be a geometric point, let G be the Tannaka dual with respect to the fiber functor defined by $\bar{y}$ of the tensor category $\langle \mathscr{V} \rangle_\otimes$ generated by $\mathscr{V}$, and let $U \subset G$ be the unipotent radical of G. Let V denote $\mathscr{V}_{\bar{y}}$ and define a filtration $F^\cdot$ on V inductively by setting F^n equal to the preimage in V of $(V/F^{n-1})^U$. Then V^{ss} is defined to be the lisse sheaf associated to the G/U-representation $\oplus_n F^n/F^{n-1}$.

If $\gamma : \mathscr{V} \to \mathscr{W}$ is a morphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on Y then there is an induced morphism $\gamma^{ss} : \mathscr{V}^{ss} \to \mathscr{W}^{ss}$.

If $K \in D^b_c(Y, \overline{\mathbb{Q}}_\ell)$ is a complex with lisse cohomology sheaves define the semisimplification K^{ss} to be the semisimplifcation of the lisse sheaf $\oplus_i \mathscr{H}^i(K)$. If $K, K' \in D^b_c(Y, \overline{\mathbb{Q}}_\ell)$ are two complexes with lisse cohomology sheaves and $\gamma : K \to K'$ is a morphism let $\mathscr{H}^*(\gamma) : \oplus_i \mathscr{H}^i(K) \to \oplus_i \mathscr{H}^i(K')$ be the map which in degree i is $(-1)^i$ times the map $\mathscr{H}^i(\gamma)$, and let $\gamma^{ss} : K^{ss} \to K'^{ss}$ be the map $\mathscr{H}^*(\gamma)^{ss}$.

1.3. Let $X/\mathbb{F}_q$ be a smooth proper geometrically connected curve. Let $I \subset X(\mathbb{F}_q)$ be a finite subset of points, and let V be a lisse irreducible sheaf on $X - I$ with trivial determinant and connected monodromy group, such that V has unipotent local monodromy around the points of I.

Theorem 1.4. *After possibly extending the ground field $\mathbb{F}_q$, there exists a proper smooth morphism $b : B \to X - I$, a lisse sheaf W on $X - I$ all of whose irreducible constituents have rank less than the rank of V, and a correspondence $\Xi \in A^d(B \times_X B)_{\overline{\mathbb{Q}}}$, where d is the dimension of B, such that if*

$$\xi : Rb_*\overline{\mathbb{Q}}_\ell \to Rb_*\overline{\mathbb{Q}}_\ell$$

is the induced endomorphism and $q : V \to V \oplus W$ is the inclusion given by the identity map to the first factor, then we have an inclusion $\iota : V \oplus W \hookrightarrow (Rb_*\overline{\mathbb{Q}}_\ell)^{ss}$ and a morphism $r : (Rb_*\overline{\mathbb{Q}}_\ell)^{ss} \to V \oplus W$ such that the diagram

$$\begin{tikzcd}
V \arrow[r, hook, "q"] \arrow[dr, hook, "q"'] & V \oplus W \arrow[r, hook, "\iota"] & (Rb_*\overline{\mathbb{Q}}_\ell)^{ss} \arrow[dl, "r"'] \arrow[d, "\xi^{ss}"] \\
 & V \oplus W \arrow[r, hook, "\iota"] & (Rb_*\overline{\mathbb{Q}}_\ell)^{ss}
\end{tikzcd}$$

commutes.

Remark 1.5. Note that since b is proper and smooth the cohomology sheaves $R^ib_*\overline{\mathbb{Q}}_\ell$ are lisse sheaves on $X - I$.

Remark 1.6. Here the map ξ is defined as follows. If $Z \subset X \times_B X$ is an irreducible scheme of dimension d, then we get a commutative diagram

$$\begin{tikzcd}
 & Z \arrow[dl, "z_1"'] \arrow[dd] \arrow[dr, "z_2"] & \\
B \arrow[dd, "b"] & & B \arrow[dd, "b"] \\
 & X - I \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\
X - I & & X - I.
\end{tikzcd}$$

Pushing forward the standard action of Z on $\overline{\mathbb{Q}}_{\ell,B}$ we therefore get an endomorphism of $Rb_*\overline{\mathbb{Q}}_{\ell,B}$. Extending this linearly any cycle $\Xi \in A^d(B \times_X B)_{\overline{\mathbb{Q}}}$ therefore defines an endomorphism of $Rb_*\overline{\mathbb{Q}}_{\ell,B}$.

2. Representations of Hecke algebras

In this subsection we review for the convenience of the reader some basic facts about Hecke algebras as they arise in Lafforgue's work. More details can be found in [10].

2.1. Let $\mathbb{F}_q$ be a finite field of characteristic p, and let $X/\mathbb{F}_q$ be a smooth proper geometrically connected curve. Let X_{cl} denote the set of closed points of X, and for $x \in X_{\mathrm{cl}}$ let F_x denote the completion of the function field F of X with respect to the discrete valuation defined by x and let $\mathscr{O}_x \subset F_x$ be the ring of integers. Let $\mathbb{A} = \prod'_{x \in X_{\mathrm{cl}}} F_x$ denote the ring of adeles of F, and let $\mathscr{O}_{\mathbb{A}}$ denote $\prod_{x \in X_{\mathrm{cl}}} \mathscr{O}_x$.

Fix an integer r and let $G = GL_r$. Let $\mathcal{H}^r$ denote the $\mathbb{Q}$-vector space of locally constant functions $f : G(\mathbb{A}) \to \mathbb{Q}$ with compact support. Let K denote $G(\mathscr{O}_{\mathbb{A}})$, and fix a Haar measure $\boldsymbol{\mu}$ on $G(\mathbb{A})$ such that $\boldsymbol{\mu}(K) = 1$. Then $\mathcal{H}^r$ becomes a $\mathbb{Q}$-algebra with multiplication defined by convolution:

$$f * g(x) := \int_{G(\mathbb{A})} f(y) g(y^{-1}x) d\boldsymbol{\mu}(y).$$

For a finite subscheme $I = \mathrm{Spec}(\mathscr{O}_I) \subset X_{\mathrm{cl}}$ let K_I denote the kernel of the restriction map $K \to G(\mathscr{O}_I)$. Let $f_I \in \mathcal{H}^r$ denote the characteristic function of K_I. The element f_I is idempotent, the subalgebra

$$f_I \mathcal{H}^r f_I \subset \mathcal{H}^r$$

is the subalgebra $\mathcal{H}^r_I \subset \mathcal{H}^r$ of functions which are left and right K_I-invariant, and

$$\mathcal{H}^r = \cup_I \mathcal{H}^r_I,$$

where the union is taken over finite subschemes $I \subset X$.

2.2. For a right $\mathcal{H}^r$-module M, let M_I denote the right $\mathcal{H}^r_I$-module $M \cdot f_I$. If L is a field of characteristic 0 and M is a right $\mathcal{H}^r$-module over L, then M is called *admissible* (see [10, IV, §3 (a), Définition 1]) if each M_I is finite dimensional over L and $M = \cup_I M_I$.

2.3. Let Aut^r denote the space of all cuspidal automorphic forms $f : G(\mathbb{A}) \to \mathbb{C}$, as defined for example in [9, VI.1d]. For an element $a \in \mathbb{A}^*$ of positive degree let $\mathrm{Aut}^r_a \subset \mathrm{Aut}^r$ denote the subspace of functions f for which $f(ga) = f(a)$ for all $g \in G(\mathbb{A})$. Then Aut^r_a is an admissible right $\mathcal{H}^r$-module with action defined by convolution.

An *irreducible automorphic cuspidal representation of* $G(\mathbb{A})$ is an irreducible representation M of $G(\mathbb{A})$ over $\mathbb{C}$ which occurs as a direct summand of Aut^r, or equivalently as a direct summand of Aut^r_a for some $a \in \mathbb{A}^*$ of positive degree. See [8, §1.1] for further discussion.

Lemma 2.4. *Let $L \subset \mathbb{C}$ be an algebraically closed subfield, and let M be an irreducible automorphic cuspidal representation defined over L. Let $I \subset X$ be a finite subscheme such that $M_I \neq 0$, and let $\overline{\mathbb{Q}} \subset L$ be the algebraic closure of $\mathbb{Q}$ in L. Then there exists an element $f \in \mathcal{H}^r_I \otimes \overline{\mathbb{Q}}$ and a 1-dimensional subspace $T \subset M_I$ such that the action of f on M_I splits the inclusion $T \hookrightarrow M_I$ (i.e. f maps M_I to T and restricts to the identity on T).*

Proof. By [10, IV, §4, Proposition 4] (see also the discussion in the proof of [9, Théorème VII.6]), the representation M is defined over $\overline{\mathbb{Q}}$, so it suffices to consider the case of $L = \overline{\mathbb{Q}}$. By [10, IV, §3, Proposition 2], the finite dimensional representation M_I of $\mathcal{H}^r_I$ is irreducible.

Let $\mathcal{A} \subset \mathrm{End}_L(M_I)$ denote the image of $\mathcal{H}_I^r$. Then M_I is an irreducible representation of $\mathcal{A}$, so by Burnside's lemma [11, Chapter 3, 7.3] the map

$$\mathcal{A} \to \mathrm{End}_L(M_I)$$

is an isomorphism. □

Lemma 2.5. *Let $L \subset \mathbb{C}$ be an algebraically closed subfield, and let $M_0, M_1, \ldots, M_t$ distinct irreducible automorphic cuspidal representations defined over L. Let $I \subset X$ be a finite subscheme such that $M_{j,I} \neq 0$ for all j. Then there exists an element $\alpha \in \mathcal{H}_I^r \otimes \overline{\mathbb{Q}}$ such that α acts as the identity on $M_{0,I}$ and as the zero map on $M_{j,I}$ for $j \geq 1$.*

Proof. As in the proof of 2.4 it suffices to consider the case when $L = \overline{\mathbb{Q}}$. Let $\mathfrak{m}_j \subset \mathcal{H}_I^r \otimes \overline{\mathbb{Q}}$ denote the kernel of the map

$$\mathcal{H}_I^r \otimes \overline{\mathbb{Q}} \to \mathrm{End}_L(M_{j,I}).$$

As in the proof of 2.4 this map is surjective so $\mathfrak{m}_j$ is a maximal 2-sided ideal and these maximal ideals are distinct. We can therefore choose an element $\alpha \in \mathcal{H}_I^r \otimes \overline{\mathbb{Q}}$ such that $\alpha \equiv 1 \pmod{\mathfrak{m}_0}$ and $\alpha \in \mathfrak{m}_1 \cdots \mathfrak{m}_t$ (indeed for each $i = 1, \ldots, t$ we have $1 \in \mathfrak{m}_0 + \mathfrak{m}_i$ so we can find $\alpha_i \in \mathfrak{m}_i$ with $\alpha_i \equiv 1 \pmod{\mathfrak{m}_0}$; now multiply the α_i together). □

3. The stack of chtoucas

3.1. Let $\mathbb{F}_q$ be a finite field of characteristic p, and let $X/\mathbb{F}_q$ be a smooth proper geometrically connected curve over $\mathbb{F}_q$. For a $\mathbb{F}_q$-scheme U let X_U denote the fiber product $X \times U$. For a locally free sheaf $\mathscr{E}$ on X_U let ${}^\tau\mathscr{E}$ denote the locally free sheaf $(\mathrm{id}_X \times F_U)^*\mathscr{E}$, where F_U denotes the q-power Frobenius morphism on U. A *chtouca of rank r over U* is a pair of inclusions of locally free sheaves on X_U of rank r

$$\mathscr{E} \xhookrightarrow{j} \mathscr{E}' \xhookleftarrow{t} {}^\tau\mathscr{E} \tag{3.1.1}$$

such that the cokernels $\mathscr{E}'/\mathscr{E}$ and $\mathscr{E}'/{}^\tau\mathscr{E}$ are isomorphic to pushforwards of locally free sheaves on U along sections $s_0 : U \to X_U$ and $s_\infty : U \to X_U$ respectively. Let Cht^r denote the stack over $\mathbb{F}_q$ which to any U associates the groupoid of chtoucas of rank r over U. By [10, I, §3, Corollaire 6] Cht^r is a smooth Deligne-Mumford stack locally of finite type over $\mathbb{F}_q$. There is a morphism

$$\pi : \mathrm{Cht}^r \to X \times X \tag{3.1.2}$$

sending a chtouca over U to the pair of sections (s_0, s_∞). This morphism is smooth of relative dimension $2r - 2$ by [10, I, §2, Théorème 9].

The stack Cht^r is a disjoint union of substacks $\mathrm{Cht}^{r,d}$ $(d \in \mathbb{Z})$, where $\mathrm{Cht}^{r,d}$ classifies rank r chtoucas of rank r and with $\deg(\mathscr{E}) = d$.

3.2. Let U be an $\mathbb{F}_q$-scheme and let (3.1.1) be a chtouca of rank r over U. Let $I \subset X$ be a finite subscheme such that the supports of $\mathscr{E}'/\mathscr{E}$ and $\mathscr{E}'/{}^\tau\mathscr{E}$ do not meet I. Then we get isomorphisms of locally free sheaves over $I_U := I \times U$

$$\mathscr{E}_{I_U} \xrightarrow{j} \mathscr{E}'_{I_U} \xrightarrow{t^{-1}} {}^\tau\mathscr{E}_{I_U},$$

where $\mathscr{E}_{I_U}$ denotes the restriction of $\mathscr{E}$ to I_U. Write also ${}^\tau\mathscr{E}_{I_U}$ for the locally free sheaf $(\mathrm{id}_I \times F_U)^*\mathscr{E}_{I_U}$. There is a canonical isomorphism ${}^\tau(\mathscr{O}^r_{I_U}) \simeq \mathscr{O}^r_{I_U}$. In particular, an isomorphism $\lambda : \mathscr{O}^r_{I_U} \simeq \mathscr{E}_{I_U}$ defines an isomorphism ${}^\tau\lambda : \mathscr{O}^r_{I_U} \simeq {}^\tau\mathscr{E}_{I_U}$.

Definition 3.3. An *I-level structure* on the chtouca (3.1.1) over U is an isomorphism $\lambda : \mathscr{O}^r_{I_U} \to \mathscr{E}_{I_U}$ such that the diagram

(3.3.1)
$$\begin{tikzcd} \mathscr{E}_{I_U} \arrow[rr, "t^{-1}\circ j"] & & {}^\tau\mathscr{E}_{I_U} \\ & \mathscr{O}^r_{I_U} \arrow[ul, "\lambda"] \arrow[ur, "{}^\tau\lambda"'] & \end{tikzcd}$$

commutes.

3.4. For a fixed chtouca (3.1.1) over U the functor on the category of U-schemes sending U'/U to the set of I-level structures on the pullback of the chtouca to U' is a torsor under $GL_r(\mathscr{O}_I)$. We therefore obtain a morphism of stacks

$$\chi : \mathrm{Cht}^r \times_{X^2} (X - I)^2 \to BGL_r(\mathscr{O}_I).$$

Let

$$\mathrm{Cht}^r_I \to \mathrm{Cht}^r \times_{X^2} (X - I)^2$$

denote the corresponding $GL_r(\mathscr{O}_I)$-torsor, and let $\mathrm{Cht}^{r,d}_I$ denote the preimage of $\mathrm{Cht}^{r,d}$.

3.5. Let $I \subset X$ be a finite subscheme, and consider the double cosets

(3.5.1)
$$F^*\backslash\mathbb{A}^*/\mathrm{Ker}(\mathscr{O}^*_{\mathbb{A}} \to \mathscr{O}^*_I).$$

There is a map $\alpha : \mathbb{A}^* \to \mathrm{Pic}(X)$ defined as follows. For a point $x \in X_{\mathrm{cl}}$ let I_x denote the ideal sheaf of x, and let ν_x denote the valuation on F_x. Then define α by sending $(f_x)_x \in \mathbb{A}^*$ to the line bundle $\otimes_{x\in X_{\mathrm{cl}}} I_x^{-\nu_x(f_x)}$. Note that the elements f_x^{-1} define trivializations of the stalk at each x of the line bundle $\alpha((f_x))$, and elements of $F^* \subset \mathbb{A}^*$ map to trivial line bundles. It follows that α defines an isomorphism between the double cosets (3.5.1) and the set of isomorphism classes of pairs (L, σ), where L is a line bundle on X and $\sigma : L_I \to \mathscr{O}_I$ is a trivialization of the restriction of L to I.

3.6. Fix an element $a \in \mathbb{A}^*$ of degree 1, and let (L, σ) be the corresponding line bundle with trivialization over I. Then we get an automorphism

$$a : \mathrm{Cht}^r_I \to \mathrm{Cht}^r_I$$

sending a chtouca (3.1.1) with trivalization $\lambda : \mathscr{O}^r_I \to \mathscr{E}_I$ such that the diagram (3.3.1) commutes, to the chtouca (note that ${}^\tau\mathscr{E} \otimes L \simeq {}^\tau(\mathscr{E} \otimes L)$)

$$\mathscr{E} \otimes L \xhookrightarrow{j} \mathscr{E}' \otimes L \xhookleftarrow{t} {}^\tau\mathscr{E} \otimes L$$

with the trivialization $\lambda \otimes \sigma : \mathscr{O}^r_I \to \mathscr{E}_I \otimes L_I$. Note that this automorphism of Cht^r_I sends $\mathrm{Cht}^{r,d}_I$ to $\mathrm{Cht}^{r,d+r}_I$. We write also

$$f : \mathrm{Cht}^r_I/a^{\mathbb{Z}} \to (X - I)^2$$

for the morphism induced by the map (3.1.2).

3.7. The stack $\mathrm{Cht}^{r,d}$ is not of finite type unless $r=1$. As explained in [9, Proposition I.3], however, it is a filtering union of finite type substacks $\mathrm{Cht}^{r,d;\leq p}$, where p runs over certain piecewise linear functions $p:[0,r]\to\mathbb{R}_{\geq 0}$. We denote by $\mathrm{Cht}^{r;\leq p}\subset \mathrm{Cht}^r$ the substack $\coprod_d \mathrm{Cht}^{r,d;\leq p}$, and by $\mathrm{Cht}_I^{r,d;\leq p}\subset \mathrm{Cht}_I^{r,d}$ (resp. $\mathrm{Cht}_I^{r;\leq p}\subset \mathrm{Cht}_I^r$) the preimage of $\mathrm{Cht}^{r,d;\leq p}$ (resp. $\mathrm{Cht}^{r;\leq p}$).

Assuming that I is reduced, let $\tilde{\pi}:\widetilde{\mathrm{Cht}}_I^{r,d;\leq p}\to (X-I)^2$ denote one of the compactifications of $\pi:\mathrm{Cht}_I^{r,d;\leq p}\to (X-I)^2$ constructed in [9, p. 86]. So $\tilde{\pi}$ is proper and smooth, the complement $D^{r,d;\leq p}$ of $\mathrm{Cht}_I^{r,d;\leq p}$ in $\widetilde{\mathrm{Cht}}_I^{r,d;\leq p}$ is a divisor with normal crossings relative to $(X-I)^2$, and each irreducible component of this divisor is smooth over $(X-I)^2$. This implies in particular that the sheaves $R^a\pi_*\overline{\mathbb{Q}}_\ell$ and $R^a\pi_!\overline{\mathbb{Q}}_\ell$ on $(X-I)^2$ are lisse sheaves.

The substacks $\mathrm{Cht}_I^{r;\leq p}$ are stable under the action of $a^{\mathbb{Z}}$, so we get quasi-compact substacks

$$\mathrm{Cht}_I^{r;\leq p}/a^{\mathbb{Z}}\hookrightarrow \mathrm{Cht}_I^r/a^{\mathbb{Z}}.$$

Let

$$f^{\leq p}:\mathrm{Cht}_I^{r;\leq p}/a^{\mathbb{Z}}\to (X-I)^2$$

denote the restriction of f.

4. Negligible sheaves

4.1. Throughout this section we work with a finite field $\mathbb{F}_q$ of characteristic p, an algebraic closure $\mathbb{F}_q\hookrightarrow k$, a smooth proper geometrically connected curve $X/\mathbb{F}_q$, and a reduced finite subscheme $I\subset X$. We write X_{cl} for the set of closed points of X.

4.2. Recall from [9, Définition VI.14] that a lisse sheaf V on $(X-I)^2$ is r-*negligible* if every irreducible subquotient of V is direct summand of a sheaf of the form $L_1\boxtimes L_2$ for lisse sheaves L_1 and L_2 on $X-I$ of rank $<r$.

As discussed in [9, Remarque following VI.14], if L_1 and L_2 are irreducible lisse sheaves on $X-I$, the sheaf $L_1\boxtimes L_2$ is semisimple but need not be irreducible. The irreducible factors are indexed by unramified rank 1 sheaves M such that $L_1\otimes M^{-1}\simeq L_1$ and $L_2\otimes M\simeq L_2$. Notice that the monodromy group of such an M is finite so if L_1 has connected arithmetic monodromy group then no such nontrivial M exist and $L_1\boxtimes L_2$ is irreducible.

4.3. If $\mathbb{F}_q\hookrightarrow \mathbb{F}_{q'}$ is a field extension, and if V is an r-negligible sheaf on $(X-I)^2$, then the pullback of V to $(X-I)^2\otimes\mathbb{F}_{q'}$ is also r-negligible.

4.4. With notation as in 3.7, for $s\geq 0$ we have a decomposition $(R^sf_!^{\leq p}\overline{\mathbb{Q}}_\ell)^{ss}\simeq E^{s,\leq p}\oplus R^{s,\leq p}$, where $R^{s,\leq p}$ is r-negligible and no subquotient of $E^{s,\leq p}$ is r-negligible. By [9, Proposition VI.20 (ii)], we have $(R^sf_!^{\leq p}\overline{\mathbb{Q}}_\ell)^{ss}=R^{s,\leq p}$ for $s\neq 2r-2$. Furthermore, by [9, Corollaire VI.21] for $p\leq p'$ the natural map

$$(R^{2r-2}f_!^{\leq p}\overline{\mathbb{Q}}_\ell)^{ss}\to (R^{2r-2}f_!^{\leq p'}\overline{\mathbb{Q}}_\ell)^{ss}$$

induces an isomorphism $E^{2r-2,\leq p}\to E^{2r-2,\leq p'}$.

The action of $a^{\mathbb{Z}}$ on $\coprod_d \mathrm{Cht}_I^{r,d;\leq p}$ extends to an action on $\widetilde{\mathrm{Cht}}_I^{r;\leq p}:=\coprod_d\widetilde{\mathrm{Cht}}_I^{r,d;\leq p}$, and we can consider

$$\tilde{f}^{\leq p}:\widetilde{\mathrm{Cht}}_I^{r;\leq p}/a^{\mathbb{Z}}\to (X-I)^2,$$

and we can again consider the decomposition of $(R^s\tilde{f}_!^{\leq p}\overline{\mathbb{Q}}_\ell)^{ss} = \widetilde{E}^{s,\leq p}\oplus\widetilde{R}^{s,\leq p}$ of $(R^s\tilde{f}_!^{\leq p}\overline{\mathbb{Q}}_\ell)^{ss}$ into the essential and negligible part. By [9, VI.15 and p. 181 Remarque] we again have $\widetilde{E}^{s,\leq p} = 0$ for $s \neq 2r-2$ and the natural map

$$(R^{2r-2}f_!^{\leq p}\overline{\mathbb{Q}}_\ell)^{ss} \to (R^s\tilde{f}_!^{\leq p}\overline{\mathbb{Q}}_\ell)^{ss}$$

induces an isomorphism $E^{2r-2,\leq p} \to \widetilde{E}^{2r-2,\leq p}$.

5. Hecke operators

We continue with the notation and assumptions of 4.1.

5.1. Let E^{2r-2} denote the lisse sheaf $\varinjlim_p E^{2r-2,\leq p}$ on $X - I$. For any p sufficiently convex the map $E^{2r-2,\leq p} \to E^{2r-2}$ is an isomorphism. There is an action of the Hecke algebra $\mathcal{H}_I^r$ on E^{2r-2} constructed as follows. For an element $f \in \mathcal{H}_I^r$ Lafforgue constructs in [9, I, §1 (c)] and [10, I, §4 (c)] a finite subset $T_f \subset X_{\mathrm{cl}}$ and a cycle

$$\Gamma_I^r(f) = \sum_i \lambda_i\Gamma_I^r(g_i) \in Z^{2r}((\mathrm{Cht}_I^r/a^{\mathbb{Z}} \times \mathrm{Cht}_I^r/a^{\mathbb{Z}}) \times_{X^2\times X^2} ((X-T_f)^2 \times (X-T_f)^2)) \otimes \mathbb{Q},$$

where the two projections $\Gamma_I^r(g_i) \to \mathrm{Cht}_I^r/a^{\mathbb{Z}} \times_{X^2} (X-T_f)^2$ are finite and étale. This cycle defines an endomorphism of E^{2r-2} as follows.

The open substacks $\mathrm{Cht}_I^{r;\leq p}/a^{\mathbb{Z}} \subset \mathrm{Cht}_I^r/a^{\mathbb{Z}}$ are not stable under the correspondences $\Gamma_I^r(f)$. Let $\Gamma_I^r(f)^{(1)_p}$ denote the preimage of $\mathrm{Cht}^{r;\leq p}$ under the first projection, and let $\Gamma_I^r(f)^{\leq p}$ denote the restriction of $\Gamma_I^r(f)$ to

$$\mathrm{Cht}_I^{r;\leq p}/a^{\mathbb{Z}} \times \mathrm{Cht}_I^{r;\leq p}/a^{\mathbb{Z}}.$$

Since $\Gamma_I^r(f)^{(1)_p}$ is quasi-compact, there exists $q \geq p$ such that the second projection $\Gamma_I^r(f)^{(1)_p} \to \mathrm{Cht}_I^r/a^{\mathbb{Z}}$ factors through $\mathrm{Cht}_I^{r;\leq q}/a^{\mathbb{Z}}$, and the resulting correspondence

$$\begin{tikzcd} & \Gamma_I^r(f)^{(1)_p} \arrow[dl] \arrow[dr] & \\ \mathrm{Cht}_I^{r;\leq p}/a^{\mathbb{Z}} & & \mathrm{Cht}_I^{r;\leq q}/a^{\mathbb{Z}} \end{tikzcd}$$

defines a morphism $E^{2r-2,\leq p} \to E^{2r-2,\leq q}$. Upon passing to the limit these maps defines the action of f on E^{2r-2}, and as shown in [9, VI, §3 (a)] this defines an action of $\mathcal{H}_I^r$ on E^{2r-2}.

5.2. Fix $p \leq q$ as above, and consider the compactification $\mathrm{Cht}^{r;\leq q}/a^{\mathbb{Z}} \hookrightarrow \widetilde{\mathrm{Cht}}^{r;\leq q}/a^{\mathbb{Z}}$. Let $\widetilde{\Gamma}_I^r(g_i)^{\leq q}$ be any compactification of $\Gamma_I^r(g_i)^{\leq q}$ to a correspondence

$$\begin{tikzcd} & \widetilde{\Gamma}_I^r(g_i)^{\leq q} \arrow[dl] \arrow[dr] & \\ \widetilde{\mathrm{Cht}}^{r;\leq q}/a^{\mathbb{Z}} & & \widetilde{\mathrm{Cht}}^{r;\leq q}/a^{\mathbb{Z}}, \end{tikzcd}$$

with $\Gamma^r_I(g_i) \hookrightarrow \widetilde{\Gamma}^r_I(g_i)$ a dense open substack. This correspondence defines an endomorphism $\tilde{g}_i : \widetilde{E}^{2r-2;\leq q} \to \widetilde{E}^{2r-2;\leq q}$. Since the preimage in $\widetilde{\Gamma}^r_I(g_i)^{\leq q}$ of $\mathrm{Cht}^{r;\leq p}_I/a^{\mathbb{Z}}$ under the first projection is equal to $\Gamma^r_I(g_i)^{(1)_p}$, we have a commutative diagram

$$\begin{tikzcd} E^{2r-2;\leq p} \arrow[r, "g_i"] \arrow[d] & E^{2r-2;\leq q} \arrow[d] \\ \widetilde{E}^{2r-2;\leq q} \arrow[r, "\tilde{g}_i"] & \widetilde{E}^{2r-2;\leq q}, \end{tikzcd}$$

where the vertical arrows denote the morphisms induced by inclusion. It follows that for any element $f \in \mathcal{H}^r_I$, the action of f on $\widetilde{E}^{2r-2,\leq q}$ induced by the isomorphism

$$E^{2r-2} \simeq \widetilde{E}^{2r-2,\leq q} \tag{5.2.1}$$

defined by the inclusions

$$\mathrm{Cht}^r_I/a^{\mathbb{Z}} \hookleftarrow \mathrm{Cht}^{r;\leq q}_I/a^{\mathbb{Z}} \hookrightarrow \widetilde{\mathrm{Cht}}^{r;\leq q}/a^{\mathbb{Z}}$$

is given by a correspondence $\widetilde{\Gamma}^r_I(f)$ on $\widetilde{\mathrm{Cht}}^r_I/a^{\mathbb{Z}}$.

6. Partial Frobenius morphisms

6.1. For a chtouca (3.1.1) over an $\mathbb{F}_q$-scheme U, let $\mathscr{E}''$ denote the pushout of the diagram

$$\begin{tikzcd} {}^\tau\mathscr{E} \arrow[r, hook, "t"] \arrow[d, hook, "{}^\tau j"] & \mathscr{E}' \\ {}^\tau\mathscr{E}' & \end{tikzcd}$$

so we have inclusions

$$\mathscr{E}' \xhookrightarrow{j'} \mathscr{E}'' \xhookleftarrow{t'} {}^\tau\mathscr{E}'. \tag{6.1.1}$$

If the sections (s_0, s_∞) are furthermore disjoint, then this is again a chtouca with image in $X \times X$ equal to $(F^*_U(s_0), s_\infty)$. Let T denote the $GL(\mathscr{O}_I)$-torsor of I-level structures on (3.1.1) and let T' denote the $GL(\mathscr{O}_I)$-torsor of I-level structures on (6.1.1). The map $j_I : \mathscr{E}_I \simeq \mathscr{E}'_I$ defines an isomorphism $T \simeq T'$. Therefore an I-level structure on (3.1.1) defines an I-level structure on (6.1.1) and we obtain a morphism

$$F_0 : \mathrm{Cht}^r_I \times_{(X-I)^2} ((X-I)^2 - \Delta) \to \mathrm{Cht}^r_I$$

over the morphism $F_X \times \mathrm{id} : X \times X \to X \times X$. This morphism is compatible with the action of $a^{\mathbb{Z}}$ and induces a morphism, which we denote by the same letter,

$$F_0 : \mathrm{Cht}^r_I/a^{\mathbb{Z}} \times_{(X-I)^2} ((X-I)^2 - \Delta) \to \mathrm{Cht}^r_I/a^{\mathbb{Z}}.$$

6.2. Similarly for a chtouca (3.1.1) we can consider the fiber product $\mathscr{F}$ of the diagram

$$\begin{tikzcd} & \mathscr{E} \arrow[d, hook, "j"] \\ {}^\tau\mathscr{E} \arrow[r, hook] & \mathscr{E}'. \end{tikzcd}$$

Let $a : \mathscr{F} \hookrightarrow {}^\tau\mathscr{E}$ and $b : \mathscr{F} \hookrightarrow \mathscr{E}$ be the inclusions. If the sections s_0 and s_∞ are disjoint then

$$\mathscr{F} \xhookrightarrow{a} {}^\tau\mathscr{E} \xhookleftarrow{{}^\tau b} {}^\tau\mathscr{F}$$

is a chtouca. As above this construction induces morphisms

$$F_\infty : \mathrm{Cht}_I^r \times_{(X-I)^2} ((X-I)^2 - \Delta) \to \mathrm{Cht}_I^r,$$

$$F_\infty : \mathrm{Cht}_I^r/a^{\mathbb{Z}} \times_{(X-I)^2} ((X-I)^2 - \Delta) \to \mathrm{Cht}_I^r/a^{\mathbb{Z}}$$

over $\mathrm{id} \times F_X : X \times X \to X \times X$.

6.3. The morphisms F_0 and F_∞ induce isomorphisms

$$\gamma_0 : (F_X \times \mathrm{id})^* E^{2r-2} \to E^{2r-2}$$

and

$$\gamma_\infty : (\mathrm{id} \times F_X)^* E^{2r-2} \to E^{2r-2}$$

which commute with each other and the action of $\mathcal{H}_I^r$, and such that $\gamma_0 \circ \gamma_\infty$ is the Frobenius morphism on E^{2r-2}.

6.4. Let

$$\begin{tikzcd} & \Gamma_\infty \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\ \mathrm{Cht}_I^r/a^{\mathbb{Z}} \times_{(X-I)^2} ((X-I)^2 - \Delta) & & \mathrm{Cht}_I^r/a^{\mathbb{Z}} \end{tikzcd}$$

denote the correspondence defined by F_∞ so α_1 is an isomorphism, and $\alpha_2 \circ \alpha_1^{-1} = F_\infty$, and let $\Gamma_\infty^{(1)_p}$ denote $\alpha_1^{-1}(\mathrm{Cht}^{r,\leq p}/a^{\mathbb{Z}})$. Then $\Gamma_\infty^{(1)_p}$ is quasi-compact so has image under α_2 contained in $\mathrm{Cht}^{r;\leq q}/a^{\mathbb{Z}}$ for some $q \geq p$.

Any choice of compactification of this correspondence to a correspondence

$$\begin{tikzcd} & \widetilde{\Gamma}_\infty \arrow[dl, "\tilde{\alpha}_1"'] \arrow[dr, "\tilde{\alpha}_2"] & \\ \widetilde{\mathrm{Cht}}_I^{r;\leq q}/a^{\mathbb{Z}} & & \widetilde{\mathrm{Cht}}_I^{r;\leq q}/a^{Z} \end{tikzcd}$$

induces on $\widetilde{E}^{2r-2,\leq q}$ an isomorphism $\tilde{\gamma}_\infty : (\mathrm{id} \times F_X)^* \widetilde{E}^{2r-2,\leq q} \to \widetilde{E}^{2r-2,\leq q}$ which under the isomorphism (5.2.1) agrees with γ_∞.

7. Proof of 1.4

7.1. Since V has unipotent local monodromy at the points of I, it corresponds via the Langlands correspondence to a representation π of $\mathcal{H}_I^r$ (see for example [4, §9.7]). The semisimplification of E^{2r-2} in the category of lisse sheaves with $\mathcal{H}_I^r$-action then has π-factor of the form

$$V \boxtimes V^\vee \boxtimes \pi.$$

Lemma 7.2. *There exists a closed point $x \in X - I$ such that the Frobenius automorphism F_x of $V_{\bar{x}}$ has an eigenvalue that occurs with multiplicity one.*

Proof. After replacing $\mathbb{F}_q$ by a field extension we may assume we have a point $x_0 \in X - I$. Let G be the monodromy group, which is connected by assumption. Let $V_{\bar{x}}$ be the irreducible representation of G corresponding to V. Since the set of Frobenius elements in G are dense and the condition that an element $g \in G$ has an eigenvalue that occurs with multiplicity one is an open condition it suffices to exhibit a single element of G which has an eigenvalue occuring with multiplicity one.

For this fix a Borel subgroup $B \subset G$ containing a maximal torus T. Let $v \in V_{\bar{x}}$ be a highest weight vector and let χ be the corresponding character of T. Then the χ-eigenspace of $V_{\bar{x}}$ is 1-dimensional, and therefore there exists an element $u \in T$ such that the $\chi(u)$-eigenspace of u acting on $V_{\bar{x}}$ is 1-dimensional. □

Replacing $\mathbb{F}_q$ by a field extension we now choose a point $x \in X - I$ as in 7.2.

Let $b : B \to X - I$ be the restriction of $\tilde{f}^{\leq p} : \widetilde{\mathrm{Cht}}_I^{r;\leq p}/a^{\mathbb{Z}} \to (X-I)^2$ to $(X - I) \times \{x\}$. The endomorphism γ_∞ restricts to an endomorphism of $(Rb_*\overline{\mathbb{Q}}_\ell)^{ss}$ which is defined by an algebraic cycle. Furthermore, the action of $\mathcal{H}_I^r$ restricts to an action on this sheaf, with operators acting by algebraic cycles.

Composing the operators $\gamma_\infty - \lambda$ for all but one of the multiplicity one eigenvalues of Frobenius on $V_{\bar{x}}^\vee$, we get an operator whose image when restricted to $V \boxtimes V_{\bar{x}}^\vee \boxtimes \pi$ is contained in $V \boxtimes L \boxtimes \pi$, where $L \subset V_{\bar{x}}^\vee$ is a 1-dimensional subspace. Using 2.4 and 2.5 choose an element $f \in \mathcal{H}_I^r$ such that f is the projection onto a 1-dimensional subspace $T \subset \pi$ and such that f acts by 0 on the other irreducible representations of $\mathcal{H}_I^r$ occurring in E^{2r-2}. Then composing our operator with f we get a projection

$$(Rb_*\overline{\mathbb{Q}}_\ell)^{ss} \to (V \boxtimes L \boxtimes T) \oplus W,$$

compatible with the inclusion $V \boxtimes L \boxtimes T \hookrightarrow (Rb_*\overline{\mathbb{Q}}_\ell)^{ss}$ and where the irreducible constituents of W have rank less than the rank of V. This completes the proof of 1.4. □

8. An equivariant generalization

8.1. Let $X/\mathbb{F}_q$ be a smooth projective curve, and assume further given a finite order automorphism $\gamma : X \to X$ of the curve. Let n be the order of γ and let $\langle\gamma\rangle$ denote the group of automorphisms generated by γ, so we have an action of $\langle\gamma\rangle$ on X. Let $\pi : X \to Y$ denote the quotient $X/\langle\gamma\rangle$.

Let $I \subset X(\mathbb{F}_q)$ be a finite $\langle\gamma\rangle$-invariant subset and let V be a lisse irreducible sheaf with trivial determinant on $X - I$, connected geometric monodromy group, and unipotent local monodromy at the points of I. Let $J \subset Y(\mathbb{F}_q)$ be the image of I so $I = \pi^{-1}(J)$ and we have a finite map $\pi' : X - I \to Y - J$. After shrinking on Y and extending the ground field $\mathbb{F}_q$ we may assume that π' is a finite étale Galois cover of $Y - J$ with group $\langle\gamma\rangle$. Assume further given a $\langle\gamma\rangle$-linearization $\sigma : \gamma^*V \to V$ and let $\overline{V}$ denote the corresponding sheaf on $Y - J$.

Theorem 8.2. *After possibly making a field extension of $\mathbb{F}_q$, there exists a proper smooth morphism $b : B \to X - I$, an integer s, and cycles $\Xi \in A^d(B \times_{b,X,b} B)_{\overline{\mathbb{Q}}}$ and $\Gamma \in A^d(B \times_{\gamma b,X,b} B)$, where d is the dimension of B, such that if*

$$\xi : Rb_*\overline{\mathbb{Q}}_\ell \to Rb_*\overline{\mathbb{Q}}_\ell, \quad \sigma : \gamma^* Rb_*\overline{\mathbb{Q}}_\ell \to Rb_*\overline{\mathbb{Q}}_\ell$$

are the induced endomorphisms and $q : V \hookrightarrow V \oplus W$ the inclusion given by the identity map on the first factor, then there exists a lisse sheaf W on $X - I$ all of whose irreducible constituents have rank less than the rank of V, an inclusion $\iota : V \oplus W \hookrightarrow (Rb_*\overline{\mathbb{Q}}_\ell)^{ss}$ and a morphism $r : (Rb_*\overline{\mathbb{Q}}_\ell)^{ss} \to V \oplus W$ such that the diagrams

$$\begin{tikzcd} V \arrow[r, hook, "q"] \arrow[dr, hook, "q"'] & V \oplus W \arrow[r, hook, "\iota"] & (Rb_*\overline{\mathbb{Q}}_\ell)^{ss} \arrow[dl, "r"'] \arrow[d, "\xi^{ss}"] \\ & V \oplus W \arrow[r, hook, "\iota"] & (Rb_*\overline{\mathbb{Q}}_\ell)^{ss} \end{tikzcd}$$

and

$$\begin{tikzcd} \gamma^*V \arrow[r, hook, "\gamma^* q"] \arrow[ddr, "\sigma"] & \gamma^*(V \oplus W) \arrow[r, hook, "\gamma^*\iota"] & \gamma^*(Rb_*\overline{\mathbb{Q}}_\ell)^{ss} \arrow[d, "\gamma^{ss}"] \\ & & (Rb_*\overline{\mathbb{Q}}_\ell)^{ss} \arrow[d, "r"] \\ & V \arrow[r, hook, "q"] & V \oplus W \end{tikzcd}$$

commute.

The rest of this section is devoted to the proof.

8.3. First note that since γ is defined over $\mathbb{F}_q$ for a locally free sheaf $\mathscr{E}$ on $X \times S$ we have $\gamma^*({}^\tau\mathscr{E}) \simeq {}^\tau(\gamma^*\mathscr{E})$. This defines a functor

$$\tilde{\gamma} : \mathrm{Cht}^r \to \mathrm{Cht}^r$$

over the map $\gamma \times \gamma : X \times X \to X \times X$ sending a chtouca 3.1.1 over a scheme U to the chtouca

$$\gamma^*\mathscr{E} \xhookrightarrow{j} \gamma^*\mathscr{E}' \xhookleftarrow{\gamma^* t} \gamma^*({}^\tau\mathscr{E}) \simeq {}^\tau(\gamma^*\mathscr{E})$$

We have $\tilde{\gamma}^n = \mathrm{id}$, so this is a lifting of the diagonal action of $\langle\gamma\rangle$ on $X \times X$ to an action on Cht^r.

Similarly, pullback along γ defines a lifting of the diagonal action of $\langle\gamma\rangle$ on $(X-I)\times(X-I)$ to an action on Cht^r_I. We write also $\tilde{\gamma} : \mathrm{Cht}^r_I \to \mathrm{Cht}^r_I$ for this lifting. Also the condition that a chtouca (3.1.1) lies in $\mathrm{Cht}^{r,d;\le p} \subset \mathrm{Cht}^r$ is invariant under pullback by γ, and so we get also automorphisms $\tilde{\gamma} : \mathrm{Cht}^{r,d;\le p}_I \to \mathrm{Cht}^{r,d;\le p}_I$.

The automorphism γ induces an automorphism of the function field F of X, and in turn an automorphism

$$\gamma^* : F^*\backslash\mathbb{A}^*/\mathrm{Ker}(\mathscr{O}_{\mathbb{A}^*} \to \mathscr{O}_I^*) \to F^*\backslash\mathbb{A}/\mathrm{Ker}(\mathscr{O}_{\mathbb{A}^*} \to \mathscr{O}_I^*).$$

In terms of line bundles this sends the class of a pair (L, σ) to the pullback $(\gamma^*L, \gamma^*\sigma)$. It follows that for $a \in \mathbb{A}^*$ the map $\tilde{\gamma}$ on Cht^r_I induces a morphism

$$\tilde{\gamma} : \mathrm{Cht}^r_I/a^{\mathbb{Z}} \to \mathrm{Cht}^r_I/\gamma^*a^{\mathbb{Z}}.$$

Let $b \in F^*\backslash\mathbb{A}^*/\mathrm{Ker}(\mathscr{O}_{\mathbb{A}^*} \to \mathscr{O}_I^*)$ denote a/γ^*a. This degree 0 element corresponds to the line bundle $M := L \otimes \gamma^*L^{-1}$ with trivialization along I defined by $\sigma \otimes \gamma^*\sigma^{-1}$. Let

$$B : \mathrm{Cht}^r_I \to \mathrm{Cht}^r_I$$

denote the map defined by b. The map B descends to a morphism

$$\mathrm{Cht}_I^r/\gamma^* a^{\mathbb{Z}} \to \mathrm{Cht}_I^r/a^{\mathbb{Z}}$$

which we again denote by B. Let

$$\Gamma : \mathrm{Cht}_I^r/a^{\mathbb{Z}} \to \mathrm{Cht}_I^r/a^{\mathbb{Z}}$$

denote the composition $B \circ \tilde{\gamma}$, so Γ is a lifting to $\mathrm{Cht}_I^r/a^{\mathbb{Z}}$ of the map $\gamma \times \gamma : X \times X \to X \times X$.

Lemma 8.4. *The map Γ^n is the identity.*

Proof. It suffices to prove the corresponding statement for the automorphisms of Cht_I^r, before passing to the quotient by $a^{\mathbb{Z}}$. For $i \in \mathbb{Z}/(n)$ let $B_i : \mathrm{Cht}_I^r \to \mathrm{Cht}_I^r$ be the automorphism defined by $\gamma^{i*}M$ (with its trivialization along I). Then we have $B_i \circ \tilde{\gamma} = \tilde{\gamma} \circ B_{i-1}$ for all i which implies that

$$\Gamma^n = \tilde{\gamma}^n \circ B_{-n} \circ B_{-n+1} \circ \cdots \circ B_{-2} \circ B_{-1}.$$

Since $\tilde{\gamma}^n = \mathrm{id}$ it therefore suffices to show that

$$\otimes_{i=0}^{n-1} \gamma^{i*} M = \otimes_{i=0}^{n-1} \gamma^{i*}(L \otimes \gamma^* L^{-1})$$

is trivial, which is immediate. □

Lemma 8.5. *The map Γ commutes with the partial Frobenius endomorphisms F_0 and F_∞ in 6.*

Proof. This is immediate from the construction. □

8.6. Let $\mathcal{H}^r$ denote the Hecke algebra of X, and let M denote the representation of $\mathcal{H}_I^r$ corresponding to V. Recall that $\mathcal{H}^r$ is the algebra of locally constant functions on $G(\mathbb{A})$ with compact support. The automorphism γ of X defines an action of $\mathbb{Z}/(n)$ on $\mathcal{H}^r$ by sending a function $f : G(\mathbb{A}) \to \mathbb{Q}$ to the function f^γ sending g to $f(\gamma^* g)$. We write also $\gamma^* : \mathcal{H}^r \to \mathcal{H}^r$ for this action.

It follows from the construction of the action of $\mathcal{H}_I^r$ on Cht_I^r that in the algebra of correspondences $\mathrm{Cor}(\mathrm{Cht}_I^r/a^{\mathbb{Z}})$, defined as in [9, Chapter I, §4 (b)] the actions of $\mathcal{H}_I^r$ and Γ commute, in the sense that we have $\Gamma \circ h = \gamma^*(h) \circ \Gamma$ for all $h \in \mathcal{H}_I^r$.

The pullback representation $\gamma^* M$ corresponds via the Langlands correspondence to $\gamma^* V$ (this follows for example by compatibility with the local Langlands correspondence), and therefore there is an isomorphism $\rho_M : \gamma^* M \to M$. Fix one such isomorphism. After possibly rescaling our choice we may assume that $\rho_M^n = \mathrm{id}$.

We can then generalize 2.4 and 2.5 as follows, incorporating the group action.

Lemma 8.7. *There exists an element $f \in \mathcal{H}_I^r \otimes \overline{\mathbb{Q}}$ and a 1-dimensional subspace $T \subset M_I$ stable under ρ such that the action of f on M_I splits the inclusion $T \hookrightarrow M_I$ (i.e. f maps M_I to T and restricts to the identity on T).*

Proof. The representation M is defined over $\overline{\mathbb{Q}}$ (and for the rest of the proof we write M for a $\overline{\mathbb{Q}}$-model), and since ρ is chosen to be of finite order the map ρ is also defined over $\overline{\mathbb{Q}}$. As in the proof of 2.4, let $\mathcal{A} \subset \mathrm{End}_{\overline{\mathbb{Q}}}(M_I)$ denote the image of $\mathcal{H}_I^r$. Then M_I is an irreducible representation of $\mathcal{A}$, so by Burnside's lemma [11, Chapter 3, 7.3] the map

$$\mathcal{A} \to \mathrm{End}_{\overline{\mathbb{Q}}}(M_I)$$

is an isomorphism. Choose an isomorphism $M_I \simeq \overline{\mathbb{Q}}^e$ diagonalizing the action of ρ, and let $f \in \mathcal{H}_I^r$ be an element mapping to the projection onto the first basis vector. $\square$

Lemma 8.8. *Let $L \subset \mathbb{C}$ be an algebraically closed subfield, and let $M_0, M_1, \ldots, M_t$ distinct irreducible automorphic cuspidal representations defined over L, such that each $M_{j,I} \neq 0$ is nonzero and none of the M_j are isomorphic to M. Then there exists an element $\alpha \in \mathcal{H}_I^r \otimes \overline{\mathbb{Q}}$, invariant under γ^*, such that α acts as the identity on M_I and as the zero map on $M_{j,I}$ for $j \geq 0$.*

Proof. By 2.5 there exists an element $\alpha \in \mathcal{H}_I^r$ such that α acts as the identity on M_I and as the zero map on $\gamma^{i*}M_{j,I}$ for all i and j. Then $\gamma^{i*}\alpha$ also has the same property, whence

$$\frac{1}{n}\sum_{i=1}^{n} \gamma^{i*}\alpha \in \mathcal{H}_I^r$$

is the desired element. $\square$

8.9. Consider the stack $\tilde{f}^{\leq p} : \widetilde{\mathrm{Cht}}_I^{\leq p}/a^{\mathbb{Z}} \to (X - I)^2$. This compactification is not $\langle\gamma\rangle$-equivariant. However, viewing the action of γ as a correspondence

$$\begin{tikzcd} & \mathrm{Cht}_I^{\leq p}/a^{\mathbb{Z}} \arrow[dl, "\Gamma"'] \arrow[dr, "\mathrm{id}"] & \\ \mathrm{Cht}_I^{\leq p}/a^{\mathbb{Z}} & & \mathrm{Cht}_I^{\leq p}/a^{\mathbb{Z}} \end{tikzcd}$$

we can, by compatifying this correspondence find a correspondence

$$\begin{tikzcd} & P^{\leq p} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \widetilde{\mathrm{Cht}}_I^{\leq p}/a^{\mathbb{Z}} & & \widetilde{\mathrm{Cht}}_I^{\leq p}/a^{\mathbb{Z}}, \end{tikzcd}$$

with p_1 and p_2 proper and $P^{\leq p}$ a union of irreducible stacks of the same dimension as $\widetilde{\mathrm{Cht}}_I^{\leq p}/a^{\mathbb{Z}}$, such that $p_1^{-1}(\mathrm{Cht}_I^{\leq p}/a^{\mathbb{Z}}) = p_2^{-1}(\mathrm{Cht}_I^{\leq p}/a^{\mathbb{Z}}) = \mathrm{Cht}_I^{\leq p}/a^{\mathbb{Z}}$.

Lemma 8.10. *Let Ω be an algebraically closed field of characteristic 0, and let $\widetilde{G}/\Omega$ be an algebraic group which sits in an extension*

$$1 \longrightarrow G \longrightarrow \widetilde{G} \xrightarrow{\pi} \mathbb{Z}/(n) \longrightarrow 1$$

with G connected and reductive. Let $\widetilde{G}^{(1)} \subset \widetilde{G}$ denote $\pi^{-1}(1)$, and let V be a representation of $\widetilde{G}$ such that the restriction to G is irreducible. Then there exists a dense open subset $U^{(1)} \subset \widetilde{G}^{(1)}$ such that for any $g \in U^{(1)}$ there exists an eigenvalue of g acting on V which occurs with multiplicity one and does not differ from any of the other eigenvalues by an n-th root of unity.

Proof. The condition that an element $g \in \widetilde{G}^{(1)}$ has an eigenvalue of multiplicity one which is not equal to an n-th root of unity times any other eigenvalue is an open condition in $\widetilde{G}^{(1)}$, so it suffices to show that there exists a single element with this property.

Fix an element $\sigma \in \widetilde{G}^{(1)}$ and let $B \subset G$ be a Borel subgroup with maximal torus $T \subset B$. Then $\sigma^{-1}B\sigma \subset G$ is also a Borel subgroup, so there exists an element $g \in G$ with $g^{-1}Bg = \sigma^{-1}B\sigma$. Changing our choice of σ by σg^{-1} we may therefore assume that $\sigma^{-1}B\sigma = B$. Let $v \in V$ be a highest weight vector. Then for $b \in B$ we have

$$b\sigma(v) = \sigma(\sigma^{-1}b\sigma)v = \chi(\sigma^{-1}b\sigma)\sigma(v),$$

where $\chi : B \to \mathbb{G}_m$ is the character of B acting on v. In particular $\sigma(v)$ is an eigenvector for B acting on V, whence in the line spanned by v. It follows that v is an eigenvector of σ.

We claim that there exists $u \in T$ such that the eigenvalue of $u\sigma$ on v occurs with multiplicity one and is not equal to an n-th root of unity times any other eigenvalue of $u\sigma$ on V. To see this let $U \subset B$ denote the unipotent radical so $B/U = T$. The action of σ on B induces an action of σ on T and hence also an action on the character group $\mathbb{X}$ of T. For $\chi \in X$ let χ^σ denote the image of χ under this map. Let Fil_V be the filtration on V defined by U. Then σ preserves this filtration and induces an automorphism $\bar{\sigma}$ of the associated graded $\mathrm{gr}(V)$. Let $\{\chi_1, \dots, \chi_r\}$ be the characters of T acting on V, and say χ_1 is the highest weight so it occurs with multiplicity one. Then if $v \in \mathrm{gr}(V)$ is in the χ_i-th eigenspace then $u\sigma \cdot v = \chi_i^\sigma(u)\sigma(v)$. It follows that the eigenvalues of $u\sigma$ acting on V are given by the set of elements $\{\chi_1^\sigma(u)\delta_j, \dots, \chi_r^\sigma(u)\delta_j\}_{j=1}^t$, where $\delta_1, \dots, \delta_t$ are the eigenvalues of σ acting on V. From this it follows that there exists a dense open subset of T of points with the desired properties. $\square$

8.11. Fix a geometric point $\bar{x} \to X - I$ with image $\bar{y} \to Y - J$. Let G_X (resp. G_Y) denote the monodromy group of V (resp. $\overline{V}$) so we have an extension

$$1 \to G_X \to G_Y \to H \to 1,$$

where H is a quotient of $\mathbb{Z}/(n)$. Pulling this extension back along $\mathbb{Z}/(n) \to H$ we get an extension

$$1 \to G_X \to G \to \mathbb{Z}/(n) \to 1.$$

Note that the representation $\pi_1(Y, \bar{y}) \to G_Y$ lifts to a homomorphism $\rho : \pi_1(Y, \bar{y}) \to G$. By 8.10 there exists a dense open subset $U^{(1)} \subset G^{(1)}$ such that for any $g \in U^{(1)}$ there exists an eigenvalue of g acting on $V_{\bar{x}}$ which occurs with multiplicity one and is not equal to an n-th root of unity times any of the other eigenvalues.

Lemma 8.12. *There exists a closed point $x' \in X - I$ and an integer s such that $F_X^s\gamma(x') = x'$ and such that the endomorphism of $V_{\bar{x}'}$ induced by $\varphi_{x'}^s\gamma$ is given by an element of $U^{(1)}$.*

Proof. Note that if x' is such a fixed point then the image $y' \in Y - J$ is fixed by F_Y^s. Conversely given a fixed point y' of F_Y^s such that the image of $F_{y'}$ in $\mathbb{Z}/(n)$ is equal to 1, any lifting $x' \in X - I$ satisfies $F_X^s\gamma(x') = x'$ and the Frobenius element $F_{y'}$ is given by $\varphi_{x'}^s\gamma$. Since the Frobenius elements of points of Y are dense we can choose x' to be a lifting of any y' with Frobenius element in $U^{(1)}$. $\square$

8.13. Let $x' \in X - I$ be a closed point as in 8.12. Let L be the field of definition of x'. Let $b : B \to (X - I)_L$ be the pullback of $\tilde{f}^{\leq p} : \widetilde{\mathrm{Cht}}_I^{r;\leq p}/a^{\mathbb{Z}} \to (X - I)^2$ to $(X - I) \times \{x'\}$. There is a commutative diagram

$$\begin{tikzcd} X_L \arrow[rr, "\gamma"] \arrow[d, "x\mapsto(x,x')"'] & & X_L \arrow[d, "x\mapsto(x,x')"] \\ X\times X \arrow[r, "\gamma\times\gamma"] & X\times X \arrow[r, "\mathrm{id}\times F_X^s"] & X\times X. \end{tikzcd}$$

In particular the endomorphism of $V\boxtimes V_{\bar{x}'}$ obtained by $\gamma\boxtimes(F_X^s\gamma)_{\bar{x}'}$ is given by an algebraic cycle on B. Let Γ be such a cycle.

To complete the proof of 8.2 choose Ξ as in 8.2, but γ-invariant using 8.7 and 8.8. $\square$

9. Local terms for curves with automorphisms

9.1. Let $X/\mathbb{F}_q$ be a smooth proper curve with an automorphism $\gamma : X\to X$ of some order n, and let $I\subset X$ be a finite set of closed points which is γ-invariant. Let $\pi : X\to\overline{X}$ denote the quotient of X by the γ-action, and let $J\subset\overline{X}$ be the image of I. For a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\overline{V}$ on $\overline{X}-J$, the pullback V of $\overline{V}$ to $X-I$ has a natural γ-linearization $u:\gamma^*V\to V$. We view this as an action of the correspondence

$$\begin{tikzcd} & X-I \arrow[dl, "\gamma"'] \arrow[dr, "\mathrm{id}"] & \\ X-I & & X-I \end{tikzcd}$$

on V.

Fix an isomorphism $\iota:\overline{\mathbb{Q}}_\ell\to\mathbb{C}$ and assume that $\overline{V}$ is pure of weight 0. Let ℓ' be a second prime and fix an isomorphism $\iota':\overline{\mathbb{Q}}_{\ell'}\to\mathbb{C}$. Let $\overline{V}'$ be the pure weight 0 lisse $\overline{\mathbb{Q}}_{\ell'}$-sheaf on $\overline{X}-J$ such that $\overline{V}$ and $\overline{V}'$ are compatible (see [3, Theorem 1.1]) . Let V' denote the pullback of $\overline{V}'$ to $X-I$ and let $u':\gamma^*V'\to V'$ be the natural γ-linearization.

Let $j:X-I\hookrightarrow X$ be the inclusion, so we get an action j_*u (resp. j_*u') of the correspondence

$$\begin{tikzcd} & X \arrow[dl, "\gamma"'] \arrow[dr, "\mathrm{id}"] & \\ X & & X \end{tikzcd}$$

on j_*V (resp. j_*V'), where as usual we write j_* for the derived functor Rj_*.

Theorem 9.2. *For every fixed point $z\in\mathrm{Fix}(\gamma^{(m)})$ the local term $\mathrm{lt}_z(j_*V, j_*u^{(m)})$ is in $\overline{\mathbb{Q}}$ and*

$$\iota(\mathrm{lt}_z(j_*V,u^{(m)}))=\iota'(\mathrm{lt}_z(j_*V',u'^{(m)})).$$

The proof is by induction on the rank r of $\overline{V}$ and occupies the remainder of this subsection. The base case $r=0$ is trivial, so we assume that the theorem is true for all sheaves of rank less than r and prove that the theorem then also holds for V of rank r.

9.3. Let $z\in\mathrm{Fix}(\gamma^{(m)})$ be a fixed point lying in $X-I$. Then $\gamma^{(m)}$ defines an automorphism $\gamma_z^{(m)*}:V_{\bar{z}}\to V_{\bar{z}}$ and we have (this is a special case of [15, 8.9])

$$\mathrm{lt}_z(V,u^{(m)})=\mathrm{tr}(\gamma_z^{(m)*}|V_{\bar{z}})\cdot(\Delta.\Gamma_{\gamma^{(m)}})_z, \tag{9.3.1}$$

where $\Gamma_{\gamma^{(m)}}$ denotes the graph of $\gamma^{(m)}$ and $(\Delta.\Gamma_{\gamma^{(m)}})_z$ denotes the part of the refined intersection supported at z. The automorphism $\gamma_z^{(m)} : V_{\bar{z}} \to V_{\bar{z}}$ has the following interpretation. Let $y \in \overline{X}$ be the image of z. Then since z is a fixed point of $\gamma^{(m)}$ the point y is defined over $\mathbb{F}_{q^m}$ and via the canonical isomorphism $V_{\bar{z}} \simeq \overline{V}_{\bar{y}}$ the map $\gamma^{(m)}$ is identified with the Frobenius automorphism

$$F_y : \overline{V}_{\bar{y}} \to \overline{V}_{\bar{y}}.$$

From the formula 9.3.1 and the following lemma, it follows that it suffices to prove 9.2 for V after replacing I by a bigger γ-invariant set, and also that 9.2 holds for fixed points in $X - I$, since we chose $\overline{V}$ and $\overline{V}'$ to be compatible.

Lemma 9.4. *Let $U \subset X$ be a γ-invariant open subset and let $\alpha : \gamma^* j_* \overline{\mathbb{Q}}_{\ell,U} \to j_* \overline{\mathbb{Q}}_{\ell,U}$ be the induced action. Then there exists an integer m_0 such that for all $m \geq m_0$ and $z \in \mathrm{Fix}(\gamma^{(m)})$ the number*

$$\mathrm{lt}_z(j_* \overline{\mathbb{Q}}_{\ell,U}, \alpha^{(m)})$$

is nonzero, in $\mathbb{Q}$, and independent of ℓ.

Proof. The rationality and independence of ℓ follows from B.3. To see that this local term is nonzero for m sufficiently big, observe that by [16, 2.2.4 (b) and 2.1.3] there exists an integer m_0' such that for $m \geq m_0'$ he local term $\mathrm{lt}_z(j_* \overline{\mathbb{Q}}_{\ell,U}, \alpha^{(m)})$ is equal to the naive local term at z, and the naive local term is given by taking the trace of γ_z on $(j_* \overline{\mathbb{Q}}_{\ell,U})_z$ composed with the m-th power of Frobenius. Since the degree 0 part of this complex is $\overline{\mathbb{Q}}_\ell$ and the degree 1 part is $\overline{\mathbb{Q}}_\ell(1)$ it follows that for a high enough twist by Frobenius the local term is nonzero. □

Lemma 9.5. *There exists a finite Galois cover $\pi : Y \to X$ and a lifting $\tilde{\gamma} : Y \to Y$ of γ to a finite order automorphism of Y such that $\pi^* V$ on $Y - \pi^{-1}(I)$ has connected monodromy group and unipotent local monodromy at all points of $\pi^{-1}(I)$.*

Proof. Let K be the function field of X, and let K_0 be the function field of $\overline{X}$. Let L/K_0 be a Galois extension containing K such that if Y is the curve corresponding to L then the pullback of V to the preimage of $X - I$ in Y has geometric connected monodromy group and unipotent local monodromy at all boundary points. Then the map on Galois groups $\mathrm{Gal}(L/K_0) \to \mathrm{Gal}(K/K_0)$ is surjective and we can lift γ to an automorphism $\tilde{\gamma}$ of Y. □

9.6. Fix $Y \to X$ as in 9.5. Let H denote the kernel of $\mathrm{Gal}(Y/\overline{X}) \to \mathrm{Gal}(X/\overline{X})$. After possibly enlarging I we may assume that $Y \to X$ is étale outside of I and we have a morphism

$$[Y/H] \to X$$

which is an isomorphism outside of I. Note that conjugation by $\tilde{\gamma}$ induces an automorphism $\rho : H \to H$ and therefore $\tilde{\gamma}$ induces an automorphism of $[Y/H]$ over γ, which we again denote by $\tilde{\gamma}$, and we have an equivariant inclusion $\tilde{j} : X - I \hookrightarrow [Y/H]$. If $q : [Y/H] \to X$ denotes the projection we have $j_* V = q_* \tilde{j}_* V$, and therefore by the Grothendieck trace formula it suffices to prove that the local terms of $\tilde{j}_* V$ for the action of $\tilde{\gamma}^{(m)}$ are in $\overline{\mathbb{Q}}$ and equal to the local terms of $\tilde{j}_* V'$. The local terms of $[Y/H]$ are in turn computed by the local terms of $h\tilde{\gamma}$ for various $h \in H$. For such an endomorphism $h\tilde{\gamma}$ of Y let $\overline{Y}_h$ denote the quotient of Y by the action of $h\tilde{\gamma}$ so we have a commutative square

$$\begin{array}{ccc} \overline{Y}_h & \longleftarrow & Y \\ \downarrow & & \downarrow \\ \overline{X} & \longleftarrow & X \end{array}$$

It then suffices to prove 9.2 for the pullback of $\overline{V}$ to $\overline{Y}_h$. This therefore reduces the proof to the case when V has connected monodromy group and unipotent local monodromy at all boundary points, which we assume for the rest of the proof.

9.7. By the induction hypothesis we may furthermore assume that $\overline{V}$ is irreducible. We can reduce even further to the case when V is also irreducible. For if V is reducible then we can write V as a direct sum of sheaves $\gamma^{i*}V_0$ for some $V_0 \subset V$ and we have $\gamma^{i*}V_0 \cap \gamma^{i+1*}V_0 = 0$. In this case all the local terms are therefore 0. This reduces us further to proving the theorem in the case when V is irreducible with connected monodromy group and unipotent local monodromy.

9.8. Let $b : B \to X - I$, W, Γ, and Ξ be as in 8.2. Using A.3 we can further arrange that B extends to a proper morphism $\bar{b} : \overline{B} \to X$ with $\overline{B}$ smooth and $\overline{B} - B$ a divisor with normal crossings in $\overline{B}$. Let $\eta : B \hookrightarrow \overline{B}$ be the inclusion and let $\overline{\Gamma} \in A^d(\overline{B} \times_{\gamma\bar{b},X,\bar{b}} \overline{B})$ (resp. $\overline{\Xi} \in A^d(\overline{B} \times_{b,X,b} \overline{B})$) be any extension of Γ (resp. Ξ).

The composition of Γ with the projector Ξ defines a morphism $\gamma^*(V \oplus W) \to V \oplus W$ of the form $\kappa := u \oplus e$ for some map $e : \gamma^*W \to W$. Similarly on the $\overline{\mathbb{Q}}_{\ell'}$-side we have a sheaf W' such that Γ and Ξ defines a morphism $\kappa' : \gamma^*(V' \oplus W') \to V' \oplus W'$.

Lemma 9.9. *(i) The map $\gamma^*V' \to V'$ defined by κ' is equal to u'.*

(ii) If $e' : \gamma^*W' \to W'$ denotes the map defined by the second component of κ' then (W, e) and (W', e') form a compatible system of γ-linearized sheaves on $X - I$.

Proof. Note that the map u' is characterized by the property that the two systems (V, u) and (V', u') form a compatible system of γ-linearized sheaves on $X - I$. By [12, 6.23] it therefore suffices to show that the system $(V \oplus W, \kappa)$ and $(V' \oplus W', \kappa')$ is a compatible system. The local terms of these systems are given by the local terms of $Rb_*\overline{\mathbb{Q}}_\ell$ with action defined by the cycles Ξ and Γ. By the Grothendieck trace formula these systems are compatible whence the result. □

9.10. For any fixed point $z \in \mathrm{Fix}(\gamma^{(m)})$ the local terms $\mathrm{lt}_z(j_*(V \oplus W), \kappa)$ and $\mathrm{lt}_z(j_*(V' \oplus W'), \kappa')$ are in $\overline{\mathbb{Q}}$ and equal. This follows from the Grothendieck trace formula which implies that these local terms are given by local terms of cycles acting on $\eta_*\overline{\mathbb{Q}}_\ell$ and $\eta_*\overline{\mathbb{Q}}_{\ell'}$, which are in $\overline{\mathbb{Q}}$ and independent of ℓ by B.3.

Let $W = \oplus_{s \in S} L_s^{n_s}$ (resp. $W' = \oplus_{s' \in S'} L_{s'}^{n_{s'}}$) be the decomposition of W (resp. W') into irreducible factors. Since the sheaves W and W' are compatible (in the sense that the traces of Frobenius at all points are equal) there exists a unique bijection $\delta : S \simeq S'$ such that L_s and $L_{\delta(s)}$ are compatible and $n_s = n_{\delta(s)}$. We identify S and S' via δ and write L'_s for $L_{\delta(s)}$. Let I (resp. J) be the subset of elements $s \in S$ for which $\gamma^{r*}L_s$ is isomorphic (resp. is not isomorphic) to some $L_{s'}$ for another $s' \in S$ for all $r \geq 0$. We then have a decomposition $S = I \cup J$. Note that again using compatibility of Frobenius traces we get the same decomposition of S using either W or W'. There is a map $\gamma_I : I \to I$ sending $i \in I$ to the unique $j \in I$ for which $\gamma^* L_i$ is isomorphic to L_j.

For an orbit $O \subset I$ of the γ-action, let M_O (resp. M'_O) denote $\oplus_{i\in O} L_i$ (resp. $\oplus_{i\in I} L'_i$).

Lemma 9.11. *For each orbit $O \subset I$ of the γ-action, there exist compatible lisse sheaves $\overline{M}_O$ and $\overline{M}'_O$ on $\overline{X} - J$ and isomorphisms $\pi^*\overline{M}_O \simeq M_O$ and $\pi^*\overline{M}'_O \simeq M'_O$.*

Proof. It suffices to show that M_O descends to $\overline{X} - J$ which is clear. □

9.12. Fix compatible lisse sheaves $\overline{M}_O$ and $\overline{M}'_O$ as above and isomorphisms $W \simeq \oplus_O \pi^* \overline{M}_O^{n_O}$ and $W' \simeq \oplus_O \pi^* \overline{M}_O'^{n_O}$. With these choices if $i \in I$ lies in the orbit O we can write the i-th factor $\gamma^* L_i^{n_O} \to L_{\gamma_I(i)}^{n_O}$ (resp. $\gamma^* L_i'^{n_O} \to L_{\gamma_I(i)}'^{n_O}$) of κ (resp. κ') as $u_i \otimes A_i$ (resp. $u'_i \otimes A'_i$) where $A_i \in GL_{n_O}(\overline{\mathbb{Q}}_\ell)$ (resp. $A'_i \in GL_{n_O}(\overline{\mathbb{Q}}_{\ell'})$) and $u_i : \gamma^* L_i \to L_{\gamma_I(i)}$ (resp. $u'_i : \gamma^* L'_i \to L'_{\gamma_I(i)}$) is the action coming from the descent of M_O (resp. M'_O) to $\overline{X} - J$.

For any fixed point $z \in \mathrm{Fix}(\gamma_r^{(m)})$ we can calculate the local terms at z as follows. For every orbit O let $W_O \subset W$ (resp. $W'_O \subset W'$) denote the sum of $L_i^{n_i}$ (resp. $L_i'^{n_i}$) with $i \in O$, and let $\kappa_O : \gamma^* W_O \to W_O$ (resp. $\kappa'_O : \gamma^* W'_O \to W'_O$) be the restriction of κ. The choice of an element $i \in O$ gives an identification $O \simeq \mathbb{Z}/(s)$ for some integer s. If s does not divide r then κ_O does not fix any of the terms $L_i^{n_i}$ and the local term is 0. If $s|r$ then we have

$$\mathrm{lt}_z(j_* W_O, \kappa_{O,r}^{(m)}) = \sum_{j=0}^{s-1} (\mathrm{lt}_z(j_* L_j, u_r^{j(m)}) \cdot \mathrm{tr}(\prod_{j=0}^{r-1} A_{i+j})), \tag{9.12.1}$$

and similarly for W'_O.

Now observe that for all $z \in \mathrm{Fix}(\gamma_r^{(m)})$ lying in $X - I$ we already know that $\mathrm{lt}_z(V, u_r^{(m)})$ is in $\overline{\mathbb{Q}}$ and that

$$\iota(\mathrm{lt}_z(V, u_r^{(m)})) = \iota'(\mathrm{lt}_z(V', u_r'^{(m)})).$$

Let $\widetilde{A}_i$ denote $\iota^{-1}\iota'(A'_i) \in GL_{n_i}(\overline{\mathbb{Q}}_\ell)$, and let $\tilde{\kappa}_i : \gamma^* L_i^{n_O} \to L_{\gamma_I(i)}^{n_O}$ denote $u_i \otimes \widetilde{A}_i$. Taking the sum of the $\tilde{\kappa}_i$ we get another γ-structure $\tilde{\kappa}$ on W which we can extend (say by acting by 0 on J-factors) to a γ-structure $\tilde{\kappa}$ on $V \oplus W$ which restricts to u on V. We thus get two γ-structures κ and $\tilde{\kappa}$ on $V \oplus W$ such that $(V \oplus W, \kappa)$ and $(V \oplus W, \tilde{\kappa})$ are compatible on $X - I$.

Lemma 9.13. *With $i \in O$ as above, for any $s|r$ we have*

$$\iota(\mathrm{tr}(\prod_{j=0}^{r-1} A_{i+j})) = \iota'(\mathrm{tr}(\prod_{j=0}^{r-1} A'_{i+j})).$$

Proof. Consider the compatible systems $(V \oplus W, \kappa_r)$ and $V \oplus W, \tilde{\kappa}_r)$ with γ_r-structure on $X - I$. By [12, 6.23] applied to this system it follows that for each $j = 0, \dots, r-1$ the systems (L_j, κ_r^j) and $(L_j, \tilde{\kappa}_r^j)$ are compatible. This implies that for every $z \in \mathrm{Fix}(\gamma_r^{(m)})$ lying in $X - I$ we have

$$\mathrm{lt}_z(L_j, u_r^{j(m)}) \cdot \mathrm{tr}(\prod_{j=0}^{r-1} A_{i+j}) = \mathrm{lt}_z(L_j, u_r^{j(m)}) \cdot \mathrm{tr}(\prod_{j=0}^{r-1} \widetilde{A}_{i+j}).$$

To prove the lemma it therefore suffices to show that there exists an integer m such that $\mathrm{lt}_z(L_j, u_r^{j(m)}) \neq 0$. This follows from the Cebotarev density theorem applied to the sheaf $\overline{L}_j$ on $X/\langle\gamma_r\rangle$. □

9.14. From this and the formula 9.12.1 it follows that the system (j_*W, j_*e) and (j_*W', j_*e') is a compatible system of sheaves with γ-structure, and hence (j_*V, j_*u) and (j_*V', j_*u') also form a compatible system. This completes the proof of 9.2. □

Appendix A. Trace maps and cycles

A.1. Let k be an algebraically closed field, and let $\pi : Y \to X$ be a proper, generically finite, dominant morphism of smooth k-schemes of dimension d, with generic degree r. For a coefficient ring Λ we then have a canonical isomorphism $\pi^!\Lambda_X \simeq \Lambda_Y$ which by adjunction defines a morphism

$$t_\pi : \pi_*\Lambda_Y \to \Lambda_X$$

such that the composite map

$$\Lambda_X \longrightarrow \pi_*\Lambda_Y \xrightarrow{t_\pi} \Lambda_X$$

is equal to multiplication by r.

A.2. Since X and Y are smooth, the morphism

$$\pi : Y \times Y \to X \times X$$

is a local complete intersection morphism. Let $\widetilde{\Delta} \in A^d(Y \times_X Y)$ denote the cycle $(\pi \times \pi)^![X]$ obtained from the cartesian diagram

$$\begin{tikzcd}
Y \times_X Y \arrow[r] \arrow[d] & Y \times Y \arrow[d, "\pi\times\pi"] \\
X \arrow[r, "\Delta_X"] & X \times X.
\end{tikzcd}$$

Let $\tilde{\delta} : \Lambda_{Y\times_X Y} \to \mathrm{pr}_2^!\Lambda_Y \simeq \Omega_{Y\times_X Y}(-d)[-2d]$ denote the map defined by $\widetilde{\Delta}$ (see for example [13, 3.9]), and consider the map

$$\pi_*\tilde{\delta} : \pi_*\Lambda_Y \to \pi_*\Lambda_Y$$

induced by pushforward from $\tilde{\delta}$.

Proposition A.3. *The map $\pi_*\tilde{\delta}$ is equal to the composite map*

$$\pi_*\Lambda_Y \xrightarrow{t_\pi} \Lambda_X \longrightarrow \pi_*\Lambda_Y. \tag{A.3.1}$$

Proof. It suffices to show that the corresponding (via adjunction) maps $\Lambda_Y \to \pi^!\pi_*\Lambda_Y \simeq \mathrm{pr}_{1*}\mathrm{pr}_2^!\Lambda_Y$ are equal, or equivalently that the corresponding classes in $H^0(Y \times_X Y, \mathrm{pr}_2^!\Lambda_Y)$ are equal. Let $[Y]$ denote the map $\Lambda_Y \to \pi^!\Lambda_X$ defined by the fundamental class of Y (so $[Y]$ is the adjoint of t_π). By base change applied to the cartesian square

$$\begin{tikzcd}
Y \times_X Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & Y \arrow[d, "\pi"] \\
Y \arrow[r, "\pi"] & X
\end{tikzcd}$$

we get a map

$$\mathrm{pr}_1^*\Lambda_Y \xrightarrow{\mathrm{pr}_1^*[Y]} \mathrm{pr}_1^*\pi^!\Lambda_X \longrightarrow \mathrm{pr}_2^!\Lambda_Y.$$

To prove the proposition it suffices to show that the corresponding class in $H^0(Y\times_X Y, \mathrm{pr}_2^!\Lambda_Y)$ is equal to the class of $\widetilde{\Delta}$. This follows from consideration of the commutative diagram

$$\begin{tikzcd}
Y\times_X Y \arrow[r] \arrow[d, "\mathrm{pr}_1"] \arrow[rr, bend left, "\mathrm{pr}_2"] & Y\times Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{id}\times\pi"] & Y \arrow[d, "\pi"] \\
Y \arrow[r, "\mathrm{id}\times\pi"] \arrow[d, "\pi"] & Y\times X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\pi\times\mathrm{id}"] & X \\
X \arrow[r, "\Delta_X"] & X\times X. &
\end{tikzcd}$$

□

Appendix B. A calculation of local terms

Let k be an algebraically closed field.

B.1. Let B be a smooth k-scheme of dimension d, let $\sigma : \Sigma \to B\times B$ and $\gamma : \Gamma \to B\times B$ be morphisms with Σ and Γ irreducible of dimension d. Let F denote $\Sigma \times_{B\times B} \Gamma$ so we have a cartesian square

$$\begin{tikzcd}
F \arrow[r, "b"] \arrow[d, "a"] & \Sigma \arrow[d, "\sigma"] \\
\Gamma \arrow[r, "\gamma"] & B\times B
\end{tikzcd}.$$

Let

$$z : \Lambda_\Gamma \to \Omega_\Gamma(-d)[-2d] \simeq \gamma^!\Lambda_{B\times B}(d)[2d]$$

be the map corresponding to the fundamental class of Γ, which we also think of as a map

$$z : \Omega_{B\times B} \to \gamma_*\Omega_\Gamma(d)[2d].$$

Applying $\sigma^!$ to this map we get a morphism

$$\Omega_\Sigma \to b_*\Omega_F(d)[2d]$$

which upon composition with the fundamental class $[\Sigma] : \Lambda_\Sigma(d)[2d] \to \Omega_\Sigma$ gives an element of $H^0(F, \Omega_F)$. By [13, 3.35] this class is the class of the localized intersection product $\Gamma\cdot\Sigma$.

B.2. Let $j : U \hookrightarrow B$ be an open subset with complement $i : Z \hookrightarrow B$ a divisor with simple normal crossings. Assume further that $\sigma_2^{-1}(U) \subset \sigma_1^{-1}(U)$ and that $\gamma_1^{-1}(U) \subset \gamma_2^{-1}(U)$.

We then have actions

$$\alpha : \gamma_2^*(j_*\Lambda_U) \to \gamma_1^!(j_*\Lambda_U)$$

and

$$\beta : \sigma_1^*(j_*\Lambda_U) \to \sigma_2^!(j_*\Lambda_U)$$

induced by the natural maps over U. We can think of these actions as sections

$$\alpha \in H^0(\Gamma, \gamma^!((j_*\Lambda_U) \boxtimes D(j_*\Lambda_U))$$

and

$$\beta \in H^0(\Sigma, \sigma^!(D(j_*\Lambda_U) \boxtimes j_*\Lambda_U)),$$

and we can also think of α as a map

$$D(j_*\Lambda_U) \boxtimes j_*\Lambda_U \to \gamma_*\Omega_\Gamma.$$

Applying $\sigma^!$ to this map gives a morphism

$$\sigma^!(D(j_*\Lambda_U) \boxtimes j_*\Lambda_U) \to b_*\Omega_F.$$

Evaulating β under this map we get a global section in $H^0(F, \Omega_F)$ which is the local term $\langle \alpha, \beta \rangle$.

Proposition B.3. *There exists an algebraic cycle $\nu \in A_0(F)_{\mathbb{Q}}$, independent of ℓ, such that the class of ν is equal to $\langle \alpha, \beta \rangle$.*

Proof. Consider the adjoint (and Tate twisted) maps

$$\alpha^t : \gamma_1^*(j_!\Lambda_U) \to \gamma_2^!(j_!\Lambda_U)$$

and

$$\beta^t : \sigma_2^*(j_!\Lambda_U) \to \sigma_1^!(j_!\Lambda_U).$$

Let Ξ denote the fiber product $\Gamma \times_{\gamma_2, B, \sigma_2} \Sigma$, so we have a commutative diagram

$$\begin{tikzcd}
 & & \Xi \arrow[dl] \arrow[dr] \arrow[ddll, bend right, "\xi_1"'] \arrow[ddrr, bend left, "\xi_2"] & & \\
 & \Gamma \arrow[dl, "\gamma_1"'] \arrow[dr, "\gamma_2"] & & \Sigma \arrow[dl, "\sigma_2"'] \arrow[dr, "\sigma_1"] & \\
B & & B & & B.
\end{tikzcd}$$

Notice that $\xi_1^{-1}(U) \subset \xi_2^{-1}(U)$.

By [13, 2.18] we have

$$\langle \alpha^t, \beta^t \rangle = \langle \alpha, \beta \rangle,$$

and (notation as in [13, 2.2.2])

$$\langle \alpha^t, \beta^t \rangle = \mathrm{Tr}(j_!\Lambda_U, \beta^t \alpha^t).$$

Pulling back to Z we also get actions α_Z^t and β_Z^t on $i_*\Lambda_Z$, and from the exact sequence (see [16, 5.3.3])

$$0 \to j_!\Lambda_U \to \Lambda_B \to i_*\Lambda_Z \to 0$$

we get

$$\mathrm{Tr}(j_!\Lambda_U, \beta^t \alpha^t) = \mathrm{cl}(\Sigma \cdot \Gamma) + \mathrm{Tr}(i_*\Lambda_Z, \beta_Z^t \alpha_Z^t).$$

Now the action $\beta_Z^t \alpha_Z^t$ is given by a perfect complex (since it is pulled back from an action given by a perfect complex), so by [14, 6.3] the term $\mathrm{Tr}(i_*\Lambda_Z, \beta_Z^t \alpha_Z^t)$ is the class of an algebraic cycle independent of ℓ. This implies the proposition. $\square$

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