KUMMER COVERINGS AND SPECIALIZATION

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Abstract. We generalize various classical results on specialization of fundamental groups to log schemes in the sense of Fontaine and Illusie. The key technical result relates the category of finite Kummer étale covers of a fs log scheme over a complete noetherian local ring to the Kummer étale coverings of its reduction, generalizing earlier results of Hoshi.

1. Introduction

1.1. Let $\widehat{A}$ be a complete noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, and let $f : X_{\widehat{A}} \to \mathrm{Spec}(\widehat{A})$ be a proper morphism with closed fiber X_k. Then it follows from the Grothendieck Existence theorem that the pullback functor

$$\mathrm{Fet}(X_{\widehat{A}}) \to \mathrm{Fet}(X_k)$$

is an equivalence of categories (see for example [SGA1, Exposé X, Théorème 2.1]), where for a scheme Y we write $\mathrm{Fet}(Y)$ for the category of finite étale Y–schemes.

This was generalized to the logarithmic setting by Hoshi [H, Corollary 1 on p. 83] under assumptions. The purpose of this note is to prove a more general version of Hoshi's result removing the assumptions in his results, and to deduce various consequences with an eye towards future applications to fundamental groups.

Let (S, M_S) be an fs log scheme with $S = \mathrm{Spec}(\widehat{A})$ a complete noetherian local ring as above, and let $(f, f^b) : (X, M_X) \to (S, M_S)$ be a morphism of fs log schemes with underlying morphism of schemes $X \to S$ proper. The main result of this paper is the following:

Theorem 1.2. *The restriction functor*

$$\mathrm{Fet}(X, M_X) \to \mathrm{Fet}(X_k, M_{X_k}) \tag{1.2.1}$$

is an equivalence of categories, where for an fs log scheme (Y, M_Y) *we write* $\mathrm{Fet}(Y, M_Y)$ *for the category of fs log schemes over* (Y, M_Y) *which are finite and Kummer étale.*

Remark 1.3. The log structure M_S on S plays no role in the statement of 1.2 and there is no loss of generality in assuming that $M_S = \mathscr{O}_S^*$.

Remark 1.4. Using Artin approximation we also prove a variant of 1.2 with $\widehat{A}$ replaced by a henselian local ring; see 5.1.

Using 1.2 we generalize two classical results on fundamental groups to log schemes.

1.5. Let (B, M_B) be an fs log scheme and let $(f, f^b) : (X, M_X) \to (B, M_B)$ be a morphism of fs log schemes with underlying morphism $f : X \to B$ proper. For a log geometric point

$\bar{b}^{\log} = (\bar{b}, M_{\bar{b}}) \to (B, M_B)$ we consider the following category, introduced by Hoshi in [H],

$$\mathrm{Fet}((X, M_X)_{(\bar{b}^{\log})}) := \mathrm{colim}_\lambda \ \mathrm{Fet}((X, M_X) \times_{(B,M_B)} b_\lambda^{\log}),$$

where the colimit is taken over fs log structures $M_\lambda \subset M_{\bar{b}^{\log}}$ containing the image of $M_{B,\bar{b}}$ and we write $\bar{b}_\lambda^{\log}$ for the log scheme $(\bar{b}, M_\lambda)$. This category should be viewed as the category of covers of the fiber of (X, M_X) over $\bar{b}^{\log}$, though this does not make literal sense because $M_{\bar{b}}$ is not fine.

Theorem 1.6. *Let $h : \bar{b}'^{\log} \to \bar{b}^{\log}$ be a morphism of log geometric points over (B, M_B). Then the pullback functor*

$$\mathrm{Fet}((X, M_X)_{(\bar{b}^{\log})}) \to \mathrm{Fet}((X, M_X)_{(\bar{b}'^{\log})}) \tag{1.6.1}$$

is an equivalence of categories.

Remark 1.7. For a noetherian fs log scheme (Y, M_Y) the category $\mathrm{Fet}(Y, M_Y)$ is a Galois category by [H, Theorem B.1]. We can therefore talk about an object $(U, M_U) \in \mathrm{Fet}(Y, M_Y)$ being Galois. Let $\mathrm{Fet}'(Y, M_Y) \subset \mathrm{Fet}(Y, M_Y)$ be the full subcategory of objects which can be written as quotients $(U, M_U)/H$ for some Galois object (U, M_U) of degree (a locally constant function on Y) invertible in Y. Then we have variants of 1.2 and 1.6 with $\mathrm{Fet}(-)$ replaced by $\mathrm{Fet}'(-)$.

A second application concerns the variation of the category of covers of the fiber in log smooth proper families:

Theorem 1.8. *Let V denote either a complete discrete valuation ring or a field and let $(B, M_B) \to (\mathrm{Spec}(V), \mathscr{O}^*_{\mathrm{Spec}(V)})$ be a log smooth morphism of fs log schemes.*

Let $f : (X, M_X) \to (B, M_B)$ be a log smooth morphism with underlying morphism of schemes proper. Then for any two log geometric points

$$\bar{b}_i^{\log} \to (B, M_B), \quad i = 1, 2$$

the categories $\mathrm{Fet}'((X, M_X)_{(\bar{b}_1^{\log})})$ and $\mathrm{Fet}'((X, M_X)_{(\bar{b}_2^{\log})})$ are equivalent, where p denotes the residue characteristic of V (which could be 0) and $\mathrm{Fet}'(-)$ denotes the categories of prime-to-p Kummer étale covers.

Remark 1.9. In the proof of 1.8 we also make precise how to relate the two categories using specialization and cospecialization functors.

Remark 1.10. Note that a special case of 1.8 is when $\bar{b}_2^{\log} \to (B, M_B)$ factors through $\bar{b}_1^{\log} \to (B, M_B)$, where the result follows from 1.6. Therefore even though we are in the log regular setting in 1.8 it is not immediately clear to us how to deduce 1.8 using Hoshi's result.

Remark 1.11. In the analytic context the analogue of 1.8 follows in certain cases from the stronger topological results proven by Nakayama and Ogus in [NO, Theorem 5.1].

Remark 1.12. Mattia Talpo suggested an alternate proof of theorem 1.2 based on his result with Vistoli [TV, 6.21]. The basic idea is to show that the category $\mathrm{Fet}(X, M_X)$ is equivalent to the category of finite étale covers of the infinite root stack associated to (X, M_X), and this latter category is then equivalent to the colimit of the categories of finite étale covers of the finite-level root stacks.

1.13. **Conventions.** We assume that the reader is familiar with the basics of log geometry as developed [K] as well as algebraic stacks as developed in [LMB].

We use the notion of *log geometric point* of an fs log scheme (X, M_X) introduced in [N, 2.5]. Recall from loc. cit. that this a morphism of log schemes $\bar{b}^{\log} = (b, M_b) \to (X, M_X)$, where b is the spectrum of a separably closed field k and M_b is an integral log structure such that for every integer $n > 0$ prime to char(k) the multiplication by n map on $\overline{M}_b$ is bijective. A morphism of log geometric points of (X, M_X) is defined to be a morphism of log schemes over (X, M_X).

If k is a separably closed field of characteristic p (possibly 0) and P is a sharp fs monoid, then we can consider the monoid $P_{\mathbb{Z}_p}$ defined to be the saturation of P inside $P^{\mathrm{gp}} \otimes \mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ away from p. Writing simply $k^* \oplus P$ for the log structure on Spec(k) given by the map $k^* \oplus P \to k$ sending all nonzero elements of P to 0, we get for any morphism of log schemes $(\mathrm{Spec}(k), k^* \oplus P) \to (X, M_X)$ a log geometric point of (X, M_X) by considering the induced morphism

$$(\mathrm{Spec}(k), k^* \oplus P_{\mathbb{Z}_{(p)}}) \to (X, M_X).$$

Every log geometric point of (X, M_X) can be written as a limit of log geometric points of this form. Indeed for any log geometric point $\bar{b}^{\log} = (b, M_b) \to (X, M_X)$ write $\overline{M}_b = \mathrm{colim}_\lambda P_\lambda$, where P_λ is a sharp fs submonoid of $\overline{M}_b$ containing the image of $\underline{M_X}$. Then for each λ the inclusion $P_\lambda \hookrightarrow \overline{M}_b$ extends uniquely to an inclusion $P_{\lambda, \mathbb{Z}_{(p)}} \hookrightarrow \overline{M}_b$. Let M_λ denote the preimage in M_b of $P_{\lambda, \mathbb{Z}_{(p)}}$. Then $(\mathrm{Spec}(k), M_\lambda)$ is noncanonically isomorphic to $(\mathrm{Spec}(k), k^* \oplus P_{\mathbb{Z}_{(p)}})$ and

$$\bar{b}^{\log} = \lim_{\lambda} (\mathrm{Spec}(k), M_\lambda)$$

in the category of log geometric points over (X, M_X), as well as in the category of log schemes.

Similarly for any morphism of log geometric points $f : \bar{b}'^{\log} \to \bar{b}^{\log}$ of (X, M_X), with underlying morphism of schemes an isomorphism, we can present the log geometric points as limits

$$\bar{b}'^{\log} = \lim_{\lambda} (\mathrm{Spec}(k), M'_\lambda), \quad \bar{b}^{\log} = \lim_{\lambda} (\mathrm{Spec}(k), M_\lambda),$$

where $M'_\lambda \simeq k^* \oplus Q_{\lambda, \mathbb{Z}_{(p)}}$ and $M_\lambda \simeq k^* \oplus P_{\lambda, \mathbb{Z}_{(p)}}$ for fs monoids P_λ and Q_λ, and f is induced by morphisms

$$f_\lambda : (\mathrm{Spec}(k), k^* \oplus Q_\lambda) \to (\mathrm{Spec}(k), k^* \oplus P_\lambda)$$

defined by maps of monoids $h_\lambda : P_\lambda \to Q_\lambda$.

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2. Kummer coverings and root stacks

2.1. Recall [N, Definition 2.1.2] that a morphism of fs log schemes $f : (Y, M_Y) \to (X, M_X)$ is *of Kummer type* if for every geometric point $\bar{y} \to Y$ the induced map

$$h : \overline{M}_{X, f(\bar{y})} \to \overline{M}_{Y, \bar{y}}$$

is injective and for every element $\bar{m} \in \overline{M}_{Y, \bar{y}}$ there exists an integer $N > 0$ such that $N\bar{m}$ is in the image of h. Note that such a morphism is exact.

If Y is quasi-compact then since $\overline{M}_X$ is constructible this implies that there exists an integer $N > 0$ such that the map of sheaves

$$\cdot N : f^{-1}\overline{M}_X \to f^{-1}\overline{M}_X$$

given by multiplication by N factors as

$$f^{-1}\overline{M}_X \xrightarrow{f^b} \overline{M}_Y \xrightarrow{f^\sharp_N} f^{-1}\overline{M}_X \tag{2.1.1}$$

for a morphism of sheaves of monoids $f^\sharp_N$ as indicated. Note also that we have a commutative diagram

$$\begin{array}{ccccc} f^{-1}\overline{M}_X & \xrightarrow{f^b} & \overline{M}_Y & \xrightarrow{f^\sharp_N} & f^{-1}\overline{M}_X \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ f^{-1}\overline{M}_X^{\mathrm{gp}}\otimes\mathbb{Q} & \xrightarrow{f^{b,\mathrm{gp}}} & \overline{M}_Y^{\mathrm{gp}}\otimes\mathbb{Q} & \xrightarrow{f^{\sharp,\mathrm{gp}}_N} & f^{-1}\overline{M}_X^{\mathrm{gp}}\otimes\mathbb{Q}, \end{array}$$

where the vertical morphisms are injective, since the sheaves of monoids are saturated, and the morphisms $f^{b,\mathrm{gp}}$ and $f^{b,\mathrm{gp}}\circ f^{\sharp,\mathrm{gp}}_N$ are isomorphisms. From this it follows that f^b and $f^\sharp_N$ are both injective, and we can view $\overline{M}_Y$ as being contained in $\frac{1}{N}f^{-1}\overline{M}_X$ inside $(f^{-1}\overline{M}_X^{\mathrm{gp}})\otimes\mathbb{Q}$.

2.2. Fix a morphism $f : (Y, M_Y) \to (X, M_X)$ of Kummer type with Y quasi-compact, and let N be a positive integer such that we have a factorization (2.1.1). We can then describe the (X, M_X)-log scheme (Y, M_Y) as follows using just (X, M_X) and certain morphisms of stacks.

Let $\mathscr{Y}$ denote the stack over Y which to any Y-scheme $g : T \to Y$ associates the groupoid of morphisms of fs log structures

$$u : g^*M_Y \to M_T$$

such that there exists an isomorphism $\eta : g^{-1}f^{-1}\overline{M}_X \to \overline{M}_T$ such that the diagram

$$\begin{array}{ccc} g^{-1}\overline{M}_Y & \xrightarrow{f^\sharp_N} & g^{-1}f^{-1}\overline{M}_X \\ & {\scriptstyle u}\searrow & \big\downarrow{\scriptstyle \eta} \\ & & \overline{M}_T \end{array}$$

commutes. Note that the isomorphism η is unique if it exists since $\overline{M}_T$ is torsion free.

Taking $(Y, M_Y) = (X, M_X)$ we also get a stack $\mathscr{X}_N$ classifying morphisms of log structures $M_X \to M$ such that the induced map $\overline{M}_X \to \overline{M}$ identifies $\overline{M}$ with $\frac{1}{N}\overline{M}_X$ inside $\overline{M}_X^{\mathrm{gp}}\otimes\mathbb{Q}$.

The morphism f induces a functor

$$q : \mathscr{Y} \to \mathscr{X}_N.$$

Given a Y-scheme $g : T \to Y$ and an object $u : g^*M_Y \to M_T$ the X-scheme $f\circ g : T \to X$ and the morphism of log structures

$$g^*f^*M_X \xrightarrow{g^*f^b} g^*M_Y \xrightarrow{u} M_T$$

is an object of $\mathscr{X}_N(T)$ and this defines q.

Remark 2.3. This construction is a special case of the general construction of root stacks discussed in [BV, §4.2, especially 4.13].

2.4. For later use, let us explicate the local structure of these stacks, which implies in particular that they are algebraic stacks, and even tame stacks in the sense of [AOV, 3.1].

Let $\bar{y} \to Y$ be a geometric point with image $\bar{x} \to X$. Let P (resp. Q) denote the monoid $\overline{M}_{X,\bar{x}}$ (resp. $\overline{M}_{Y,\bar{y}}$) so we have a morphism of monoids $\theta : P \to Q$. By our assumptions we also have a morphism of monoids $\theta^\sharp : Q \to P$ such that the composition

$$P \xrightarrow{\theta} Q \xrightarrow{\theta^\sharp} P$$

is multiplication by N. Let $\boldsymbol{\mu}_{P/Q}$ denote the diagonalizable group scheme associated to the quotient $P^{\mathrm{gp}}/Q^{\mathrm{gp}}$, where Q^{gp} is included in P^{gp} by $\theta^\sharp$. Similarly define $\boldsymbol{\mu}_{Q/P}$ to be the diagonalizable group scheme associated to $Q^{\mathrm{gp}}/\theta(P^{\mathrm{gp}})$.

Lemma 2.5. *After replacing X by an étale neighborhood of $\bar{x}$ and Y by an fppf neighborhood of $\bar{y}$ we can find a commutative diagram*

$$\begin{array}{ccc} Q & \xrightarrow{\alpha_Y} & M_Y(Y) \\ \uparrow{\scriptstyle \theta} & & \uparrow{\scriptstyle f^b} \\ P & \xrightarrow{\alpha_X} & M_X(X), \end{array} \tag{2.5.1}$$

where α_X and α_Y are charts inducing the given identifications $Q \simeq \overline{M}_{Y,\bar{y}}$ and $P \simeq \overline{M}_{X,\bar{x}}$. If N is invertible in $k(\bar{y})$ we can find such charts étale locally on Y.

Proof. This is very similar to [O1, 2.1]. First after replacing X and Y by étale neighborhoods we can find charts α_X and α_Y inducing the isomorphisms $Q \simeq \overline{M}_{Y,\bar{y}}$ and $P \simeq \overline{M}_{X,\bar{x}}$. The diagram (2.5.1) may not commute: the failure is measured by a homomorphism $h : P^{\mathrm{gp}} \to \mathscr{O}_Y^*$. The obstruction to extending this to a homomorphism $\tilde{h} : Q^{\mathrm{gp}} \to \mathscr{O}_Y^*$ is a class in $\mathrm{Ext}^1(Q^{\mathrm{gp}}/P^{\mathrm{gp}}, \mathscr{O}_Y^*)$. Therefore after replacing Y by an fppf neighborhood of $\bar{y}$ we can lift h to a homomorphism $\tilde{h} : Q^{\mathrm{gp}} \to \mathscr{O}_Y^*$, and if N is invertible in Y we can do so étale locally. Modifying our chart α_Y by this homomorphism we then arrange that (2.5.1) commutes. □

2.6. Localizing we now assume chosen such charts α_X and α_Y. In this case for any Y-scheme $g : T \to Y$ and object $u : g^*M_Y \to M_T$ there exists a unique extension $\beta_T : P \to \overline{M}_T$ of the composition

$$Q \xrightarrow{\bar{\alpha}_Y} g^{-1}\overline{M}_Y \xrightarrow{\bar{u}} \overline{M}_T$$

and the map β_T lifts étale locally on T to a chart. This extension is defined to be the composition

$$P \xrightarrow{\bar{\alpha}_X} g^{-1}f^{-1}\overline{M}_X \xrightarrow{\bar{\eta}} \overline{M}_T.$$

From this and [O1, 5.20] we deduce that the stack $\mathscr{Y}$ in this local situation is described as the stack quotient

$$[\underline{\mathrm{Spec}}_Y(\mathscr{O}_Y \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P])/\boldsymbol{\mu}_{P/Q}],$$

where the action is induced by the natural action of $\boldsymbol{\mu}_{P/Q}$ on $\mathrm{Spec}(\mathbb{Z}[P])$ over $\mathrm{Spec}(\mathbb{Z}[Q])$.

Similarly we have a description of $\mathscr{X}_N$ as the quotient

$$[\underline{\mathrm{Spec}}_X(\mathscr{O}_X \otimes_{\mathbb{Z}[P], \cdot N} \mathbb{Z}[P])/\boldsymbol{\mu}_{P,N}],$$

where $\boldsymbol{\mu}_{P,N}$ denotes the diagonalizable group scheme associated to $P^{\mathrm{gp}} \otimes \mathbb{Z}/(N)$.

The morphism $q : \mathscr{Y} \to \mathscr{X}_N$ is the morphism of stacks induced by the natural map

$$\underline{\mathrm{Spec}}_Y(\mathscr{O}_Y \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]) \to \underline{\mathrm{Spec}}_X(\mathscr{O}_X \otimes_{\mathbb{Z}[P],\cdot N} \mathbb{Z}[P])$$

by taking quotients. Note here that there is a natural inclusion $\boldsymbol{\mu}_{P/Q} \hookrightarrow \boldsymbol{\mu}_{P,N}$.

2.7. From this we can read off a number of properties of the stacks $\mathscr{Y}$ and $\mathscr{X}_N$:

(i) The stack $\mathscr{Y}$ (resp. $\mathscr{X}_N$) is a tame algebraic stack with finite diagonal over Y (resp. X). Furthermore if N is invertible on Y (resp. X) then $\mathscr{Y}$ (resp. $\mathscr{X}_N$) is Deligne-Mumford.
(ii) The coarse space of $\mathscr{Y}$ (resp. $\mathscr{X}_N$) is Y (resp. X).
(iii) The morphism $q : \mathscr{Y} \to \mathscr{X}_N$ is representable and finite if the morphism $Y \to X$ is finite.

Remark 2.8. Note that because $\mathscr{Y}$ and $\mathscr{X}_N$ are tame stacks the formation of their coarse spaces commutes with arbitrary base change by [AOV, 3.2].

2.9. Let $p_X : \mathscr{X}_N \to X$ and $p_Y : \mathscr{Y} \to Y$ be the projections. We have tautological morphisms of log structures $p_X^*M_X \to M_{\mathscr{X}_N}$ and $p_Y^*M_Y \to M_{\mathscr{Y}}$ and an isomorphism $q^*M_{\mathscr{X}_N} \simeq M_{\mathscr{Y}}$. So we have a commutative square of log stacks

(2.9.1)
$$\begin{array}{ccc} (\mathscr{Y}, M_{\mathscr{Y}}) & \xrightarrow{q} & (\mathscr{X}_N, M_{\mathscr{X}_N}) \\ \downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle p_X} \\ (Y, M_Y) & \xrightarrow{f} & (X, M_X), \end{array}$$

where the morphism q is strict.

Remark 2.10. The vertical morphisms in (2.9.1) are log étale. This follows from [O1, 5.24].

Lemma 2.11. *The map $M_Y \to p_{Y*}^{\log}M_{\mathscr{Y}}$ is an isomorphism, where $p_{Y*}^{\log}$ denotes the pushforward in the category of log structures.*

Proof. Notice that the sheaf $\overline{M}_{\mathscr{X}_N}$ is isomorphic to $p_X^{-1}\overline{M}_X$ and therefore $\overline{M}_{\mathscr{Y}}$ descends to Y. In fact $\overline{M}_{\mathscr{Y}} \simeq p_Y^{-1}f^{-1}\overline{M}_X$.

Since Y is the coarse moduli space of $\mathscr{Y}$, we have

$$p_{Y*}^{\log}M_{\mathscr{Y}} = p_{Y*}M_{\mathscr{Y}},$$

where the right side is the pushforward in the category of sheaves. Indeed by definition $p_{Y*}^{\log}M_{\mathscr{Y}}$ is the fiber product of the diagram

$$\begin{array}{ccc} & & \mathscr{O}_Y \\ & & \downarrow{\scriptstyle \simeq} \\ p_{Y*}M_Y & \longrightarrow & p_{Y*}\mathscr{O}_{\mathscr{Y}}, \end{array}$$

where the vertical morphism is an isomorphism. To ease notation, let us write M'_Y for the log structure $p_{Y*}M_{\mathscr{Y}}$. We then have an inclusion

$$\overline{M}'_Y \hookrightarrow p_{Y*}p_Y^{-1}\overline{M}_X = \overline{M}_X,$$

and it suffices to show that this map identifies $\overline{M}'_Y$ with $\overline{M}_Y$. Here we use the fact that the map $\overline{M}_X \to p_{Y*}p_Y^{-1}\overline{M}_X$ is an isomorphism, which follows for example from the proper base change theorem [SGA4, XII, 5.1].

Now to prove the lemma we may work locally in the fppf topology on Y. Let $\bar{y} \to Y$ be a geometric point with image $\bar{x} \to X$ and choose charts as in 2.5, after possibly shrinking on X and Y in the fppf topology. Write $\mathscr{Y}_{(\bar{y})} := \mathscr{Y} \times_Y \mathrm{Spec}(\mathscr{O}_{Y,\bar{y}})$ and let $m \in \overline{M}_{X,\bar{x}} = P$ be a section. Then m lifts to $(p_{Y*}M_{\mathscr{Y}})_{\bar{y}}$ if and only if the $\mathscr{O}^*_{Y_{(\bar{y})}}$-torsor $\mathscr{L}_m$ on $\mathscr{Y}_{(\bar{y})}$ of liftings of m to $M_{\mathscr{Y}_{(\bar{y})}}$ is trivial. Now observe that with the description of $\mathscr{Y}$ in 2.6 the closed point defines a closed immersion

$$B\boldsymbol{\mu}_{P/Q} \hookrightarrow \mathscr{Y}_{(\bar{y})}.$$

The pullback of $\mathscr{L}_m$ to $B\boldsymbol{\mu}_{P/Q}$ is the torsor corresponding to the character of $\boldsymbol{\mu}_{P/Q}$ defined by m. It follows that a necessary condition for $\mathscr{L}_m$ to be trivial is that m lies in $Q = Q^{\mathrm{gp}} \cap P \subset P^{\mathrm{gp}}$ (using that Kummer morphisms are exact). It follows that $M_Y \to p_{Y*}M_{\mathscr{Y}}$ is surjective and therefore an isomorphism. □

2.12. Adding to our list in 2.7:

(iv) The morphism of log schemes $f : (Y, M_Y) \to (X, M_X)$ is obtained from the data of the representable morphism of stacks $q : \mathscr{Y} \to \mathscr{X}_N$ and the morphism of log stacks $(\mathscr{X}_N, M_{\mathscr{X}_N}) \to (X, M_X)$ by taking Y the coarse moduli space of $\mathscr{Y}$ and M_Y the pushforward $p_{Y*}q^*M_{\mathscr{X}_N}$.

2.13. Conversely we can try to reverse the preceding constructions to get a Kummer morphism from a representable morphism of stacks. Fix a representable morphism of stacks $\mathscr{Y} \to \mathscr{X}_N$ separated over X let $Y \to X$ be the coarse moduli space and M_Y the pushforward log structure $p_{Y*}^{\log}q^*M_{\mathscr{X}_N}$. Note that a priori it is not clear what good properties (e.g. fine and/or saturated) the log structure $p_{Y*}^{\log}q^*M_{\mathscr{X}_N}$ posseses.

Let $\bar{y} \to Y$ be a geometric point with image $\bar{x} \to X$. Then, following the arguments above, the stalk $\overline{M}_{Y,\bar{y}}$ can be described as follows. Let $\mathscr{Y}_{(\bar{y})}$ be the fiber product $\mathscr{Y} \times_Y \mathrm{Spec}(\mathscr{O}_{Y,\bar{y}})$. Let P denote the stalk $\overline{M}_{X,\bar{x}}$. Let $\boldsymbol{\mu}$ denote the stabilizer group scheme of $\mathscr{Y}$ over $\bar{y}$, and let $\boldsymbol{\mu}_{P,N}$ denote the diagonalizable group scheme

$$\mathrm{Hom}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N).$$

Then $\boldsymbol{\mu}_{P,N}$ is the stabilizer group scheme of $\mathscr{X}_N$ at $\bar{x}$, and since $\mathscr{Y} \to \mathscr{X}_N$ is representable we have a closed immersion $\boldsymbol{\mu} \hookrightarrow \boldsymbol{\mu}_{P,N}$. Since a closed subgroup scheme of a diagonalizable group scheme is again diagonalizable, the group scheme $\boldsymbol{\mu}$ is also diagonalizable and the inclusion $\boldsymbol{\mu} \hookrightarrow \boldsymbol{\mu}_{P,N}$ corresponds to a quotient

$$P^{\mathrm{gp}}/NP^{\mathrm{gp}} \to A,$$

or equivalently a subgroup $Q^{\mathrm{gp}} \subset P^{\mathrm{gp}}$ containing NP^{gp}. Let $Q \subset P$ denote $P \cap Q^{\mathrm{gp}}$.

Lemma 2.14. *We have* $\overline{M}_{Y,\bar{y}} = Q$ *inside* $\overline{M}_{X,\bar{x}} = P$.

Proof. With notation as in the proof of 2.11 let $m \in P$ be a section and let $\mathscr{L}_m$ denote the $\mathscr{O}^*_{\mathscr{Y}}$-torsor over $\mathscr{Y}_{(\bar{y})}$ of liftings of m to a section of $M_{\mathscr{Y}} := q^*M_{\mathscr{X}_N}$. Then $\mathscr{L}_m^{\otimes N}$ is trivial, and fixing a trivialization we get a $\boldsymbol{\mu}_N$-torsor $\widetilde{\mathscr{L}_m}$ whose pushout along $\boldsymbol{\mu}_N \subset \mathscr{O}^*_{\mathscr{Y}}$ is the torsor $\mathscr{L}_m$.

The result then follows from the following lemma, where we use the natural identifications of $H^1(B\boldsymbol{\mu}, \boldsymbol{\mu}_N)$ and $\mathrm{Hom}(\boldsymbol{\mu}, \boldsymbol{\mu}_N) \simeq A$. □

Lemma 2.15. *Pullback to $i : B\boldsymbol{\mu} \subset \mathscr{Y}_{(\bar{y})}$ defines a bijection between $H^1(\mathscr{Y}_{(\bar{y})}, \boldsymbol{\mu}_N)$ and A.*

Proof. By the general theory of tame stacks [AOV, Proposition 3.6] the stack $\mathscr{Y}_{(\bar{y})}$ can be written as a quotient $[V/\boldsymbol{\mu}]$, where V is a finite $\mathrm{Spec}(\mathscr{O}_{Y,\bar{y}})$-scheme. In particular there is a retraction $r : \mathscr{Y}_{(\bar{y})} \to B\boldsymbol{\mu}$ of i. It follows that the restriction map

$$H^1(\mathscr{Y}_{(\bar{y})}, \boldsymbol{\mu}_N) \to H^1(B\boldsymbol{\mu}, \boldsymbol{\mu}_N) = A$$

is surjective. To prove injectivity it suffices to show that a $\boldsymbol{\mu}_N$-torsor $\mathscr{L}$ which pulls back to the trivial torsor over $B\boldsymbol{\mu}$ is trivial. Using the Artin approximation theorem it suffices to prove that such a torsor is trivial after base change to the completion $\mathrm{Spec}(\widehat{\mathscr{O}}_{Y,\bar{y}})$, and then by the Grothendieck existence theorem it suffices to show that each of the reductions of $\mathscr{L}$ to infinitesimal neighborhoods of $B\boldsymbol{\mu}$ is trivial. Now the deformation theory of $\boldsymbol{\mu}_N$-torsors are governed by the groups $H^i(B\boldsymbol{\mu}, \mathrm{Lie}(\boldsymbol{\mu}_N))$. These groups vanish for $i > 0$, since $\boldsymbol{\mu}$ is linearly reductive, which proves the lemma. □

2.16. To further understand the log structure M_Y we analyze the situation locally. Replacing Y by some étale neighborhood of $\bar{y}$ we can find a morphism $\alpha_Y : Q \to \overline{M}_Y$ extending the given isomorphism $Q \simeq \overline{M}_{Y,\bar{y}}$. Furthermore, as in the proof of 2.5 we can arrange that we have a chart $\alpha_X : P \to M_X$ inducing the given isomorphism $P \simeq \overline{M}_{X,\bar{x}}$ and that the diagram (2.5.1) commutes. Let M'_Y be the log structure associated to $Q \to M_Y \to \mathscr{O}_Y$ so we have an induced morphism of log structures $M'_Y \to M_Y$. We can then apply the construction of 2.2 to get another stack $\mathscr{Y}' \to Y$ and the natural map $M'_Y \to q^*M_{\mathscr{X}_N}$ induces a morphism of stacks

$$g : \mathscr{Y} \to \mathscr{Y}'.$$

This is a morphism of algebraic stacks proper and quasi-finite over Y with finite diagonal equipped with representable morphisms to $\mathscr{X}_N$. This implies that g is a representable morphism and therefore a finite morphism.

Proposition 2.17. *Suppose for every integer m the base change of g to $\mathrm{Spec}(\mathscr{O}_{Y,\bar{y}}/\mathfrak{m}^m)$ is an isomorphism. Then there exists an étale neighborhood of $\bar{y}$ over which g is an isomorphism and $M'_Y \to M_Y$ is an isomorphism.*

Proof. Since g is finite we have $\mathscr{Y} = \underline{\mathrm{Spec}}_{\mathscr{Y}'}(g_*\mathscr{O}_{\mathscr{Y}})$. Under our assumptions in the proposition we have that the map of coherent $\mathscr{O}_{\mathscr{Y}'}$-modules

$$\mathscr{O}_{\mathscr{Y}'} \to g_*\mathscr{O}_{\mathscr{Y}} \tag{2.17.1}$$

pulls back to an isomorphism over each of the $\mathrm{Spec}(\mathscr{O}_{Y,\bar{y}}/\mathfrak{m}^m)$. By the Grothendieck existence theorem it follows that the map (2.17.1) is an isomorphism in a neighborhood of $\bar{y}$ which implies the first statement in the proposition. The statement that $M'_Y \to M_Y$ is an isomorphism over this étale neighborhood follows from 2.11. □

Remark 2.18. The formation of the stack $\mathscr{Y}$ is functorial in the log scheme (Y, M_Y). If $g : (Y, M_Y) \to (Y', M_{Y'})$ is a morphism over (X, M_X) between Kummer (X, M_X)-log schemes then there is an induced morphism of stacks

$$\tilde{g} : \mathscr{Y} \to \mathscr{Y}'$$

over $\mathscr{X}_N$, where N is chosen appropriately. The fiber of this morphism over $h : T \to Y$ is given by sending a morphism of log structures $h^*M_Y \to M_T$ defining an object of $\mathscr{Y}(T)$ to the composition

$$h^*g^*M_{Y'} \to h^*M_Y \to M_T,$$

which is an object of $\mathscr{Y}'(T)$.

2.19. We will be particularly interested in the case when $(Y, M_Y) \to (X, M_X)$ is Kummer étale. In this case consideration of the diagram (2.9.1) shows that the morphism of log stacks $(\mathscr{Y}, M_{\mathscr{Y}}) \to (\mathscr{X}_N, M_{\mathscr{X}_N})$ is strict and log étale, and therefore the underlying morphism of stacks $\mathscr{Y} \to \mathscr{X}_N$ is representable and étale.

If furthermore the underlying morphism $Y \to X$ is finite, then the representable morphism $\mathscr{Y} \to \mathscr{X}_N$ is also proper and quasi-finite, therefore a finite étale morphism.

2.20. We can use this to understand morphisms of log étale schemes better. Let

$$(Y_i, M_i) \to (X, M_X), \quad i = 1, 2,$$

be two finite Kummer étale morphisms with associated finite étale morphisms $\mathscr{Y}_i \to \mathscr{X}_N$.

Let $\mathscr{H}om_{\mathscr{X}_N}(\mathscr{Y}_1, \mathscr{Y}_2)$ be the functor over $\mathscr{X}_N$ which associates to any morphism $T \to \mathscr{X}_N$ the set of morphisms

$$\mathscr{Y}_1 \times_{\mathscr{X}_N} T \to \mathscr{Y}_2 \times_{\mathscr{X}_N} T.$$

Since $\mathscr{Y}_1$ and $\mathscr{Y}_2$ are finite étale over $\mathscr{X}_N$, this functor is representable by a finite étale morphism $h : \mathscr{H} \to \mathscr{X}_N$. Likewise define

$$\mathscr{H}om_{(X,M_X)}((Y_1, M_{Y_1}), (Y_2, M_{Y_2}))$$

to be the functor on the category of X-schemes which to any $f : T \to X$ associates the set of morphisms of log schemes

$$r : (Y_1, M_{Y_1}) \times_{(X,M_X)} (T, f^*M_X) \to (Y_1, M_{Y_1}) \times_{(X,M_X)} (T, f^*M_X) \tag{2.20.1}$$

over (T, f^*M_X).

Proposition 2.21. *The functor $\mathscr{H}om_{(X,M_X)}((Y_1, M_{Y_1}), (Y_2, M_{Y_2}))$ is representable by a scheme finite and étale over X.*

Proof. By the functoriality discussed in 2.18 any morphism (2.20.1) defines a morphism of stacks

$$t_r : \mathscr{Y}_1 \times_{\mathscr{X}_N} T \to \mathscr{Y}_2 \times_{\mathscr{X}_N} T,$$

and conversely such a morphism of stacks defines a morphism of log schemes (2.20.1) by passing to coarse moduli spaces. In this way the functor $\mathscr{H}om_{(X,M_X)}((Y_1, M_{Y_1}), (Y_2, M_{Y_2}))$ is identified with the functor H which to any X-scheme T associates the section of sections $s : \mathscr{X}_{N,T} \to \mathscr{H}_T$ of the base change of $\mathscr{H} \to \mathscr{X}_N$ to T. The result therefore follows from the following stack-theoretic lemma. □

Lemma 2.22. *Let $\mathscr{X}$ be a tame Artin stack with coarse moduli space $\pi : \mathscr{X} \to X$ and let $h : \mathscr{H} \to \mathscr{X}$ be a finite étale morphism. Let H be the functor on X-schemes sending a scheme T to the set of sections $s : \mathscr{X}_T \to \mathscr{H}_T$ of the base change $\pi : \mathscr{X}_T \to X_T$ of π to T. Then H is representable by a scheme H finite and étale over X.*

Proof. The assertion is étale local on X so by [AOV, 3.2] we may assume that $\mathscr{X} = [U/G]$, where U is a finite X-scheme and G is a finite linearly reductive group scheme acting on U over X. Let $h_U : V \to U$ be the fiber product $\mathscr{H} \times_{\mathscr{X}} U$, so h is finite and étale and there is an action of G on V such that $\mathscr{H} = [V/G]$. Then giving a section of $\mathscr{H} \to \mathscr{X}$ is equivalent to giving a G-equivariant closed subscheme $\Gamma \subset U \times_X V$ such that the projection $\Gamma \to U$ is an isomorphism.

Since U is finite over X we can after further localizing on X arrange that V is a trivial étale cover of U. In this case H is representable by the set of G-invariant elements of $\pi_0(V)$. □

For later use we also record the following result:

Proposition 2.23. *Let $S = \mathrm{Spec}(A)$ be the spectrum of a noetherian integral domain and let (X, M_X) be an fs log scheme over S with flat structure morphism $f : X \to S$. Let $(U, M_U), (V, M_V) \in \mathrm{Fet}(X, M_X)$ be two Kummer étale coverings of (X, M_X) and let $g_K : (U_K, M_{U_K}) \to (V_K, M_{V_K})$ be a (X_K, M_{X_K})-morphism over the generic point of S, where K denotes the field of fractions of A. Then g_K extends uniquely to an (X, M_X)-morphism $(U, M_U) \to (V, M_V)$.*

Proof. The uniqueness statement is immediate.

For the existence part, let $H \to X$ be the finite étale X-scheme classifying morphisms $(U, M_U) \to (V, M_V)$ as in 2.21. Then the morphism g_K corresponds to a section $s_K : X_K \to H_K$. Let $\Gamma \subset H$ be the scheme-theoretic closure of $s_K(X_K)$. Then the claim is that the projection $\Gamma \to X$ is an isomorphism. This can be verified étale locally on X when H is a trivial covering of X, where it is immediate. □

Remark 2.24. The various functors considered above can also be studied using the general methods in [O2]. However, in our cases we need slightly stronger results, under stronger hypotheses, than what we get directly from the results in [O2].

3. An example

3.1. Let (b, M_b) be a log point with $b = \mathrm{Spec}(k)$ the spectrum of a separably closed field and M_b an fs log structure. Let Q denote the monoid $\overline{M}_b$. Choosing a section of the projection $M_b \to \overline{M}_b$ we get a decomposition $M_b = k^* \oplus Q$ with the map to k given by sending all nonzero elements of Q to 0. For $N > 0$ let $\boldsymbol{\mu}_{Q,N}$ denote the finite flat group scheme

$$\mathrm{Hom}(Q^{\mathrm{gp}}, \boldsymbol{\mu}_N).$$

Let $\mathcal{B}_N$ denote the associated N-th root stack over $\mathrm{Spec}(k)$. The stack $\mathcal{B}_N$ can be described as the stack quotient

$$\mathcal{B}_N = [\mathrm{Spec}(k \otimes_{k[Q],N} k[Q])/\boldsymbol{\mu}_{Q,N}].$$

In particular there is a closed immersion defined by a nilpotent ideal

$$j_N : B\boldsymbol{\mu}_{Q,N} \hookrightarrow \mathcal{B}_N.$$

This enables us to completely describe the category of finite étale covers of $\mathcal{B}_N$.

Let $U \to \mathcal{B}_N$ be a finite étale morphism with U connected. Then $U_0 := j_N^* U$ is connected and finite étale over $B\boldsymbol{\mu}_{Q,N}$ and therefore isomorphic to

$$B\boldsymbol{\mu}_{U_0} \to B\boldsymbol{\mu}_{Q,N}$$

for some closed subgroup $\boldsymbol{\mu}_{U_0} \subset \boldsymbol{\mu}_{Q,N}$. Such a subgroup is given by a quotient $z : Q^{\mathrm{gp}}/NQ^{\mathrm{gp}} \to A$. By the invariance of the étale site under infinitesimal thickenings it follows that $U \to \mathcal{B}_N$ is isomorphic to the quotient

$$[\mathrm{Spec}(k \otimes_{k[Q],N} k[Q])/\boldsymbol{\mu}_{U_0}]$$

with its natural map to $\mathcal{B}_N$. Let $Q' \subset Q$ be the set of elements $q' \in Q$ for which $z(q') = 0$. Then the log scheme obtained from U by the construction 2.13 is the scheme

$$\mathrm{Spec}(k \otimes_{k[Q} k[Q'])$$

with the natural log structure $M_{Q'}$ induced by Q'.

Observe also that the projection $Q^{\mathrm{gp}} \to A$ induces an isomorphism $Q^{\mathrm{gp}}/Q'^{\mathrm{gp}} \simeq A$. Indeed it is clear that Q'^{gp} is in the kernel of the map to A, and if $q \in Q^{\mathrm{gp}}$ is an element in the kernel then there exists $y \in Q$ such that $q' := q + Ny \in Q \cap \mathrm{Ker}(Q^{\mathrm{gp}} \to A) = Q'$. Therefore $q = q' - Ny$ lies in Q'^{gp}.

It follows that the finite étale cover $U \to \mathcal{B}_N$ is the stack obtained by the construction of 2.2 from the morphism of log schemes

$$(\mathrm{Spec}(k \otimes_{k[Q} k[Q']), M_{Q'}) \to (b, M_b).$$

3.2. If $N|M$ then there is a natural morphism of stacks

$$\pi_{M,N} : \mathcal{B}_M \to \mathcal{B}_N.$$

In fact if $q_N : \mathcal{B}_N \to b$ is the structure morphism and $\alpha : q_N^* M_b \to M_{\mathcal{B}_N}$ the tautological morphism of log structures over $\mathcal{B}_N$, then we can consider the stakc $\mathcal{B}_{N,M/N}$ over $\mathcal{B}_N$ classifying M/N-th roots of $M_{\mathcal{B}_N}$, and it follows immediately from the construction that composition with α defines an isomorphism of stacks

$$\mathcal{B}_{N,M/N} \to \mathcal{B}_M,$$

and in particular the projection $\pi_{M,N}$. Furthermore, it follows from the preceding discussion that if $U \to \mathcal{B}_N$ is a finite étale morphism with associated Kummer morphism $(c, M_c) \to (b, M_b)$ then the base change $U \times_{\mathcal{B}_N} \mathcal{B}_M$ is the finite étale morphism over $\mathcal{B}_M$ also corresponding to (c, M_c). Since every object of $\mathrm{Fet}((b, M_b))$ is obtained by this construction for some N prime to the characteristic of k we obtain an equivalence of categories

$$\mathrm{Fet}((b, M_b)) \simeq \mathrm{colim}_{N \in \mathbb{N}'} \mathrm{Fet}(\mathcal{B}_N),$$

where the colimit on the right is taken with respect to the morphisms $\pi_{M,N}$ for $N|M$ for N prime to p.

4. Proof of Theorem 1.2

4.1. We may without loss of generality assume that the log structure M_S is trivial. For a $\widehat{A}$-algebra B write (X_B, M_{X_B}) for the base change of (X, M_X) to $\mathrm{Spec}(B)$, so $(X_{\widehat{A}}, M_{X_{\widehat{A}}}) = (X, M_X)$. Let $\mathfrak{m} \subset \widehat{A}$ be the maximal ideal, and write A_n for the quotient $\widehat{A}/\mathfrak{m}^n$. First we show the full faithfulness of the functor (1.2.1):

Proposition 4.2. *For any two objects* $(U, M_U), (V, M_V) \in \mathrm{Fet}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ *the map*

$$\mathrm{Hom}_{(X_{\widehat{A}}, M_{X_{\widehat{A}}})}((U, M_U), (V, M_V)) \to \mathrm{Hom}_{(X_k, M_{X_k})}((U_k, M_{U_k}), (V_k, M_{V_k})) \tag{4.2.1}$$

is bijective.

Proof. Let $H_{\widehat{A}} \to X_{\widehat{A}}$ be the finite étale scheme classifying morphisms $(U, M_U) \to (V, M_V)$ over $(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ as in 2.21. We then want to show that a section of $H_k \to X_k$ lifts uniquely to a section of $H_{\widehat{A}} \to X_{\widehat{A}}$. This follows from the fact that the reduction functor

$$\mathrm{Fet}(X_{\widehat{A}}) \to \mathrm{Fet}(X_k)$$

is an equivalence of categories [SGA1, Exposé X, Théorème 2.1]. □

4.3. To complete the proof of 1.2 it suffices to show that every object $(U_k, M_{U_k}) \in \mathrm{Fet}(X_k, M_{X_k})$ is in the essential image of (1.2.1). To see this, for each $n \geq 1$ let $(U_n, M_{U_n}) \in \mathrm{Fet}(X_{A_n}, M_{X_{A_n}})$ be the unique lifting of (U_k, M_{U_k}) so we have a compatible system $\{(U_n, M_{U_n})\}_n$ of log schemes. Let $N > 0$ be an integer as in 2.1. For each n we then obtain as in 2.9 a commutative square of log stacks

$$\begin{array}{ccc} (\mathscr{U}_n, M_{\mathscr{U}_n}) & \xrightarrow{q_n} & (\mathscr{X}_{A_n,N}, M_{\mathscr{X}_{A_n,N}}) \\ \downarrow{\scriptstyle p_{U_n}} & & \downarrow{\scriptstyle p_{X_{A_n}}} \\ (U_n, M_{U_n}) & \xrightarrow{f} & (X_{A_n}, M_{X_{A_n}}). \end{array}$$

The morphisms $q_n : \mathscr{U}_n \to \mathscr{X}_{A_n,N}$ are strict and étale as noted in 2.19.

By the Grothendieck existence theorem for stacks [O2, A.1] the system $\{q_n : \mathscr{U}_n \to \mathscr{X}_{A_n,N}\}$ of finite étale morphisms is uniquely algebraizable to a finite étale morphism $q : \mathscr{U} \to \mathscr{X}_{\widehat{A},N}$. Let $U \to X_{\widehat{A}}$ be the coarse moduli space of $\mathscr{U}$ and let M_U be the log structure on U given by $p_{U*}^{\log} q^* M_{\mathscr{X}_{\widehat{A},N}}$, where $p_U : \mathscr{U} \to U$ is the projection. We claim that (U, M_U) is an object of $\mathrm{Fet}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ reducing to (U_k, M_{U_k}).

Note first of all that since $\mathscr{U}$ is a tame Deligne-Mumford stack the formation of its coarse moduli space commutes with arbitrary base change [AOV, 3.3]. This implies that U reduces to the system $\{U_n\}$ over the A_n.

The log structure M_U is an fs log structure. Indeed since U is proper over $\widehat{A}$ it suffices to show that every geometric point $\bar{u} \to U$ of the closed fiber admits an étale neighborhood over which M_U is fine and saturated. This follows from 2.17 and the fact that $\mathscr{U}$ reduces to the $\mathscr{U}_n$ by definition.

Furthermore the log structure M_U reduces to M_{U_n} over U_n. Indeed there is a natural map $M_U|_{U_n} \to M_{U_n}$ and to verify that this is an isomorphism it suffices to show that for every geometric point $\bar{u} \to U$ of the closed fiber this map induces an isomorphism $\overline{M}_{U,\bar{u}} \to \overline{M}_{U_n,\bar{u}}$. This follows from 2.14.

Thus $(U, M_U) \to (X_{\widehat{A}}, M_{X_{\widehat{A}}})$ is a finite morphism of fs log schemes reducing to the system of morphisms $(U_n, M_{U_n}) \to (X_{A_n}, M_{X_{A_n}})$, and therefore log étale and Kummer. This completes the proof of theorem 1.2. □

5. Artin approximation and étale covers

As in [A], theorem 1.2 can be generalized to the case when A is not necessarily complete but henselian. We explain how to do so in this section. The main result is the following:

Proposition 5.1 (Log variant of [A, 3.1]). *Let A be a henselian local ring with residue field k, and let $f : (X_A, M_{X_A}) \to \mathrm{Spec}(A)$ be an fs log scheme over A with underlying morphism $X_A \to \mathrm{Spec}(A)$ proper and locally of finite presentation. Then the pullback functor*

$$\mathrm{Fet}(X_A, M_{X_A}) \to \mathrm{Fet}(X_k, M_{X_k}) \tag{5.1.1}$$

is an equivalence of categories.

Proof. By a standard reduction as in the proof of [A, 3.1] it suffices to consider the case when A is the strict henselization of a finite type affine $\mathbb{Z}$-scheme $\mathrm{Spec}(S)$, and (X_A, M_{X_A}) is obtained by base change from a morphism of fs log schemes $f : (X, M_X) \to \mathrm{Spec}(S)$ with underlying morphism of schemes proper. Let k denote the residue field and let $\widehat{A}$ denote the completion of A. Let $\mathfrak{m} \subset \widehat{A}$ denote the maximal ideal and for an integer $n \geq 1$ let A_n denote the quotient $\widehat{A}/\mathfrak{m}^n$, so $A_1 = k$.

For an S-algebra B write (X_B, M_{X_B}) for the base change of (X, M_X) to $\mathrm{Spec}(B)$. By 1.2 it then suffices to show that the functor

$$\mathrm{Fet}(X_A, M_{X_A}) \to \mathrm{Fet}(X_{\widehat{A}}, M_{X_{\widehat{A}}}) \tag{5.1.2}$$

is an equivalence of categories.

To prove this statement we first show that every object of $\mathrm{Fet}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ is in the essential image of (5.1.2). For this consider the functor

$$F : \mathrm{Alg}_S \to \mathrm{Set}$$

sending an S-algebra B to the set of isomorphism classes of objects in $\mathrm{Fet}(X_B, M_{X_B})$. This functor is limit preserving; that is, for any filtering inductive system of S-algebras $\{B_i\}$ with $B = \mathrm{colim}_i B_i$ the natural map

$$\mathrm{colim}_i F(B_i) \to F(B)$$

is bijective. Indeed the functor sending an S-algebra R to isomorphism classes of finite X_R-schemes is limit preserving by [EGA, IV, 8.5.2 and 8.5.5], and since the stacks $\mathcal{L}og_{(X,M_X)}$ introduced in [O1] are locally of finite type we further get that the functor sending an S-algebra to isomorphism classes of morphism of log schemes $(Y, M_Y) \to (X_R, M_{X_R})$ with $Y \to X_R$ finite is limit preserving. It therefore suffices to observe that the property of being Kummer étale is a condition locally of finite presentation which is immediate.

By the Artin approximation theorem [A, 1.12] it follows that given $(\widehat{U}, M_{\widehat{U}}) \in \mathrm{Fet}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ there exists an object $(U, M_U) \in \mathrm{Fet}(X_A, M_{X_A})$ such that $(\widehat{U}, M_{\widehat{U}})$ and (U, M_U) map to isomorphic objects in $\mathrm{Fet}(X_k, M_{X_k})$. By the bijectivity of (4.2.1) it follows that in fact $(\widehat{U}, M_{\widehat{U}})$ is isomorphic to the image of (U, M_U) under the functor (5.1.2).

It remains to show that given two objects $(U, M_U), (V, M_V) \in \mathrm{Fet}(X_A, M_{X_A})$ with induced objects $(\widehat{U}, M_{\widehat{U}}), (\widehat{V}, M_{\widehat{V}}) \in \mathrm{Fet}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ the map

$$\mathrm{Hom}_{(X_A, M_{X_A})}((U, M_U), (V, M_V)) \to \mathrm{Hom}_{(X_{\widehat{A}}, M_{X_{\widehat{A}}})}((\widehat{U}, M_{\widehat{U}}), (\widehat{V}, M_{\widehat{V}})) \tag{5.1.3}$$

is bijective. Let $H_A \to X_A$ be the finite étale X_A-scheme classifying morphisms $(U, M_U) \to (V, M_V)$ as in 2.21. Then the bijectivity of (5.1.3) is equivalent to the statement that base change gives a bijection between the set of sections of $H_A \to X_A$ and the set of sections of $H_{\widehat{A}} \to X_{\widehat{A}}$. This follows from [A, 3.1]. This completes the proof of 5.1. □

6. Proof of theorem 1.6

We first prove two special cases of the theorem, and then deduce the general result.

Special Case 6.1. The morphism h is strict, and therefore given by an inclusion of separably closed fields $k(\bar{b}) \subset k(\bar{b}')$.

The category of Kummer étale coverings form a stack for the fppf topology. From this it follows immediately that (1.6.1) is an equivalence when $k(\bar{b}) \subset k(\bar{b}')$ is a purely inseparable algebraic extension. Indeed in this case every object of $\mathrm{Fet}((X, M_X)_{(\bar{b}'^{\log})})$ has unique descent data since the kernel of the surjection

$$k(\bar{b}') \otimes_{k(b)} k(\bar{b}') \otimes_{k(\bar{b})} \cdots \otimes_{k(\bar{b})} k(\bar{b}') \to k(\bar{b}')$$

is nilpotent. It follows that in order to show that (1.6.1) is an equivalence in the case when h is strict we may assume that $k(\bar{b})$ is algebraically closed.

Next let us show that (1.6.1) is fully faithful. Consider two objects $(U, M_U), (V, M_V) \in \mathrm{Fet}((X, M_X)_{(\bar{b}^{\log})})$, and let $(U', M_{U'}), (V', M_{V'}) \in \mathrm{Fet}((X, M_X)_{(\bar{b}'^{\log})})$ be their base changes. We show that the natural map

$$\text{(6.1.1)} \quad \mathrm{Hom}_{\mathrm{Fet}((X,M_X)_{(\bar{b}^{\log})})}((U, M_U), (V, M_V)) \to \mathrm{Hom}_{\mathrm{Fet}((X,M_X)_{(\bar{b}'^{\log})})}((U', M_{U'}), (V', M_{V'}))$$

is bijective. The injectivity is clear by faithfully flat descent. For surjectivity, let $f' : (U', M_{U'}) \to (V', M_{V'})$ be a morphism. By spreading out, we can find an integral finite type $k(\bar{b})$-scheme T with function field $k(\bar{b}')$ and an extension of the morphism f' to a morphism $f_T : (U_T, M_{U_T}) \to (V_T, M_{V_T})$ between the base changes of (U, M_U) and (V, M_V) to T. Restricting f_T to the fiber over a $k(\bar{b})$-point of T, using that $k(\bar{b})$ is algebraically closed, we get a morphism $f : (U, M_U) \to (V, M_V)$ whose base change to T agrees with f_T at a point. By 4.2 and the injectivity of (6.1.1) already shown it follows that f_T agrees with the map f everywhere. This completes the proof of the full faithfulness.

To show essential surjectivity let $(U', M_{U'}) \in \mathrm{Fet}((X, M_X)_{(\bar{b}'^{\log})})$ be an object which we show is obtained by base change from an object of $\mathrm{Fet}((X, M_X)_{(\bar{b}^{\log})})$. Spreading out and looking at a $k(\bar{b})$-point as in the proof of full faithfulness we find a finite type $k(\bar{b})$-scheme T, an extension $(U'_T, M_{U'_T})$ of $(U', M_{U'})$ to T and an object $(U, M_U) \in \mathrm{Fet}((X, M_X)_{(\bar{b}^{\log})})$ whose base change to T is isomorphic to $(U', M_{U'})$ at a point. Then by 4.2 there exists an extension of $k(\bar{b}')$ over which $(U', M_{U'})$ becomes isomorphic to the base change of (U, M_U). By the full faithfulness already shown this implies that $(U', M_{U'})$ is isomorphic to the base change of (U, M_U). This concludes the proof of special case 6.1.

Special Case 6.2. The underlying map of geometric points $\bar{b}' \to \bar{b}$ is an isomorphism, there exist fine saturated torsion free monoids Q and P such that the log structure $M_{\bar{b}'}$ (resp. $M_{\bar{b}}$) is given by a map $k(\bar{b}')^* \oplus P_{\mathbb{Z}_{(p)}} \oplus Q_{\mathbb{Z}_{(p)}} \to k(\bar{b}')$ (resp. $k(\bar{b})^* \oplus P_{\mathbb{Z}_{(p)}} \to k(\bar{b})$) sending all nonzero elements of $P_{\mathbb{Z}_{(p)}} \oplus Q_{\mathbb{Z}_{(p)}}$ (resp. $P_{\mathbb{Z}_{(p)}}$) to 0, and the map h is induced by the graph $\gamma_\theta : P \to P \oplus Q$ of a map of monoids $\theta : P \to Q$.

Let A denote the ring $k(\bar{b}')[[Q]]$ and let M_A denote the log structure on $\mathrm{Spec}(A)$ given by the map of monoids

$$P \oplus Q \to A$$

sending all nonzero elements of P to 0 and the natural map on Q. We then have a commutative diagram

$$\begin{array}{ccc} P \oplus Q & \longrightarrow & A \\ {\scriptstyle \gamma_\theta}\uparrow & & \uparrow \\ P & \longrightarrow & k(\bar{b}) \end{array}$$

Similarly, for an integer $n > 0$ prime to p let Q_n denote the fs monoid $\frac{1}{n}Q \subset Q^{\mathrm{gp}} \otimes \mathbb{Z}_{(p)}$, and define P_n similarly. Let A_n denote $k(\bar{b}')[[Q_n]]$ and define M_{A_n} to be the log structure associated to the map of monoids $P_n \oplus Q_n \to A_n$ defined as above. We then have a commutative diagram of log schemes

$$\begin{array}{ccc} \bar{b}'^{\log} & \hookrightarrow & \lim_n \ (\mathrm{Spec}(A_n), M_{A_n}) \\ & {\scriptstyle h}\searrow & \downarrow \\ & & \bar{b}^{\log}. \end{array}$$

We therefore get a factorization

$$\begin{array}{ccc} \mathrm{Fet}((X, M_X)_{(\bar{b}^{\log})}) & \xrightarrow{a} & \mathrm{colim}_n \ \mathrm{Fet}((X, M_X) \times_{(B, M_B)} (\mathrm{Spec}(A_n), M_{A_n})) \\ & {\scriptstyle h^*}\searrow & \downarrow{\scriptstyle b} \\ & & \mathrm{Fet}((X, M_X)_{(\bar{b}'^{\log})}), \end{array}$$

where the functor b is an equivalence by 1.2. Let K_n denote the field of fractions of A_n and let Ω be an algebraic closure of $\mathrm{colim}_n K_n$. Let M_Ω denote the log structure induced by taking the colimit of the pullback of the log structures M_{A_n}. Let

$$\Omega^{\log} := (\mathrm{Spec}(\Omega), M_\Omega) \to \lim_n \ (\mathrm{Spec}(A_n), M_{A_n})$$

be the resulting log geometric point. Then the composite morphism

$$\Omega^{\log} \to \lim_n \ (\mathrm{Spec}(A_n), M_{A_n}) \to \bar{b}^{\log}$$

is strict. By the first special case 6.1 the pullback functor

$$\mathrm{Fet}((X, M_X)_{(\bar{b}^{\log})}) \to \mathrm{Fet}((X, M_X)_{(\Omega^{\log})})$$

is an equivalence. Composing a quasi-inverse for this functor with the pullback functor

$$\tilde{c} : \mathrm{colim}_n \ \mathrm{Fet}((X, M_X) \times_{(B, M_B)} (\mathrm{Spec}(A_n), M_{A_n})) \to \mathrm{Fet}((X, M_X)_{(\Omega^{\log})})$$

we get a functor

$$c : \mathrm{colim}_n \ \mathrm{Fet}((X, M_X) \times_{(B, M_B)} (\mathrm{Spec}(A_n), M_{A_n})) \to \mathrm{Fet}((X, M_X)_{(\bar{b}^{\log})})$$

such that $c \circ a$ is isomorphic to the identity functor.

To prove 1.6 in this special case it therefore suffices to show that the functor $\tilde{c}$ is fully faithful. So let (U, M_U) and (V, M_V) be objects of $\mathrm{Fet}'((X, M_X) \times_{(B, M_B)} (\mathrm{Spec}(A_n), M_{A_n}))$ for some n, and let

$$g_\Omega : \tilde{c}(U, M_U) \to \tilde{c}(V, M_V)$$

be a morphism in $\mathrm{Fet}((X, M_X)_{(\Omega^{\log})})$. We must show that g_Ω is induced by a unique morphism in $\mathrm{colim}_n\ \mathrm{Fet}((X, M_X) \times_{(B,M_B)} (\mathrm{Spec}(A_n), M_{A_n}))$. To see this, note that by definition of $\mathrm{Fet}((X, M_X)_{(\Omega^{\log})})$ we can, after possibly enlarging n, find a finite extension K of K_n and a morphism

$$g_K : (U_K, M_{U_K}) \to (V_K, M_{V_K})$$

over $\mathrm{Spec}(K)$ with the log structure M_K induced by the log structure M_{K_n} on K_n. Let R be the integral closure of A_n in K and let M_R be the log structure on $\mathrm{Spec}(R)$ induced by M_{A_n}. By 2.23, the morphism g_K then extends to a morphism $g_R : (U_R, M_{U_R}) \to (V_R, M_{V_R})$ over R. The reduction of this morphism to the closed fiber then lifts uniquely, by 1.2 amd the first special case, to a morphism over $(\mathrm{Spec}(A_n), M_{A_n})$. The pullback of this morphism to R then must agree with g_R since it agrees on the closed fiber. This proves that g_Ω is induced by a morphism $\mathrm{colim}_n\ \mathrm{Fet}((X, M_X) \times_{(B,M_B)} (\mathrm{Spec}(A_n), M_{A_n}))$, and the uniqueness is immediate (using flatness).

General Case 6.3. Let us now prove 1.6 in general. Let $h : \bar{b}'^{\log} \to \bar{b}^{\log}$ be a morphism of log geometric points over (B, M_B). Then h factors as

$$\bar{b}'^{\log} \xrightarrow{a} \bar{b}''^{\log} \xrightarrow{b} \bar{b}^{\log},$$

where b is strict and given by an extension of separably closed fields. Using 6.1 we are therefore reduced to the case when h is an isomorphism on underlying schemes. We may further assume (see discussion in 1.13) that we have a morphism $\theta : P \to Q$ of fine saturated sharp monoids such that h is given by the map of log schemes

$$(\mathrm{Spec}(k), k^* \oplus Q_{\mathbb{Z}_{(p)}}) \to (\mathrm{Spec}(k), k^* \oplus P_{\mathbb{Z}_{(p)}})$$

induced by θ. Note that θ necessarily sends all nonzero elements of P to nonunits. For a positive integer n prime to p let Q_n (resp. P_n) denote $\frac{1}{n}Q \subset Q^{\mathrm{gp}} \otimes \mathbb{Q}$ (resp. $\frac{1}{n}P \subset P^{\mathrm{gp}} \otimes \mathbb{Q}$), so that θ also defines a morphism $\theta_n : P_n \to Q_n$. For such n let A_n denote the ring $k[[P_n]]$ with log structure M_{A_n} induced by the map $P_n \oplus Q_n \to A_n$ induced by the natural map on P_n and the map on Q_n sending all nonzero elements to 0. Let $\gamma_\theta : P \to P \oplus Q$ be the graph of θ, and define γ_{θ_n} similarly. Let

$$\gamma_n : (\mathrm{Spec}(A_n), M_{A_n}) \to (\mathrm{Spec}(k), k^* \oplus P_n)$$

be the map of log schemes given by the natural inclusion $k \hookrightarrow k[[P_n]] = A_n$ and the map of monoids γ_{θ_n}. If K_n denotes the field of fractions of A_n we then have a commutative diagram

$$\begin{array}{ccccc}
(\mathrm{Spec}(K_n), M_{K_n}) & \hookrightarrow & (\mathrm{Spec}(A_n), M_{A_n}) & \hookleftarrow & (\mathrm{Spec}(k), k^* \oplus P_n \oplus Q_n) \\
\downarrow & \searrow & \downarrow & \swarrow{\scriptstyle \gamma_{\theta_n}} & \\
(\mathrm{Spec}(k), k^* \oplus Q_n) & & (\mathrm{Spec}(k), k^* \oplus P_n), & &
\end{array}$$

where for a field field k and sharp monoid M we write $k^* \oplus M$ for the log structure on $\mathrm{Spec}(k)$ induced by the map $k^* \oplus M \to k$ sending all nonzero elements of M to 0. As in 6.2 choose a separably closure Ω of $\mathrm{colim}\ K_n$ and let $\Omega^{\log}$ be the resulting log geometric point. Passing to the limit over n we get a commutative diagram of functors

$$\begin{array}{ccccc} \mathrm{Fet}((X,M_X)_{(\Omega^{\log})}) & \xleftarrow{\gamma} & \mathrm{colim}_n\ \mathrm{Fet}((X,M_X)\times_{(B,M_B)}(\mathrm{Spec}(A_n),M_{A_n})) & \xrightarrow{\delta} & \mathrm{Fet}((X,M_X)_{(\bar{c}^{\log})}) \\ \uparrow{\scriptstyle\alpha} & \nwarrow & \uparrow & \nearrow{\scriptstyle\beta} & \\ \mathrm{Fet}((X,M_X)_{(\bar{b}'^{\log})}) & & \mathrm{Fet}((X,M_X)_{(\bar{b}^{\log})}), & & \end{array}$$

where we write $\bar{c}^{\log}$ for $\lim_n\ (\mathrm{Spec}(k), k^*\oplus P_n\oplus Q_n)$. By the second special case 6.2 and the argument used for its proof, the functors α, γ, and δ are all equivalences. Furthermore, by 6.2 the functor β is an equivalence. Since the pullback functor

$$\mathrm{Fet}((X,M_X)_{(\bar{b}^{\log})})\to \mathrm{Fet}((X,M_X)_{(\bar{b}'^{\log})})$$

can be described as the composition $\alpha^{-1}\circ\gamma\circ\delta^{-1}\circ\beta$ composed with an autoequivalence we obtain the theorem. $\square$

7. Proof of theorem 1.8

We proceed with the notation of 1.8.

7.1. We say that a log geometric point $\bar{b}^{\log}\to(B,M_B)$ is *quasi-strict* if the map

$$\overline{M}_{B,\bar{b}}\to\overline{M}_{\bar{b}}$$

induces an isomorphism $\overline{M}_{B,\bar{b},\mathbb{Z}_{(p)}}\to\overline{M}_{\bar{b}}$, where p is the residue charcteristic of $\bar{b}$.

For any log geometric point $\bar{b}^{\log}\to(B,M_B)$ there exists a morphism of log geometric points of (B,M_B)

(7.1.1)
$$\begin{array}{ccc} \bar{b}^{\log} & \longrightarrow & \bar{b}'^{\log} \\ & \searrow & \downarrow \\ & & (B,M_B), \end{array}$$

where $\bar{b}'^{\log}\to(B,M_B)$ is quasi-strict. Indeed by definition of log geometric point we get an induced map $\overline{M}_{B,\bar{b},\mathbb{Z}_{(p)}}\to\overline{M}_{\bar{b}}$. Choose a lifting $\overline{M}_{B,\bar{b},\mathbb{Z}_{(p)}}\to M_{\bar{b}}$ of this map, and let $M'_{\bar{b}}$ denote the associated log structure on $\bar{b}$. Setting $\bar{b}'^{\log}:=(\bar{b},M'_{\bar{b}})$ we get the desired factorization (7.1.1).

In particular, by this discussion and 1.6 it suffices to prove 1.8 in the case when $\bar{b}^{\log}\to(B,M_B)$ is quasi-strict.

7.2. Let $\bar{s}^{\log}\to(B,M_B)$ and $\bar{b}^{\log}\to(B,M_B)$ be two log geometric points of (B,M_B). A *specialization* $\bar{s}^{\log}\rightsquigarrow\bar{b}^{\log}$ is a commutative diagram

$$\begin{array}{ccccc} \bar{b}^{\log} & \xrightarrow{\alpha} & (E,M_E) & \xleftarrow{\beta} & \bar{s}^{\log} \\ & \searrow & \downarrow & \swarrow & \\ & & (B,M_B), & & \end{array}$$

where α and β are strict and (E,M_E) is log strictly local in the sense of [N, 2.8 (6)] with residue field given by $\bar{b}$.

Given such a specialization we get an induced cospecialization functor

$$\text{cosp} : \text{Fet}((X, M_X)_{(\bar{b}^{\log})}) \to \text{Fet}((X, M_X)_{(\bar{s}^{\log})}) \tag{7.2.1}$$

defined as follows.

Write $\bar{b}^{\log} = (\bar{b}, M_{\bar{b}})$, and $M_{\bar{b}} = \text{colim}_\lambda\, M_{\bar{b},\lambda}$, where $M_{\bar{b},\lambda}$ is a fs log structure contained in $M_{\bar{b}}$ and containing the image of $M_{B,\bar{b}}$. Let $\bar{b}_\lambda^{\log}$ denote $(\bar{b}, M_{\bar{b},\lambda})$ so $\bar{b}^{\log} = \lim_\lambda\, \bar{b}_\lambda^{\log}$. Let $M_{E,\lambda} \subset M_E$ be the sublog structure induced by the submonoid $M_{E,\bar{b}} \times_{\overline{M}_{E,\bar{b}}} \overline{M}_{\bar{b},\lambda}$, so $M_E = \text{colim}_\lambda\, M_{E,\lambda}$, and let $M_{\bar{s},\lambda} \subset M_{\bar{s}}$ denote the log structure defined by the image of $M_{\bar{b},\lambda}$. Setting $\bar{s}_\lambda^{\log} := (\bar{s}, M_{\bar{s},\lambda})$ we then have a commutative diagram for all λ

$$\begin{array}{ccccc}
\bar{b}^{\log} & \hookrightarrow & (E, M_E) & \hookleftarrow & \bar{s}^{\log} \\
\downarrow & & \downarrow & & \downarrow \\
\bar{b}_\lambda & \hookrightarrow & (E, M_{E,\lambda}) & \hookleftarrow & \bar{s}_\lambda^{\log}
\end{array}$$

We then get restriction functors

$$\text{Fet}((X, M_X)_{\bar{b}_\lambda^{\log}}) \xleftarrow{u} \text{Fet}((X, M_X)_{(E, M_{E,\lambda})}) \xrightarrow{v} \text{Fet}((X, M_X)_{\bar{s}_\lambda^{\log}}),$$

where u is an equivalence by 5.1 and 1.2. The functor $v \circ u^{-1}$ then is a functor

$$\text{Fet}((X, M_X)_{\bar{b}_\lambda^{\log}} \to \text{Fet}((X, M_X)_{\bar{s}_\lambda^{\log}}.$$

Passing to the limit over λ we get the functor (7.2.1).

7.3. Since (B, M_B) is log regular the log structure M_B is trivial over some dense open subset of B. Let $\bar{\eta} \to (B, M_B)$ be the log geometric point defined by choosing a separable closure of the function field $k(B)$ and endowing it with the trivial log structure. Then for any quasi-strict log geometric point $\bar{b}^{\log} \to (B, M_B)$ there exists a specialization $\bar{\eta} \rightsquigarrow \bar{b}^{\log}$. To see this let P denote the fs monoid $\overline{M}_{B,\bar{b}}$ and choose a lifting $P \to M_{B,\bar{b}}$. We can then extend this to a map $P_{\mathbb{Z}_{(p)}} \to M_{\bar{b}}$ inducing an isomorphism $P_{\mathbb{Z}_{(p)}} \simeq \overline{M}_{\bar{b}}$, where p is the residue characteristic. For each n prime to p let A_n denote $\mathscr{O}_{B,\bar{b}} \otimes_{\mathbb{Z}[P]} \mathbb{Z}[\frac{1}{n}P]$, and let A denote $\text{colim}_n\, A_n$. Let M_A denote the natural log structure on $\text{Spec}(A)$ and choose an extension $A \to k(\bar{\eta})$ of $\mathscr{O}_{B,\bar{b}} \to k(\bar{\eta})$. We then get a specialization $\bar{\eta} \rightsquigarrow \bar{b}^{\log}$ with (E, M_E) given by $(\text{Spec}(A), M_A)$.

7.4. To prove 1.8 it suffices to show that for such specializations $\bar{\eta} \rightsquigarrow \bar{b}^{\log}$ the cospecialization functor

$$\text{cosp} : \text{Fet}'((X, M_X)_{(b^{\log})}) \to \text{Fet}'((X, M_X)_{\bar{\eta}})$$

is an equivalence of categories.

Write $k(\bar{\eta})$ as a filtered colimit of fields $k(\bar{\eta}) = \text{colim}_\epsilon F_\epsilon$, where F_ϵ is a finite Galois extension of the function field $k(B)$. Let η_ϵ denote $\text{Spec}(F_\epsilon)$ so $\bar{\eta} = \lim_\epsilon\ \eta_\epsilon$. Then

$$\text{Fet}'((X, M_X)_{\bar{\eta}}) = \text{colim}_\epsilon\ \text{Fet}'((X, M_X) \times_B \eta_\epsilon).$$

We show that for a given ϵ and object $(U_\epsilon, M_{U_\epsilon}) \in \text{Fet}'((X, M_X) \times_B \eta_\epsilon)$ there exists an object of $(U_0, M_{U_0}) \in \text{Fet}'((X, M_X)_{(\bar{b}^{\log})})$ whose cospecialization is isomorphic to the image of $(U_\epsilon, M_{U_\epsilon})$ in $\text{Fet}'((X, M_X) \times_B \bar{s}^{\log})$. By [dJ, 6.5] we can, after replacing V by a finite extension find a log regular scheme $(B_\epsilon, M_{B_\epsilon})$ proper and generically finite over B and with function field containing F_ϵ. Using the log purity theorem [M, 3.3] we can further arrange that $(U_\epsilon, M_{U_\epsilon})$ extends to an object $(\widetilde{U}, M_{\widetilde{U}})$ of $\mathrm{Fet}'((X, M_X) \times_{(B,M_B)} (B_\epsilon, M_{B_\epsilon}))$. Let $\bar{b}_\epsilon^{\log} \to (B_\epsilon, M_{B_\epsilon})$ be a log geometric point over $\bar{b}^{\log} \to (B, M_B)$, and let $(\widehat{B}_\epsilon, M_{\widehat{B}_\epsilon})$ be the completion of $(B_\epsilon, M_{B_\epsilon})$ at this point. Let $(\widehat{U}, M_{\widehat{U}})$ denote the base change of $(\widetilde{U}, M_{\widetilde{U}})$ to $\widehat{B}_\epsilon$. We then have a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{Fet}'((X, M_X)_{(\bar{b}_\epsilon^{\log})}) & \longleftarrow & \mathrm{Fet}'((X, M_X) \times_{(B,M_B)} (\widehat{B}_\epsilon, M_{\widehat{B}_\epsilon})) \\ \simeq \uparrow & & \uparrow \\ \mathrm{Fet}'((X, M_X)_{(\bar{b}^{\log})}) & \longleftarrow & \mathrm{Fet}'((X, M_X) \times_{(B,M_B)} (E_\lambda, M_{E_\lambda})) \end{array}$$

where λ is chosen such that we have a morphism of log schemes $(\widehat{B}_\epsilon, M_{\widehat{B}_\epsilon}) \to (E_\lambda, M_{E_\lambda})$. It follows that after possibly changing our choice of ϵ and λ we can find an object of $\mathrm{Fet}'((X, M_X) \times_{(B,M_B)} (E_\lambda, M_{E_\lambda}))$ mapping to $(\widehat{U}, M_{\widehat{U}})$. It follows that the image of $(U_\epsilon, M_{U_\epsilon})$ in $\mathrm{Fet}'((X, M_X)_{\bar{\eta}})$ is in the essential image of the cospecialization functor.

The full faithfulness of the cospecialization functor is shown by a similar argument as in the proof of full faithfulness in 6.2 above.

This completes the proof of 1.8. □

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