

KUMMER COVERINGS AND SPECIALIZATION

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ABSTRACT. We prove versions of various classical results on specialization of fundamental groups in the context of log schemes in the sense of Fontaine and Illusie, generalizing earlier results of Hoshi, Lepage, and Orgogozo. The key technical result relates the category of finite Kummer étale covers of a fs log scheme over a complete noetherian local ring to the Kummer étale coverings of its reduction.

1. INTRODUCTION

1.1. Let \widehat{A} be a complete noetherian local ring with maximal ideal \mathfrak{m} and residue field k , and let $f : X_{\widehat{A}} \rightarrow \mathrm{Spec}(\widehat{A})$ be a proper morphism with closed fiber X_k . Then it follows from the Grothendieck Existence theorem that the pullback functor

$$\mathrm{Fet}(X_{\widehat{A}}) \rightarrow \mathrm{Fet}(X_k)$$

is an equivalence of categories (see for example [SGA1, Exposé X, Théorème 2.1]), where for a scheme Y we write $\mathrm{Fet}(Y)$ for the category of finite étale Y -schemes.

This was generalized to the logarithmic setting by Hoshi [H, Corollary 1 on p. 83] under assumptions and Orgogozo [O]. In this paper we give a stack-theoretic proof of the logarithmic version of [SGA1, Exposé X, Théorème 2.1], and deduce various consequences with an eye towards future applications to fundamental groups.

Let (S, M_S) be an fs log scheme with $S = \mathrm{Spec}(\widehat{A})$ a complete noetherian local ring as above, and let $(f, f^b) : (X, M_X) \rightarrow (S, M_S)$ be a morphism of fs log schemes with underlying morphism of schemes $X \rightarrow S$ proper. The main purpose of this paper is to explain how to deduce the following result, originally due to Orgogozo [O] who references ideas of Gabber, from stack-theoretic considerations.

Theorem 1.2. *The restriction functor*

$$(1.2.1) \quad \mathrm{Fet}(X, M_X) \rightarrow \mathrm{Fet}(X_k, M_{X_k})$$

is an equivalence of categories, where for an fs log scheme (Y, M_Y) we write $\mathrm{Fet}(Y, M_Y)$ for the category of fs log schemes over (Y, M_Y) which are finite and Kummer étale.

Remark 1.3. The log structure M_S on S plays no role in the statement of 1.2 and there is no loss of generality in assuming that $M_S = \mathcal{O}_S^*$.

Remark 1.4. Using Artin approximation we also prove a variant of 1.2 with \widehat{A} replaced by a henselian local ring; see 5.1. This variant over a henselian local ring has also been obtained by Lepage [L, 1.8].

Using 1.2 we generalize two classical results on fundamental groups to log schemes.

1.5. For a noetherian fs log scheme (Y, M_Y) the category $\text{Fet}(Y, M_Y)$ is a Galois category by [H, Theorem B.1]. We can therefore talk about an object $(U, M_U) \in \text{Fet}(Y, M_Y)$ being Galois. For a prime p let $\text{Fet}^{(p)}(Y, M_Y) \subset \text{Fet}(Y, M_Y)$ be the full subcategory of objects which can be written as quotients $(U, M_U)/H$ for some Galois object (U, M_U) of degree (a locally constant function on Y) prime to p and H a finite group of automorphisms of (U, M_U) over (Y, M_Y) . Then we have variants of 1.2 and 1.7 with $\text{Fet}(-)$ replaced by $\text{Fet}^{(p)}(-)$. More generally, for a set of primes \mathbf{L} we can consider the category $\text{Fet}^{\mathbf{L}}(Y, M_Y)$ defined to be the intersection of the categories $\text{Fet}^{(p)}(Y, M_Y)$ for $p \in \mathbf{L}$.

1.6. Let (B, M_B) be an fs log scheme and let $(f, f^b) : (X, M_X) \rightarrow (B, M_B)$ be a morphism of fs log schemes with underlying morphism $f : X \rightarrow B$ proper. For a log geometric point $\bar{b}^{\log} = (\bar{b}, M_{\bar{b}^{\log}}) \rightarrow (B, M_B)$ we consider the following category, introduced by Hoshi in [H],

$$(1.6.1) \quad \text{Fet}((X, M_X)_{(\bar{b}^{\log})}) := \text{colim}_{\lambda} \text{Fet}((X, M_X) \times_{(B, M_B)} b_{\lambda}^{\log}),$$

where the colimit is taken over fs log structures $M_{\lambda} \subset M_{\bar{b}^{\log}}$ containing the image of $M_{B, \bar{b}}$ and we write \bar{b}_{λ}^{\log} for the log scheme (\bar{b}, M_{λ}) (here, and throughout this paper, fiber products are taken in the category of fs log schemes). This category should be viewed as the category of covers of the fiber of (X, M_X) over \bar{b}^{\log} , though this does not make literal sense because $M_{\bar{b}^{\log}}$ is not fine.

Theorem 1.7. *Let $h : \bar{b}'^{\log} \rightarrow \bar{b}^{\log}$ be a morphism of log geometric points over (B, M_B) . Then the pullback functor*

$$(1.7.1) \quad \text{Fet}^{(p)}((X, M_X)_{(\bar{b}'^{\log})}) \rightarrow \text{Fet}^{(p)}((X, M_X)_{(\bar{b}^{\log})})$$

is an equivalence of categories, where p is the residue characteristic of \bar{b}^{\log} .

Remark 1.8. Lepage [L, 2.15] had earlier obtained this result for saturated morphisms $(X, M_X) \rightarrow (B, M_B)$ and morphisms h which are isomorphisms on underlying geometric points using a different argument, which holds also for non-proper morphisms and without the prime-to- p assumption.

A second application concerns the variation of the category of covers of the fiber in log smooth proper families:

Theorem 1.9. *Let (B, M_B) be an fs log scheme with B connected and of finite type over a field or excellent Dedekind ring. Let $f : (X, M_X) \rightarrow (B, M_B)$ be a log smooth morphism with underlying morphism of schemes proper. Then for any two log geometric points*

$$\bar{b}_i^{\log} \rightarrow (B, M_B), \quad i = 1, 2$$

the categories $\text{Fet}^{(p)}((X, M_X)_{(\bar{b}_1^{\log})})$ and $\text{Fet}^{(p)}((X, M_X)_{(\bar{b}_2^{\log})})$ are equivalent, where p is the residue characteristic.

Remark 1.10. This result had previously been obtained by Lepage for “proper polystable log fibrations” [L2, Theorem 3.3].

Remark 1.11. In the proof of 1.9 we also make precise how to relate the two categories using specialization and cospecialization functors.

Remark 1.12. As in the classical case [SGA1, Exposé X, Corollaire 3.9] the prime to p assumptions in 1.9 are necessary, and arise in the proofs with the applications of the purity theorem, which in the logarithmic context is [M, 3.3].

Remark 1.13. In the analytic context the analogue of 1.9 for exact morphisms follows from the stronger topological results proven by Nakayama and Ogus in [NO, Theorem 5.1].

Remark 1.14. Mattia Talpo suggested an alternate proof of theorem 1.2 based on his result with Vistoli [TV, 6.22]. The basic idea is to show that the category $\text{Fet}(X, M_X)$ is equivalent to the category of finite étale covers of the infinite root stack associated to (X, M_X) , and this latter category is then equivalent to the colimit of the categories of finite étale covers of the finite-level root stacks. The technology of infinite root stacks may well be the “correct” language for proving 1.2, but in this paper we choose to develop the stack-theoretic tools needed directly. However, the reader familiar with [TV] may in places find more direct proofs of some of the technical results using that theory.

Example 1.15. An interesting example to consider in regards to 1.9 is the case when (B, M_B) is log smooth over a field k , and (X, M_X) is a log blowup of (B, M_B) with respect to a coherent sheaf of ideals. In this case the morphism $(X, M_X) \rightarrow (B, M_B)$ is an isomorphism over a dense open subset of B , and therefore 1.9 includes the statement that the geometric fundamental group of the fiber of a log blowup is trivial. This result is already known by work of Fujiwara and Kato [I, 6.10], and is, in fact, crucial for the argument in this paper. Since a proof of this result is not published we provide a proof in section 9 below.

1.16. **Conventions.** We assume that the reader is familiar with the basics of log geometry as developed in [K1] as well as algebraic stacks as developed in [LMB].

We use the notion of *log geometric point* of an fs log scheme (X, M_X) introduced in [N, 2.5]. Recall from loc. cit. that this a morphism of log schemes $\bar{b}^{\text{log}} = (\bar{b}, M_{\bar{b}^{\text{log}}}) \rightarrow (X, M_X)$, where \bar{b} is the spectrum of a separably closed field k and $M_{\bar{b}^{\text{log}}}$ is an integral log structure such that for every integer $n > 0$ prime to $\text{char}(k)$ the multiplication by n map on $\overline{M}_{\bar{b}^{\text{log}}}$ is bijective. A morphism of log geometric points of (X, M_X) is defined to be a morphism of log schemes over (X, M_X) .

If k is a separably closed field of characteristic p (possibly 0) and P is a sharp fs monoid, then we can consider the monoid $P_{\mathbb{Z}_p}$ defined to be the saturation of P inside $P^{\text{gp}} \otimes \mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ is the localization of \mathbb{Z} away from p . Writing simply $k^* \oplus P$ for the log structure on $\text{Spec}(k)$ given by the map $k^* \oplus P \rightarrow k$ sending all nonzero elements of P to 0, we get for any morphism of log schemes $(\text{Spec}(k), k^* \oplus P) \rightarrow (X, M_X)$ a log geometric point of (X, M_X) by considering the induced morphism

$$(\text{Spec}(k), k^* \oplus P_{\mathbb{Z}_{(p)}}) \rightarrow (X, M_X).$$

Every log geometric point of (X, M_X) can be written as a limit of log geometric points of this form. Indeed for any log geometric point $\bar{b}^{\text{log}} = (\bar{b}, M_{\bar{b}^{\text{log}}}) \rightarrow (X, M_X)$ write $\overline{M}_{\bar{b}^{\text{log}}} = \text{colim}_{\lambda} P_{\lambda}$, where P_{λ} is a sharp fs submonoid of $\overline{M}_{\bar{b}^{\text{log}}}$ containing the image of \overline{M}_X . Then for each λ the inclusion $P_{\lambda} \hookrightarrow \overline{M}_{\bar{b}^{\text{log}}}$ extends uniquely to an inclusion $P_{\lambda, \mathbb{Z}_{(p)}} \hookrightarrow \overline{M}_{\bar{b}^{\text{log}}}$. Let M_{λ} denote the preimage in $M_{\bar{b}^{\text{log}}}$ of $P_{\lambda, \mathbb{Z}_{(p)}}$. Then $(\text{Spec}(k), M_{\lambda})$ is noncanonically isomorphic to

$(\mathrm{Spec}(k), k^* \oplus P_{\mathbb{Z}(p)})$ and

$$\bar{b}^{\mathrm{log}} = \lim_{\lambda} (\mathrm{Spec}(k), M_{\lambda})$$

in the category of log geometric points over (X, M_X) , as well as in the category of log schemes.

Similarly for any morphism of log geometric points $f : \bar{b}'^{\mathrm{log}} \rightarrow \bar{b}^{\mathrm{log}}$ of (X, M_X) , with underlying morphism of schemes an isomorphism, we can present the log geometric points as limits

$$\bar{b}'^{\mathrm{log}} = \lim_{\lambda} (\mathrm{Spec}(k), M'_{\lambda}), \quad \bar{b}^{\mathrm{log}} = \lim_{\lambda} (\mathrm{Spec}(k), M_{\lambda}),$$

where $M'_{\lambda} \simeq k^* \oplus Q_{\lambda, \mathbb{Z}(p)}$ and $M_{\lambda} \simeq k^* \oplus P_{\lambda, \mathbb{Z}(p)}$ for fs monoids P_{λ} and Q_{λ} , and f is induced by morphisms

$$f_{\lambda} : (\mathrm{Spec}(k), k^* \oplus Q_{\lambda}) \rightarrow (\mathrm{Spec}(k), k^* \oplus P_{\lambda})$$

defined by maps of monoids $h_{\lambda} : P_{\lambda} \rightarrow Q_{\lambda}$.

We will use the Kummer étale site and topos of a fs log scheme. We refer to the survey article [I] and references therein for basics on the Kummer étale topology.

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2. KUMMER COVERINGS AND ROOT STACKS

2.1. Recall [N, Definition 2.1.2] that a morphism of fs log schemes $f : (Y, M_Y) \rightarrow (X, M_X)$ is of *Kummer type* if for every geometric point $\bar{y} \rightarrow Y$ the induced map

$$h : \overline{M}_{X, f(\bar{y})} \rightarrow \overline{M}_{Y, \bar{y}}$$

is injective and for every element $\bar{m} \in \overline{M}_{Y, \bar{y}}$ there exists an integer $N > 0$ such that $N\bar{m}$ is in the image of h . Note that such a morphism is exact.

If Y is quasi-compact then since \overline{M}_X is constructible this implies that there exists an integer $N > 0$ such that the map of sheaves

$$\cdot N : f^{-1}\overline{M}_X \rightarrow f^{-1}\overline{M}_X$$

given by multiplication by N factors as

$$(2.1.1) \quad f^{-1}\overline{M}_X \xrightarrow{f^b} \overline{M}_Y \xrightarrow{f_N^{\sharp}} f^{-1}\overline{M}_X$$

for a morphism of sheaves of monoids f_N^{\sharp} as indicated. Note also that we have a commutative diagram

$$\begin{array}{ccccc} f^{-1}\overline{M}_X & \xrightarrow{f^b} & \overline{M}_Y & \xrightarrow{f_N^{\sharp}} & f^{-1}\overline{M}_X \\ \downarrow & & \downarrow & & \downarrow \\ f^{-1}\overline{M}_X^{\mathrm{gp}} \otimes \mathbb{Q} & \xrightarrow{f^{b, \mathrm{gp}}} & \overline{M}_Y^{\mathrm{gp}} \otimes \mathbb{Q} & \xrightarrow{f_N^{\sharp, \mathrm{gp}}} & f^{-1}\overline{M}_X^{\mathrm{gp}} \otimes \mathbb{Q} \end{array}$$

where the vertical morphisms are injective, since the sheaves of monoids are saturated, and the morphisms $f^{b, \text{gp}}$ and $f_N^{\sharp, \text{gp}}$ are isomorphisms. From this it follows that f^b and f_N^\sharp are both injective, and we can view \overline{M}_Y as being contained in $\frac{1}{N}f^{-1}\overline{M}_X$ inside $(f^{-1}\overline{M}_X^{\text{gp}}) \otimes \mathbb{Q}$.

2.2. Fix a morphism $f : (Y, M_Y) \rightarrow (X, M_X)$ of Kummer type with Y quasi-compact, and let N be a positive integer such that we have a factorization (2.1.1). We can then describe the (X, M_X) -log scheme (Y, M_Y) as follows using just (X, M_X) and certain morphisms of stacks.

Let \mathcal{Y} denote the stack over Y which to any Y -scheme $g : T \rightarrow Y$ associates the groupoid of morphisms of fs log structures

$$u : g^* M_Y \rightarrow M_T$$

such that there exists an isomorphism $\eta : g^{-1}f^{-1}\overline{M}_X \rightarrow \overline{M}_T$ such that the diagram

$$\begin{array}{ccc} g^{-1}\overline{M}_Y & \xrightarrow{f_N^\sharp} & g^{-1}f^{-1}\overline{M}_X \\ & \searrow u & \downarrow \eta \\ & & \overline{M}_T \end{array}$$

commutes. Note that the isomorphism η is unique if it exists since \overline{M}_T is torsion free.

Taking $(Y, M_Y) = (X, M_X)$ we also get a stack \mathcal{X}_N classifying morphisms of log structures $M_X \rightarrow M$ such that the induced map $\overline{M}_X \rightarrow \overline{M}$ identifies \overline{M} with $\frac{1}{N}\overline{M}_X$ inside $\overline{M}_X^{\text{gp}} \otimes \mathbb{Q}$.

The morphism f induces a functor

$$q : \mathcal{Y} \rightarrow \mathcal{X}_N.$$

Given a Y -scheme $g : T \rightarrow Y$ and an object $u : g^* M_Y \rightarrow M_T$ the X -scheme $f \circ g : T \rightarrow X$ and the morphism of log structures

$$g^* f^* M_X \xrightarrow{g^* f^b} g^* M_Y \xrightarrow{u} M_T$$

is an object of $\mathcal{X}_N(T)$ and this defines q .

Remark 2.3. This construction is a special case of the general construction of root stacks discussed in [BV, §4.2, especially 4.13].

2.4. For later use, let us explicate the local structure of these stacks, which implies in particular that they are algebraic stacks, and even tame stacks in the sense of [AOV, 3.1].

Let $\bar{y} \rightarrow Y$ be a geometric point with image $\bar{x} \rightarrow X$. Let P (resp. Q) denote the monoid $\overline{M}_{X, \bar{x}}$ (resp. $\overline{M}_{Y, \bar{y}}$) so we have a morphism of monoids $\theta : P \rightarrow Q$. By our assumptions we also have a morphism of monoids $\theta^\sharp : Q \rightarrow P$ such that the composition

$$P \xrightarrow{\theta} Q \xrightarrow{\theta^\sharp} P$$

is multiplication by N . Let $\mu_{P/Q}$ denote the diagonalizable group scheme associated to the quotient $P^{\text{gp}}/Q^{\text{gp}}$, where Q^{gp} is included in P^{gp} by θ^\sharp . Similarly define $\mu_{Q/P}$ to be the diagonalizable group scheme associated to $Q^{\text{gp}}/\theta(P^{\text{gp}})$.

Lemma 2.5. *After replacing X by an étale neighborhood of \bar{x} and Y by an fppf neighborhood of \bar{y} we can find a commutative diagram*

$$(2.5.1) \quad \begin{array}{ccc} Q & \xrightarrow{\alpha_Y} & M_Y(Y) \\ \uparrow \theta & & \uparrow f^b \\ P & \xrightarrow{\alpha_X} & M_X(X), \end{array}$$

where α_X and α_Y are charts inducing the given identifications $Q \simeq \overline{M}_{Y,\bar{y}}$ and $P \simeq \overline{M}_{X,\bar{x}}$. If N is invertible in $k(\bar{y})$ we can find such charts étale locally on Y .

Proof. This is very similar to [O1, 2.1]. First after replacing X and Y by étale neighborhoods we can find charts α_X and α_Y inducing the isomorphisms $Q \simeq \overline{M}_{Y,\bar{y}}$ and $P \simeq \overline{M}_{X,\bar{x}}$. The diagram (2.5.1) may not commute: the failure is measured by a homomorphism $h : P^{\text{gp}} \rightarrow \mathcal{O}_Y^*$. The obstruction to extending this to a homomorphism $\tilde{h} : Q^{\text{gp}} \rightarrow \mathcal{O}_Y^*$ is a class in $\text{Ext}^1(Q^{\text{gp}}/P^{\text{gp}}, \mathcal{O}_Y^*)$. Therefore after replacing Y by an fppf neighborhood of \bar{y} we can lift h to a homomorphism $\tilde{h} : Q^{\text{gp}} \rightarrow \mathcal{O}_Y^*$, and if N is invertible in Y we can do so étale locally. Modifying our chart α_Y by this homomorphism we then arrange that (2.5.1) commutes. \square

2.6. Localizing we now assume chosen such charts α_X and α_Y . In this case for any Y -scheme $g : T \rightarrow Y$ and object $u : g^*M_Y \rightarrow M_T$ there exists a unique extension $\beta_T : P \rightarrow \overline{M}_T$ of the composition

$$Q \xrightarrow{\bar{\alpha}_Y} g^{-1}\overline{M}_Y \xrightarrow{\bar{u}} \overline{M}_T$$

and the map β_T lifts étale locally on T to a chart. This extension is defined to be the composition

$$P \xrightarrow{\bar{\alpha}_X} g^{-1}f^{-1}\overline{M}_X \xrightarrow{\bar{\eta}} \overline{M}_T.$$

From this and [O1, 5.20] we deduce that the stack \mathcal{Y} in this local situation is described as the stack quotient

$$[\underline{\text{Spec}}_Y(\mathcal{O}_Y \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]) / \mu_{P/Q}],$$

where the action is induced by the natural action of $\mu_{P/Q}$ on $\text{Spec}(\mathbb{Z}[P])$ over $\text{Spec}(\mathbb{Z}[Q])$.

Similarly we have a description of \mathcal{X}_N as the quotient

$$[\underline{\text{Spec}}_X(\mathcal{O}_X \otimes_{\mathbb{Z}[P],N} \mathbb{Z}[P]) / \mu_{P,N}],$$

where $\mu_{P,N}$ denotes the diagonalizable group scheme associated to $P^{\text{gp}} \otimes \mathbb{Z}/(N)$.

The morphism $q : \mathcal{Y} \rightarrow \mathcal{X}_N$ is the morphism of stacks induced by the natural map

$$\underline{\text{Spec}}_Y(\mathcal{O}_Y \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]) \rightarrow \underline{\text{Spec}}_X(\mathcal{O}_X \otimes_{\mathbb{Z}[P],N} \mathbb{Z}[P])$$

by taking quotients. Note here that there is a natural inclusion $\mu_{P/Q} \hookrightarrow \mu_{P,N}$.

2.7. From this we can read off a number of properties of the stacks \mathcal{Y} and \mathcal{X}_N :

- (i) The stack \mathcal{Y} (resp. \mathcal{X}_N) is a tame algebraic stack with finite diagonal over Y (resp. X). Furthermore if N is invertible on Y (resp. X) then \mathcal{Y} (resp. \mathcal{X}_N) is Deligne-Mumford.
- (ii) The coarse space of \mathcal{Y} (resp. \mathcal{X}_N) is Y (resp. X).

(iii) The morphism $q : \mathcal{Y} \rightarrow \mathcal{X}_N$ is representable and finite if the morphism $Y \rightarrow X$ is finite.

Remark 2.8. Note that because \mathcal{Y} and \mathcal{X}_N are tame stacks the formation of their coarse spaces commutes with arbitrary base change by [AOV, 3.2].

2.9. Let $p_X : \mathcal{X}_N \rightarrow X$ and $p_Y : \mathcal{Y} \rightarrow Y$ be the projections. We have tautological morphisms of log structures $p_X^* M_X \rightarrow M_{\mathcal{X}_N}$ and $p_Y^* M_Y \rightarrow M_{\mathcal{Y}}$ and an isomorphism $q^* M_{\mathcal{X}_N} \simeq M_{\mathcal{Y}}$. So we have a commutative square of log stacks

$$(2.9.1) \quad \begin{array}{ccc} (\mathcal{Y}, M_{\mathcal{Y}}) & \xrightarrow{q} & (\mathcal{X}_N, M_{\mathcal{X}_N}) \\ \downarrow p_Y & & \downarrow p_X \\ (Y, M_Y) & \xrightarrow{f} & (X, M_X), \end{array}$$

where the morphism q is strict.

Remark 2.10. The vertical morphisms in (2.9.1) are log étale. This follows from [O1, 5.24].

Lemma 2.11. *The map $M_Y \rightarrow p_{Y*}^{\log} M_{\mathcal{Y}}$ is an isomorphism, where p_{Y*}^{\log} denotes the pushforward in the category of log structures.*

Proof. Notice that the sheaf $\overline{M}_{\mathcal{X}_N}$ is isomorphic to $p_X^{-1} \overline{M}_X$ and therefore $\overline{M}_{\mathcal{Y}}$ descends to Y . In fact $\overline{M}_{\mathcal{Y}} \simeq p_Y^{-1} f^{-1} \overline{M}_X$.

Since Y is the coarse moduli space of \mathcal{Y} , we have

$$p_{Y*}^{\log} M_{\mathcal{Y}} = p_{Y*} M_{\mathcal{Y}},$$

where the right side is the pushforward in the category of sheaves. Indeed by definition $p_{Y*}^{\log} M_{\mathcal{Y}}$ is the fiber product of the diagram

$$\begin{array}{ccc} & \mathcal{O}_Y & \\ & \downarrow \simeq & \\ p_{Y*} M_Y & \longrightarrow & p_{Y*} \mathcal{O}_{\mathcal{Y}}, \end{array}$$

where the vertical morphism is an isomorphism. To ease notation, let us write M'_Y for the log structure $p_{Y*}^{\log} M_{\mathcal{Y}}$. Note that by the left exactness of p_{Y*} the natural map

$$\overline{M}'_Y \simeq (p_{Y*} M_{\mathcal{Y}}) / (p_{Y*} \mathcal{O}_{\mathcal{Y}}^*) \rightarrow p_{Y*} \overline{M}_{\mathcal{Y}}$$

is injective. We therefore have an inclusion

$$\overline{M}'_Y \hookrightarrow p_{Y*} p_Y^{-1} f^{-1} \overline{M}_X = f^{-1} \overline{M}_X,$$

and it suffices to show that this map identifies \overline{M}'_Y with \overline{M}_Y . Here we use the fact that the map $\overline{M}_X \rightarrow p_{Y*} p_Y^{-1} \overline{M}_X$ is an isomorphism, which follows for example from the proper base change theorem [SGA4, XII, 5.1].

Now to prove the lemma we may work locally in the fppf topology on Y . Let $\bar{y} \rightarrow Y$ be a geometric point with image $\bar{x} \rightarrow X$ and choose charts as in 2.5, after possibly shrinking on X and Y in the fppf topology. Write $\mathcal{Y}_{(\bar{y})} := \mathcal{Y} \times_Y \text{Spec}(\mathcal{O}_{Y, \bar{y}})$ and let $m \in \overline{M}_{X, \bar{x}} = P$ be a section. Then m lifts to $(p_{Y*} M_{\mathcal{Y}})_{\bar{y}}$ if and only if the $\mathcal{O}_{\mathcal{Y}_{(\bar{y})}}^*$ -torsor \mathcal{L}_m on $\mathcal{Y}_{(\bar{y})}$ of liftings of m

to $M_{\mathcal{Y}(\bar{y})}$ is trivial. Indeed we have $(p_{Y*}M_{\mathcal{Y}})_{\bar{y}} = H^0(\mathcal{Y}(\bar{y}), M_{\mathcal{Y}(\bar{y})})$ and the (possibly empty) set of trivializations of \mathcal{L}_m is precisely the set of global sections of $M_{\mathcal{Y}(\bar{y})}$ mapping to m . Now observe that with the description of \mathcal{Y} in 2.6 the closed point defines a closed immersion

$$B\mu_{P/Q} \hookrightarrow \mathcal{Y}(\bar{y}).$$

The pullback of \mathcal{L}_m to $B\mu_{P/Q}$ is the torsor corresponding to the character of $\mu_{P/Q}$ defined by m . It follows that a necessary condition for \mathcal{L}_m to be trivial is that m lies in $Q = Q^{\text{gp}} \cap P \subset P^{\text{gp}}$ (using that Kummer morphisms are exact). It follows that $M_Y \rightarrow p_{Y*}M_{\mathcal{Y}}$ is surjective and therefore an isomorphism. \square

2.12. Adding to our list in 2.7:

- (iv) The morphism of log schemes $f : (Y, M_Y) \rightarrow (X, M_X)$ is obtained from the data of the representable morphism of stacks $q : \mathcal{Y} \rightarrow \mathcal{X}_N$ and the morphism of log stacks $(\mathcal{X}_N, M_{\mathcal{X}_N}) \rightarrow (X, M_X)$ by taking Y the coarse moduli space of \mathcal{Y} and M_Y the pushforward $p_{Y*}q^*M_{\mathcal{X}_N}$.

2.13. Conversely we can try to reverse the preceding constructions to get a Kummer morphism from a representable morphism of stacks. Fix a representable morphism of stacks $\mathcal{Y} \rightarrow \mathcal{X}_N$ separated over X let $Y \rightarrow X$ be the coarse moduli space and M_Y the pushforward log structure $p_{Y*}^{\text{log}}q^*M_{\mathcal{X}_N}$. Note that a priori it is not clear what good properties (e.g. fine and/or saturated) the log structure $p_{Y*}^{\text{log}}q^*M_{\mathcal{X}_N}$ possesses.

Let $\bar{y} \rightarrow Y$ be a geometric point with image $\bar{x} \rightarrow X$. Then, following the arguments above, the stalk $\overline{M}_{Y, \bar{y}}$ can be described as follows. Let $\mathcal{Y}(\bar{y})$ be the fiber product $\mathcal{Y} \times_Y \text{Spec}(\mathcal{O}_{Y, \bar{y}})$. Let P denote the stalk $\overline{M}_{X, \bar{x}}$. Let μ denote the stabilizer group scheme of \mathcal{Y} over \bar{y} , and let $\mu_{P, N}$ denote the diagonalizable group scheme

$$\text{Hom}(P^{\text{gp}}, \mu_N).$$

Then $\mu_{P, N}$ is the stabilizer group scheme of \mathcal{X}_N at \bar{x} , and since $\mathcal{Y} \rightarrow \mathcal{X}_N$ is representable we have a closed immersion $\mu \hookrightarrow \mu_{P, N}$. Since a closed subgroup scheme of a diagonalizable group scheme is again diagonalizable, the group scheme μ is also diagonalizable and the inclusion $\mu \hookrightarrow \mu_{P, N}$ corresponds to a quotient

$$P^{\text{gp}}/NP^{\text{gp}} \rightarrow A,$$

or equivalently a subgroup $Q^{\text{gp}} \subset P^{\text{gp}}$ containing NP^{gp} . Let $Q \subset P$ denote $P \cap Q^{\text{gp}}$.

Lemma 2.14. *We have $\overline{M}_{Y, \bar{y}} = Q$ inside $\overline{M}_{X, \bar{x}} = P$.*

Proof. With notation as in the proof of 2.11 let $m \in P$ be a section and let \mathcal{L}_m denote the $\mathcal{O}_{\mathcal{Y}}^*$ -torsor over $\mathcal{Y}(\bar{y})$ of liftings of m to a section of $M_{\mathcal{Y}} := q^*M_{\mathcal{X}_N}$. Then $\mathcal{L}_m^{\otimes N}$ is trivial, and fixing a trivialization we get a μ_N -torsor $\widetilde{\mathcal{L}}_m$ whose pushout along $\mu_N \subset \mathcal{O}_{\mathcal{Y}}^*$ is the torsor \mathcal{L}_m . The result then follows from the following lemma, where we use the natural identifications of $H^1(B\mu, \mu_N)$ and $\text{Hom}(\mu, \mu_N) \simeq A$. \square

Lemma 2.15. *Pullback to $i : B\mu \subset \mathcal{Y}(\bar{y})$ defines a bijection between $H^1(\mathcal{Y}(\bar{y}), \mu_N)$ and A .*

Proof. By the general theory of tame stacks [AOV, Proposition 3.6] the stack $\mathcal{Y}(\bar{y})$ can be written as a quotient $[V/\mu]$, where V is a finite $\text{Spec}(\mathcal{O}_{Y, \bar{y}})$ -scheme. In particular there is a

retraction $r : \mathcal{Y}_{(\bar{y})} \rightarrow B\mu$ of i . It follows that the restriction map

$$H^1(\mathcal{Y}_{(\bar{y})}, \mu_N) \rightarrow H^1(B\mu, \mu_N) = A$$

is surjective. To prove injectivity it suffices to show that a μ_N -torsor \mathcal{L} which pulls back to the trivial torsor over $B\mu$ is trivial. Using the Artin approximation theorem it suffices to prove that such a torsor is trivial after base change to the completion $\text{Spec}(\widehat{\mathcal{O}}_{Y, \bar{y}})$, and then by the Grothendieck existence theorem it suffices to show that each of the reductions of \mathcal{L} to infinitesimal neighborhoods of $B\mu$ is trivial. Now the deformation theory of μ_N -torsors are governed by the groups $H^i(B\mu, \text{Lie}(\mu_N))$. These groups vanish for $i > 0$, since μ is linearly reductive, which proves the lemma. \square

2.16. To further understand the log structure M_Y we analyze the situation locally. Replacing Y by some étale neighborhood of \bar{y} we can find a morphism $\alpha_Y : Q \rightarrow \overline{M}_Y$ extending the given isomorphism $Q \simeq \overline{M}_{Y, \bar{y}}$. Furthermore, as in the proof of 2.5 we can arrange that we have a chart $\alpha_X : P \rightarrow M_X$ inducing the given isomorphism $P \simeq \overline{M}_{X, \bar{x}}$ and that the diagram (2.5.1) commutes. Let M'_Y be the log structure associated to $Q \rightarrow M_Y \rightarrow \mathcal{O}_Y$ so we have an induced morphism of log structures $M'_Y \rightarrow M_Y$. We can then apply the construction of 2.2 to get another stack $\mathcal{Y}' \rightarrow Y$ and the natural map $M'_Y \rightarrow q^* M_{\mathcal{X}_N}$ induces a morphism of stacks

$$g : \mathcal{Y} \rightarrow \mathcal{Y}'.$$

This is a morphism of algebraic stacks proper and quasi-finite over Y with finite diagonal equipped with representable morphisms to \mathcal{X}_N . This implies that g is a representable morphism and therefore a finite morphism.

Proposition 2.17. *Suppose for every integer m the base change of g to $\text{Spec}(\mathcal{O}_{Y, \bar{y}}/\mathfrak{m}^m)$ is an isomorphism. Then there exists an étale neighborhood of \bar{y} over which g is an isomorphism and $M'_Y \rightarrow M_Y$ is an isomorphism.*

Proof. Since g is finite we have $\mathcal{Y} = \text{Spec}_{\mathcal{O}_{Y, \bar{y}}} (g_* \mathcal{O}_{\mathcal{Y}'})$. Under our assumptions in the proposition we have that the map of coherent $\mathcal{O}_{\mathcal{Y}'}$ -modules

$$(2.17.1) \quad \mathcal{O}_{\mathcal{Y}'} \rightarrow g_* \mathcal{O}_{\mathcal{Y}}$$

pulls back to an isomorphism over each of the $\text{Spec}(\mathcal{O}_{Y, \bar{y}}/\mathfrak{m}^m)$. By the Grothendieck existence theorem it follows that the map (2.17.1) is an isomorphism in a neighborhood of \bar{y} which implies the first statement in the proposition. The statement that $M'_Y \rightarrow M_Y$ is an isomorphism over this étale neighborhood follows from 2.11. \square

Remark 2.18. The formation of the stack \mathcal{Y} is functorial in the log scheme (Y, M_Y) . If $g : (Y, M_Y) \rightarrow (Y', M_{Y'})$ is a morphism over (X, M_X) between Kummer (X, M_X) -log schemes then there is an induced morphism of stacks

$$\tilde{g} : \mathcal{Y} \rightarrow \mathcal{Y}'$$

over \mathcal{X}_N , where N is chosen appropriately. The fiber of this morphism over $h : T \rightarrow Y$ is given by sending a morphism of log structures $h^* M_Y \rightarrow M_T$ defining an object of $\mathcal{Y}(T)$ to the composition

$$h^* g^* M_{Y'} \rightarrow h^* M_Y \rightarrow M_T,$$

which is an object of $\mathcal{Y}'(T)$.

2.19. We will be particularly interested in the case when $(Y, M_Y) \rightarrow (X, M_X)$ is Kummer étale. In this case consideration of the diagram (2.9.1) shows that the morphism of log stacks $(\mathcal{Y}, M_{\mathcal{Y}}) \rightarrow (\mathcal{X}_N, M_{\mathcal{X}_N})$ is strict and log étale, and therefore the underlying morphism of stacks $\mathcal{Y} \rightarrow \mathcal{X}_N$ is representable and étale.

If furthermore the underlying morphism $Y \rightarrow X$ is finite, then the representable morphism $\mathcal{Y} \rightarrow \mathcal{X}_N$ is also proper and quasi-finite, therefore a finite étale morphism.

2.20. We can use this to understand morphisms of log étale schemes better. Let

$$(Y_i, M_i) \rightarrow (X, M_X), \quad i = 1, 2,$$

be two finite Kummer étale morphisms with associated finite étale morphisms $\mathcal{Y}_i \rightarrow \mathcal{X}_N$.

Let $\mathcal{H}om_{\mathcal{X}_N}(\mathcal{Y}_1, \mathcal{Y}_2)$ be the functor over \mathcal{X}_N which associates to any morphism $T \rightarrow \mathcal{X}_N$ the set of morphisms

$$\mathcal{Y}_1 \times_{\mathcal{X}_N} T \rightarrow \mathcal{Y}_2 \times_{\mathcal{X}_N} T.$$

Since \mathcal{Y}_1 and \mathcal{Y}_2 are finite étale over \mathcal{X}_N , this functor is representable by a finite étale morphism $h : \mathcal{H} \rightarrow \mathcal{X}_N$. Likewise define

$$\mathcal{H}om_{(X, M_X)}((Y_1, M_{Y_1}), (Y_2, M_{Y_2}))$$

to be the functor on the category of X -schemes which to any $f : T \rightarrow X$ associates the set of morphisms of log schemes

$$(2.20.1) \quad r : (Y_1, M_{Y_1}) \times_{(X, M_X)} (T, f^* M_X) \rightarrow (Y_2, M_{Y_2}) \times_{(X, M_X)} (T, f^* M_X)$$

over $(T, f^* M_X)$.

Proposition 2.21. *The functor $\mathcal{H}om_{(X, M_X)}((Y_1, M_{Y_1}), (Y_2, M_{Y_2}))$ is representable by a scheme finite and étale over X .*

Proof. By the functoriality discussed in 2.18 any morphism (2.20.1) defines a morphism of stacks

$$t_r : \mathcal{Y}_1 \times_{\mathcal{X}_N} T \rightarrow \mathcal{Y}_2 \times_{\mathcal{X}_N} T,$$

and conversely such a morphism of stacks defines a morphism of log schemes (2.20.1) by passing to coarse moduli spaces. In this way the functor $\mathcal{H}om_{(X, M_X)}((Y_1, M_{Y_1}), (Y_2, M_{Y_2}))$ is identified with the functor H which to any X -scheme T associates the set of sections $s : \mathcal{X}_{N, T} \rightarrow \mathcal{H}_T$ of the base change of $\mathcal{H} \rightarrow \mathcal{X}_N$ to T . The result therefore follows from the following stack-theoretic lemma. \square

Lemma 2.22. *Let \mathcal{X} be a tame Artin stack with coarse moduli space $\pi : \mathcal{X} \rightarrow X$ and let $h : \mathcal{H} \rightarrow \mathcal{X}$ be a finite étale morphism. Let H be the functor on X -schemes sending a scheme T to the set of sections $s : \mathcal{X}_T \rightarrow \mathcal{H}_T$ of the base change $h_T : \mathcal{H}_T \rightarrow \mathcal{X}_T$ of h to T . Then H is representable by a scheme H finite and étale over X .*

Proof. The assertion is étale local on X so by [AOV, 3.2] we may assume that $\mathcal{X} = [U/G]$, where U is a finite X -scheme and G is a finite linearly reductive group scheme acting on U over X . Let $h_U : V \rightarrow U$ be the fiber product $\mathcal{H} \times_{\mathcal{X}} U$, so h is finite and étale and there is an action of G on V such that $\mathcal{H} = [V/G]$. Then giving a section of $\mathcal{H} \rightarrow \mathcal{X}$ is equivalent to giving a G -equivariant closed subscheme $\Gamma \subset U \times_X V$ such that the projection $\Gamma \rightarrow U$ is an isomorphism.

Since U is finite over X we can after further localizing on X arrange that V is a trivial étale cover of U . In this case H is representable by the set of G -invariant sections of the projection $\pi_0(V) \rightarrow \pi_0(U)$. \square

Remark 2.23. The various functors considered above can also be studied using the general methods in [O2]. However, in our cases we need slightly stronger results, under stronger hypotheses, than what we get directly from the results in [O2].

3. AN EXAMPLE

To illustrate the constructions and results of the preceding section, we make them explicit in this section in the case of a log point. This will also make clear the connection with infinite root stacks in the sense of [TV], and the result [TV, 6.22] can be viewed as a vast generalization of the equivalence (3.2.1) constructed below.

3.1. Let (b, M_b) be a log point with $b = \text{Spec}(k)$ the spectrum of a separably closed field and M_b an fs log structure. Let Q denote the monoid \overline{M}_b . Choosing a section of the projection $M_b \rightarrow \overline{M}_b$ we get a decomposition $M_b = k^* \oplus Q$ with the map to k given by sending all nonzero elements of Q to 0. For $N > 0$ let \mathcal{B}_N denote the associated N -th root stack over $\text{Spec}(k)$. The stack \mathcal{B}_N can be described as the stack quotient

$$\mathcal{B}_N = [\text{Spec}(k \otimes_{k[Q], N} k[Q]) / \mu_{Q, N}].$$

In particular there is a closed immersion defined by a nilpotent ideal

$$j_N : B\mu_{Q, N} \hookrightarrow \mathcal{B}_N.$$

This enables us to completely describe the category of finite étale covers of \mathcal{B}_N .

Let $U \rightarrow \mathcal{B}_N$ be a finite étale morphism with U connected. Then $U_0 := j_N^* U$ is connected and finite étale over $B\mu_{Q, N}$ and therefore isomorphic to

$$B\mu_{U_0} \rightarrow B\mu_{Q, N}$$

for some closed subgroup $\mu_{U_0} \subset \mu_{Q, N}$. Such a subgroup is given by a quotient $z : Q^{\text{gp}} / NQ^{\text{gp}} \rightarrow A$. By the invariance of the étale site under infinitesimal thickenings (see for example [E, 3.41]) it follows that $U \rightarrow \mathcal{B}_N$ is isomorphic to the quotient

$$[\text{Spec}(k \otimes_{k[Q], N} k[Q]) / \mu_{U_0}]$$

with its natural map to \mathcal{B}_N . Let $Q' \subset Q$ be the set of elements $q' \in Q$ for which $z(q') = 0$. Then the log scheme obtained from U by the construction 2.13 is the scheme

$$\text{Spec}(k \otimes_{k[Q]} k[Q'])$$

with the natural log structure $M_{Q'}$ induced by Q' .

Observe also that the projection $Q^{\text{gp}} \rightarrow A$ induces an isomorphism $Q^{\text{gp}} / Q'^{\text{gp}} \simeq A$. Indeed it is clear that Q'^{gp} is in the kernel of the map to A , and if $q \in Q^{\text{gp}}$ is an element in the kernel then there exists $y \in Q$ such that $q' := q + Ny \in Q \cap \text{Ker}(Q^{\text{gp}} \rightarrow A) = Q'$. Therefore $q = q' - Ny$ lies in Q'^{gp} .

It follows that the finite étale cover $U \rightarrow \mathcal{B}_N$ is the stack obtained by the construction of 2.2 from the morphism of log schemes

$$(\text{Spec}(k \otimes_{k[Q]} k[Q']), M_{Q'}) \rightarrow (b, M_b).$$

3.2. If $N|M$ then there is a natural morphism of stacks

$$\pi_{M,N} : \mathcal{B}_M \rightarrow \mathcal{B}_N.$$

In fact if $q_N : \mathcal{B}_N \rightarrow b$ is the structure morphism and $\alpha : q_N^* M_b \rightarrow M_{\mathcal{B}_N}$ the tautological morphism of log structures over \mathcal{B}_N , then we can consider the stack $\mathcal{B}_{N,M/N}$ over \mathcal{B}_N classifying M/N -th roots of $M_{\mathcal{B}_N}$, and it follows immediately from the construction that composition with α defines an isomorphism of stacks

$$\mathcal{B}_{N,M/N} \rightarrow \mathcal{B}_M,$$

and in particular we obtain the projection $\pi_{M,N}$ by taking the inverse of this isomorphism followed by the projection $\mathcal{B}_{N,M/N} \rightarrow \mathcal{B}_N$. Furthermore, it follows from the preceding discussion that if $U \rightarrow \mathcal{B}_N$ is a finite étale morphism with associated Kummer morphism $(c, M_c) \rightarrow (b, M_b)$ then the base change $U \times_{\mathcal{B}_N} \mathcal{B}_M$ is the finite étale morphism over \mathcal{B}_M also corresponding to (c, M_c) . Since every object of $\text{Fet}((b, M_b))$ is obtained by this construction for some N prime to the characteristic of k we obtain an equivalence of categories

$$(3.2.1) \quad \text{Fet}((b, M_b)) \simeq \text{colim}_{N \in \mathbb{N}'} \text{Fet}(\mathcal{B}_N),$$

where the colimit on the right is taken with respect to the morphisms $\pi_{M,N}$ for $N|M$ for N prime to p .

4. PROOF OF THEOREM 1.2

4.1. We may without loss of generality assume that the log structure M_S is trivial. For a \widehat{A} -algebra B write (X_B, M_{X_B}) for the base change of (X, M_X) to $\text{Spec}(B)$, so $(X_{\widehat{A}}, M_{X_{\widehat{A}}}) = (X, M_X)$. Let $\mathfrak{m} \subset \widehat{A}$ be the maximal ideal, and write A_n for the quotient $\widehat{A}/\mathfrak{m}^n$. First we show the full faithfulness of the functor (1.2.1):

Proposition 4.2. *For any two objects $(U, M_U), (V, M_V) \in \text{Fet}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ the map*

$$(4.2.1) \quad \text{Hom}_{(X_{\widehat{A}}, M_{X_{\widehat{A}}})}((U, M_U), (V, M_V)) \rightarrow \text{Hom}_{(X_k, M_{X_k})}((U_k, M_{U_k}), (V_k, M_{V_k}))$$

is bijective.

Proof. Let $H_{\widehat{A}} \rightarrow X_{\widehat{A}}$ be the finite étale scheme classifying morphisms $(U, M_U) \rightarrow (V, M_V)$ over $(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ as in 2.21. We then want to show that a section of $H_k \rightarrow X_k$ lifts uniquely to a section of $H_{\widehat{A}} \rightarrow X_{\widehat{A}}$. This follows from the fact that the reduction functor

$$\text{Fet}(X_{\widehat{A}}) \rightarrow \text{Fet}(X_k)$$

is an equivalence of categories [SGA1, Exposé X, Théorème 2.1]. \square

4.3. To complete the proof of 1.2 it suffices to show that every object $(U_k, M_{U_k}) \in \text{Fet}(X_k, M_{X_k})$ is in the essential image of (1.2.1). To see this, recall the general fact that if $i : (T_0, M_{T_0}) \hookrightarrow (T, M_T)$ is an exact closed immersion of fine log schemes defined by a nilpotent ideal then the reduction functor

$$\text{Fet}(T, M_T) \rightarrow \text{Fet}(T_0, M_{T_0})$$

is an equivalence of categories. This follows from the same argument as in the classical case [SGA1, Exposé I, 8.3] combined with [K1, 3.14]. Therefore, for each $n \geq 1$ there exists a unique lifting $(U_n, M_{U_n}) \in \text{Fet}(X_{A_n}, M_{X_{A_n}})$ of (U_k, M_{U_k}) so we have a compatible system

$\{(U_n, M_{U_n})\}_n$ of log schemes. Let $N > 0$ be an integer as in 2.1. For each n we then obtain as in 2.9 a commutative square of log stacks

$$\begin{array}{ccc} (\mathcal{U}_n, M_{\mathcal{U}_n}) & \xrightarrow{q_n} & (\mathcal{X}_{A_n, N}, M_{\mathcal{X}_{A_n, N}}) \\ \downarrow p_{U_n} & & \downarrow p_{X_{A_n}} \\ (U_n, M_{U_n}) & \xrightarrow{f} & (X_{A_n}, M_{X_{A_n}}). \end{array}$$

The morphisms $q_n : \mathcal{U}_n \rightarrow \mathcal{X}_{A_n, N}$ are strict and étale as noted in 2.19.

By the Grothendieck existence theorem for stacks [O2, A.1] the system $\{q_n : \mathcal{U}_n \rightarrow \mathcal{X}_{A_n, N}\}$ of finite étale morphisms is uniquely algebraizable to a finite étale morphism $q : \mathcal{U} \rightarrow \mathcal{X}_{\widehat{A}, N}$. Let $U \rightarrow X_{\widehat{A}}$ be the coarse moduli space of \mathcal{U} and let M_U be the log structure on U given by $p_{U*}^{\log} q^* M_{\mathcal{X}_{\widehat{A}, N}}$, where $p_U : \mathcal{U} \rightarrow U$ is the projection. We claim that (U, M_U) is an object of $\text{Fet}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ reducing to (U_k, M_{U_k}) .

Note first of all that since \mathcal{U} is a tame Deligne-Mumford stack the formation of its coarse moduli space commutes with arbitrary base change [AOV, 3.3]. This implies that U reduces to the system $\{U_n\}$ over the A_n .

The log structure M_U is an fs log structure. Indeed since U is proper over \widehat{A} it suffices to show that every geometric point $\bar{u} \rightarrow U$ of the closed fiber admits an étale neighborhood over which M_U is fine and saturated. This follows from 2.17 and the fact that \mathcal{U} reduces to the \mathcal{U}_n by definition.

Furthermore the log structure M_U reduces to M_{U_n} over U_n . Indeed there is a natural map $M_U|_{U_n} \rightarrow M_{U_n}$ and to verify that this is an isomorphism it suffices to show that for every geometric point $\bar{u} \rightarrow U$ of the closed fiber this map induces an isomorphism $\overline{M}_{U, \bar{u}} \rightarrow \overline{M}_{U_n, \bar{u}}$. This follows from 2.14.

To complete the proof of 1.2 it now suffices to observe that the morphism $(U, M_U) \rightarrow (X_{\widehat{A}}, M_{X_{\widehat{A}}})$ is log étale and Kummer. Indeed the locus in U where this morphism is log étale and Kummer is an open (this statement for log étale morphisms follows from Kato's structure theorem [K1, 3.5] and the corresponding statement for schemes [SGA1, Exposé I, 4.5]; the statement that the Kummer type condition is open follows from the constructibility of the sheaves \overline{M}_U and $\overline{M}_{X_{\widehat{A}}}$), and since $X_{\widehat{A}}$, and therefore also U , is proper over $\text{Spec}(\widehat{A})$ this open set, which contains the closed fiber, must be all of U . \square

5. ARTIN APPROXIMATION AND ÉTALE COVERS

As in [A], theorem 1.2 can be generalized to the case when A is not necessarily complete but henselian. This result in the classical case is [A, 3.1] and in the log setting is due to Lepage. For the convenience of the reader we provide a proof.

Proposition 5.1 ([L, 1.8]). *Let A be a henselian local ring with residue field k , and let $f : (X_A, M_{X_A}) \rightarrow \text{Spec}(A)$ be an fs log scheme over A with underlying morphism $X_A \rightarrow \text{Spec}(A)$ proper and locally of finite presentation. Then the pullback functor*

$$(5.1.1) \quad \text{Fet}(X_A, M_{X_A}) \rightarrow \text{Fet}(X_k, M_{X_k})$$

is an equivalence of categories.

Proof. By a standard reduction as in the proof of [A, 3.1] it suffices to consider the case when A is the strict henselization of a finite type affine \mathbb{Z} -scheme $\mathrm{Spec}(S)$, and (X_A, M_{X_A}) is obtained by base change from a morphism of fs log schemes $f : (X, M_X) \rightarrow \mathrm{Spec}(S)$ with underlying morphism of schemes proper. Let k denote the residue field and let \widehat{A} denote the completion of A . Let $\mathfrak{m} \subset \widehat{A}$ denote the maximal ideal and for an integer $n \geq 1$ let A_n denote the quotient $\widehat{A}/\mathfrak{m}^n$, so $A_1 = k$.

For an S -algebra B write (X_B, M_{X_B}) for the base change of (X, M_X) to $\mathrm{Spec}(B)$. By 1.2 it then suffices to show that the functor

$$(5.1.2) \quad \mathrm{Fet}(X_A, M_{X_A}) \rightarrow \mathrm{Fet}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$$

is an equivalence of categories.

To prove this statement we first show that every object of $\mathrm{Fet}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ is in the essential image of (5.1.2). For this consider the functor

$$F : \mathrm{Alg}_S \rightarrow \mathrm{Set}$$

sending an S -algebra B to the set of isomorphism classes of objects in $\mathrm{Fet}(X_B, M_{X_B})$. This functor is limit preserving; that is, for any filtering inductive system of S -algebras $\{B_i\}$ with $B = \mathrm{colim}_i B_i$ the natural map

$$\mathrm{colim}_i F(B_i) \rightarrow F(B)$$

is bijective. Indeed the functor sending an S -algebra R to isomorphism classes of finite X_R -schemes is limit preserving by [EGA, IV, 8.5.2 and 8.5.5], and since the stacks $\mathcal{L}og_{(X, M_X)}$ introduced in [O1] are locally of finite type we further get that the functor sending an S -algebra to isomorphism classes of morphism of log schemes $(Y, M_Y) \rightarrow (X_R, M_{X_R})$ with $Y \rightarrow X_R$ finite is limit preserving. It therefore suffices to observe that the property of being Kummer étale is a condition locally of finite presentation which is immediate.

By the Artin approximation theorem [A, 1.12] it follows that given $(\widehat{U}, M_{\widehat{U}}) \in \mathrm{Fet}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ there exists an object $(U, M_U) \in \mathrm{Fet}(X_A, M_{X_A})$ such that $(\widehat{U}, M_{\widehat{U}})$ and (U, M_U) map to isomorphic objects in $\mathrm{Fet}(X_k, M_{X_k})$. By the bijectivity of (4.2.1) it follows that in fact $(\widehat{U}, M_{\widehat{U}})$ is isomorphic to the image of (U, M_U) under the functor (5.1.2).

It remains to show that given two objects $(U, M_U), (V, M_V) \in \mathrm{Fet}(X_A, M_{X_A})$ with induced objects $(\widehat{U}, M_{\widehat{U}}), (\widehat{V}, M_{\widehat{V}}) \in \mathrm{Fet}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ the map

$$(5.1.3) \quad \mathrm{Hom}_{(X_A, M_{X_A})}((U, M_U), (V, M_V)) \rightarrow \mathrm{Hom}_{(X_{\widehat{A}}, M_{X_{\widehat{A}}})}((\widehat{U}, M_{\widehat{U}}), (\widehat{V}, M_{\widehat{V}}))$$

is bijective. Let $H_A \rightarrow X_A$ be the finite étale X_A -scheme classifying morphisms $(U, M_U) \rightarrow (V, M_V)$ as in 2.21. Then the bijectivity of (5.1.3) is equivalent to the statement that base change gives a bijection between the set of sections of $H_A \rightarrow X_A$ and the set of sections of $H_{\widehat{A}} \rightarrow X_{\widehat{A}}$. This follows from [A, 3.1]. This completes the proof of 5.1. \square

6. PROOF OF 1.7 FOR INTEGRAL MORPHISMS

In this section we prove 1.7 in the case when (f, f^b) is assumed in addition to be integral [O, III, 2.5.1].

6.1. Let $h : \bar{b}'^{\log} \rightarrow \bar{b}^{\log}$ be a morphism of log geometric points over (B, M_B) . Then h factors as

$$\bar{b}'^{\log} \xrightarrow{a} \bar{b}''^{\log} \xrightarrow{b} \bar{b}^{\log},$$

where b is strict and given by an extension of separably closed fields, and a is an isomorphism on underlying fields.

6.2. By [IKN, A.4.2] any integral morphism of fs log schemes becomes saturated after a finite Kummer étale base change. Theorem 1.7 for a follows from this and [L, 2.15].

6.3. This therefore reduces the proof for integral morphisms to the case when h is strict, given by an inclusion of separably closed fields $k(\bar{b}) \subset k(\bar{b}')$.

The category of Kummer étale coverings forms a stack for the fppf topology. From this it follows immediately that (1.7.1) is an equivalence when $k(\bar{b}) \subset k(\bar{b}')$ is a purely inseparable algebraic extension. Indeed in this case every object of $\text{Fet}((X, M_X)_{(\bar{b}'^{\log})})$ has unique descent data since the kernel of the surjection

$$k(\bar{b}') \otimes_{k(\bar{b})} k(\bar{b}') \otimes_{k(\bar{b})} \cdots \otimes_{k(\bar{b})} k(\bar{b}') \rightarrow k(\bar{b}')$$

is nilpotent. It follows that in order to show that (1.7.1) is an equivalence in the case when h is strict we may assume that $k(\bar{b})$ is algebraically closed.

6.4. Next let us show that (1.7.1) is fully faithful. Consider two objects $(U, M_U), (V, M_V) \in \text{Fet}((X, M_X)_{(\bar{b}^{\log})})$, and let $(U', M_{U'}), (V', M_{V'}) \in \text{Fet}((X, M_X)_{(\bar{b}'^{\log})})$ be their base changes. We show that the natural map

$$(6.4.1) \quad \text{Hom}_{\text{Fet}((X, M_X)_{(\bar{b}^{\log})})}((U, M_U), (V, M_V)) \rightarrow \text{Hom}_{\text{Fet}((X, M_X)_{(\bar{b}'^{\log})})}((U', M_{U'}), (V', M_{V'}))$$

is bijective. The injectivity is clear by faithfully flat descent. For surjectivity, let $f' : (U', M_{U'}) \rightarrow (V', M_{V'})$ be a morphism. By spreading out, we can find an integral finite type $k(\bar{b})$ -scheme T with function field $k(\bar{b}')$ and an extension of the morphism f' to a morphism $f_T : (U_T, M_{U_T}) \rightarrow (V_T, M_{V_T})$ between the base changes of (U, M_U) and (V, M_V) to T . Restricting f_T to the fiber over a $k(\bar{b})$ -point of T , using that $k(\bar{b})$ is algebraically closed, we get a morphism $f : (U, M_U) \rightarrow (V, M_V)$ whose base change to T agrees with f_T at a point. By 4.2 and the injectivity of (6.4.1) already shown it follows that f_T agrees everywhere with the map obtained from f by base change to T . This completes the proof of the full faithfulness.

6.5. To show essential surjectivity let $(U', M_{U'}) \in \text{Fet}((X, M_X)_{(\bar{b}'^{\log})})$ be an object which we show is obtained by base change from an object of $\text{Fet}((X, M_X)_{(\bar{b}^{\log})})$. Spreading out and looking at a $k(\bar{b})$ -point as in the proof of full faithfulness we find a finite type $k(\bar{b})$ -scheme T , an extension $(U'_T, M_{U'_T})$ of $(U', M_{U'})$ to T and an object $(U, M_U) \in \text{Fet}((X, M_X)_{(\bar{b}^{\log})})$ whose base change to T is isomorphic to $(U'_T, M_{U'_T})$ at a point. Then by 4.2 there exists an extension of $k(\bar{b}')$ over which $(U', M_{U'})$ becomes isomorphic to the base change of (U, M_U) . By the full faithfulness already shown this implies that $(U', M_{U'})$ is isomorphic to the base change of (U, M_U) .

This completes the proof of 1.7 for integral morphisms.

7. VARIATIONS ON THE LOG PURITY THEOREM

7.1. Local version. In this subsection we consider variants (which follow from the original case) of Kato’s log purity theorem [M, 3.3].

We will consider two setups:

7.2. Setup 1. Let k be a field of characteristic p (possibly 0) and let M be an fs monoid with the torsion subgroup of M^{gp} of order invertible in k . Let $F \subset M$ be a face and let (A_F, M_{A_F}) denote the log scheme whose underlying scheme is $\text{Spec}(k[F])$ and whose log structure is induced by the map

$$M \rightarrow k[F], \quad m \mapsto \begin{cases} m & \text{if } m \in F \\ 0 & \text{otherwise.} \end{cases}$$

Let $A_F^\circ \subset A_F$ denote the open subset

$$\text{Spec}(k[F^{\text{gp}}]) \subset \text{Spec}(k[F]).$$

7.3. Setup 2. Let V be a discrete valuation ring with uniformizer π and residue field of characteristic p (possibly 0). Let M be an fs monoids, let $F \subset M$ be a face of M , and let $f \in F$ be an element. Let (A_F, M_{A_F}) denote the log scheme whose underlying scheme is

$$\text{Spec}(V[F]/(\pi - f)),$$

where we abusively write also $f \in V[F]$ for the “monomial” corresponding to f , and log structure M_{A_F} induced by the map of monoids

$$M \rightarrow V[F]/(\pi - f), \quad m \mapsto \begin{cases} m & \text{if } m \in F \\ 0 & \text{otherwise.} \end{cases}$$

Let $A_F^\circ \subset A_F$ denote the open subset

$$\text{Spec}(V[F^{\text{gp}}]/(\pi - f)) \subset \text{Spec}(V[F]/(\pi - f)).$$

Note that A_F° is a scheme over $K := \text{Frac}(V)$.

7.4. Considering either setup, let

$$(X, M_X) \rightarrow (A_F, M_{A_F})$$

be a strict étale morphism and let

$$(X^\circ, M_{X^\circ}) \subset (X, M_X)$$

denote the preimage of A_F° .

Theorem 7.5. *The restriction functor*

$$\text{Fet}^{(p)}(X, M_X) \rightarrow \text{Fet}^{(p)}(X^\circ, M_{X^\circ})$$

is an equivalence of categories.

Proof. Note that the underlying scheme X is normal [K2, 4.1]. This implies that if $H \rightarrow X$ is a finite étale morphism then any section $X^\circ \rightarrow H$ over X° extends uniquely to a section over X . From this and 2.21 the full faithfulness of the restriction functor follows.

For an integer $n \geq 1$ prime to p define

$$\times n : (A_{F,n}, M_{A_{F,n}}) \rightarrow (A_F, M_{A_F})$$

as follows.

Setup 1. Here we take $(A_{F,n}, M_{A_{F,n}})$ to be equal to (A_F, M_{A_F}) and the map $\times n$ to be the morphism induced by multiplication by n on M and F .

Setup 2. Here we let V_n be the extension of V obtained by adjoining an n -th root π_n of π , setting

$$A_{F,n} := \text{Spec}(V_n[F]/(\pi_n - f))$$

with log structure induced by the map $M \rightarrow V_n[F]$ as before and $\times n$ induced by the natural map $V \rightarrow V_n$, and multiplication by n on F and M .

To prove the essential surjectivity, it suffices to show that for any Kummer étale cover

$$(7.5.1) \quad (U^\circ, M_{U^\circ}) \rightarrow (X^\circ, M_{X^\circ})$$

in $\text{Fet}^{(p)}(X^\circ, M_{X^\circ})$ there exists an integer n , prime to p , such that the base change

$$(7.5.2) \quad (U^\circ, M_{U^\circ}) \times_{(A_F, M_{A_F}), \times n} (A_{F,n}, M_{A_{F,n}})$$

extends to a strict étale cover of the base change

$$(X, M_X) \times_{(A_F, M_{A_F}), \times n} (A_{F,n}, M_{A_{F,n}}).$$

Indeed there is a commutative diagram

$$\begin{array}{ccc} (A_{F,n}, M_{A_{F,n}}) & \hookrightarrow & (A_F, M_{A_F}) \times_{\text{Spec}(M \rightarrow \mathbf{Z}[M]), \times n} \text{Spec}(M \rightarrow \mathbf{Z}[M]) \\ & \searrow^{\times n} & \downarrow \\ & & (A_F, M_{A_F}), \end{array}$$

where the right vertical morphism is the Kummer étale morphism given by multiplication by n on M , and the top inclusion is a strict closed immersion defined by a nilpotent ideal. Therefore (7.5.2) extends to a strict étale cover if and only if the base change

$$(U^\circ, M_{U^\circ}) \times_{(A_F, M_{A_F})} ((A_F, M_{A_F}) \times_{\text{Spec}(M \rightarrow \mathbf{Z}[M]), \times n} \text{Spec}(M \rightarrow \mathbf{Z}[M]))$$

has this property.

For this note that by the definition of a Kummer étale morphism, we can find an integer n prime to p such that the base change (7.5.2) is a strict étale cover of the base change

$$X^\circ \times_{A_F, \times n} A_{F,n}.$$

Making such a base change, we are reduced to the case when (7.5.1) is strict and étale.

Let (Y, M_Y) be the log scheme with underlying scheme $Y = X$ and log structure M_Y induced by the map $F \rightarrow \mathcal{O}_X$, so we have a morphism of fs log schemes

$$(X, M_X) \rightarrow (Y, M_Y).$$

By the log purity theorem [M, 3.3] the cover U° of X° extends to a Kummer étale cover $(U, M_U) \rightarrow (Y, M_Y)$, and applying further base change to (X, M_X) we get the desired extension. \square

Remark 7.6. In the above we could also have considered X étale over the power series ring $k[[F]]$ or $V[[F]]/(\pi - f)$. The same result holds with the same proof (simply take $A_F = \text{Spec}(k[[F]])$ or $\text{Spec}(V[[F]]/(\pi - f))$ in the above proof).

7.7. Global version.

7.8. Let (A_F, M_{A_F}) be as in the previous section (either setup 1 or setup 2), and let

$$(f, f^b) : (X, M_X) \rightarrow (A_F, M_{A_F})$$

be a log smooth integral morphism of fs log schemes. As before let (X°, M_{X°) denote the preimage of $(A_F^\circ, M_{A_F^\circ})$. Then we have the same result:

Theorem 7.9. *The restriction functor*

$$\text{Fet}^{(p)}(X, M_X) \rightarrow \text{Fet}^{(p)}(X^\circ, M_{X^\circ})$$

is an equivalence of categories.

Proof. We will reduce the proof to the previous case. We may work étale locally on X , so by [O, Chapter IV, 3.3.1] we may assume that we have an integral morphism of fs monoids

$$\theta : M \rightarrow N$$

with

$$\theta^{\text{gp}} : M^{\text{gp}} \rightarrow N^{\text{gp}}$$

injective with torsion of $M^{\text{gp}}/N^{\text{gp}}$ of order invertible in k , and an étale morphism

$$X \rightarrow \text{Spec}(R),$$

where

$$R = \begin{cases} k[F] \otimes_{k[M]} k[N] & \text{Setup 1,} \\ (V[F]/(\pi - f)) \otimes_{V[M]} V[N] & \text{Setup 2.} \end{cases}$$

We may further assume that N^{gp} is torsion free.

Let $Z \subset \text{Spec}(R)$ be an irreducible component, viewed as a closed subscheme with the reduced structure. We can describe the coordinate ring \mathcal{O}_Z as follows.

Setup 1. Since $D(N^{\text{gp}}) := \text{Hom}(N^{\text{gp}}, \mathbf{G}_m)$ is connected, the action of $D(N^{\text{gp}})$ restricts to an action on Z , and looking at the character decomposition of the coordinate ring we find that

$$Z = Z_S := \text{Spec}(k[S])$$

for a face $S \subset N$. Furthermore, we must have $S \cap M \subset F$ since the map $Z \rightarrow \text{Spec}(k[M])$ factors through $\text{Spec}(k[F])$. In fact, $S \cap M$ is a face of M and the map factors through the closed subscheme of $\text{Spec}(k[F])$ defined by this face. Since

$$\text{Spec}(k[F] \otimes_{k[M]} k[N]) \rightarrow \text{Spec}(k[F])$$

is flat, Z dominates $\text{Spec}(k[F])$ and therefore we must have $S \cap M = F$.

Setup 2. Let $S \subset N$ be the face of elements whose image in \mathcal{O}_Z is nonzero. Then $S \cap M = F$. Indeed since θ is an integral morphism the map $Z \rightarrow A_F$ is dominant which implies that all elements of F map to nonzero elements in \mathcal{O}_Z and therefore $F \subset S \cap M$, and the reverse inclusion $S \cap M \subset F$ is immediate. Let $Z_S \subset \text{Spec}(R)$ be the closed subscheme given by the surjection

$$R \rightarrow V[S]/(\pi - f)$$

induced by the map $V[N] \rightarrow V[S]$ sending $n \in N$ to n for $n \in S$ and 0 otherwise. By definition of S , this surjection factors through \mathcal{O}_Z and therefore we get an inclusion $i : Z \hookrightarrow Z_S$. This

inclusion i is in fact an isomorphism. To see this, note that by looking at the character decomposition of the action of $D(N^{\text{gp}}/(f))$ on the ring

$$R^\circ := K[F^{\text{gp}}]/(f - \pi) \otimes_{K[M_F]} K[N_F],$$

which is the coordinate ring of $\text{Spec}(R) \times_{A_F} A_F^\circ$, one gets that the coordinate ring of $Z^\circ := Z \times_{A_F} A_F^\circ$ is given by

$$K[S']/(\pi - f),$$

where $S' \subset N_F$ is a face with $S' \cap M_F = F^{\text{gp}}$. Furthermore, since Z is flat over A_F we have $S = S' \cap N$, which in turn implies that $S' = S_F$. We conclude that i restricts to an isomorphism over A_F° , and since Z is flat over A_F this implies that i is an isomorphism.

Returning to the proof of the theorem in either setup 1 or 2, let \mathcal{S} denote the set of faces $S \subset N$ such that $S \cap M = F$, and for $S \in \mathcal{S}$ let $X_S \subset X$ be the preimage of

$$Z_S \subset A_F$$

and let M_{X_S} be the restriction of M_X to X_S . Note that for $S_1, S_2 \in \mathcal{S}$ we have

$$X_{S_1} \cap X_{S_2} = X_{S_1 \cap S_2},$$

and $S_1 \cap S_2 \in \mathcal{S}$. Then the functor induced by restriction

$$(7.9.1) \quad \text{Fet}(X, M_X) \rightarrow \lim_{S \in \mathcal{S}} \text{Fet}(X_S, M_{X_S})$$

is an equivalence, and similarly for $\text{Fet}(X^\circ, M_{X^\circ})$. In the classical case without log structures this follows from [SGA4, VIII, 9.4]. The logarithmic version can be deduced from this as follows. Consider the fibered category \mathcal{F} (resp. \mathcal{F}_S for $S \in \mathcal{S}$) over the Kummer étale site of (X, M_X) which to any (U, M_U) associates the category of finite étale U -schemes (resp. U_S -schemes, where $(U_S, M_{U_S}) := (U, M_U) \times_{(X, M_X)} (X_S, M_{X_S})$). Let \mathcal{F}^a (resp. \mathcal{F}_S^a) be the stack over the Kummer étale site of (X, M_X) associated to \mathcal{F} (resp. \mathcal{F}_S). Then it follows from [I, 3.13] that we have

$$\text{Fet}(X, M_X) \simeq \mathcal{F}^a(X, M_X), \quad \text{Fet}(X_S, M_{X_S}) = \mathcal{F}_S^a(X, M_X).$$

Furthermore, the natural functor

$$(\lim_{S \in \mathcal{S}} \mathcal{F}_S)^a \rightarrow \lim_{S \in \mathcal{S}} \mathcal{F}_S^a$$

is an equivalence (using that there are only finitely many categories in this limit). To prove that (7.9.1) is an equivalence it therefore suffices to show that the map of prestacks

$$\mathcal{F} \rightarrow \lim_{S \in \mathcal{S}} \mathcal{F}_S$$

is an equivalence, which follows from the case of ordinary schemes.

This reduces the proof to showing that each of the functors

$$\text{Fet}(X_S, M_{X_S}) \rightarrow \text{Fet}(X_S^\circ, M_{X_S^\circ})$$

is an equivalence, which follows from 7.5. □

8. PROOF OF THEOREM 1.9 IN THE CASE OF AN INTEGRAL MORPHISM

We proceed with the notation of 1.9. Since B is connected, in order to prove 1.9 it suffices to consider the case when B is irreducible. Furthermore, it suffices to consider the case when (B, M_B) is defined over a field or a complete discrete valuation ring.

8.1. We say that a log geometric point $\bar{b}^{\log} \rightarrow (B, M_B)$ is *quasi-strict* if the map

$$\overline{M}_{B, \bar{b}} \rightarrow \overline{M}_{\bar{b}^{\log}}$$

induces an isomorphism $\overline{M}_{B, \bar{b}, \mathbb{Z}(p)} \rightarrow \overline{M}_{\bar{b}^{\log}}$, where p is the residue characteristic of \bar{b} .

For any log geometric point $\bar{b}^{\log} \rightarrow (B, M_B)$ there exists a morphism of log geometric points of (B, M_B)

$$(8.1.1) \quad \begin{array}{ccc} \bar{b}^{\log} & \longrightarrow & \bar{b}'^{\log} \\ & \searrow & \downarrow \\ & & (B, M_B), \end{array}$$

where $\bar{b}'^{\log} \rightarrow (B, M_B)$ is quasi-strict. Indeed by definition of log geometric point we get an induced map $\overline{M}_{B, \bar{b}, \mathbb{Z}(p)} \rightarrow \overline{M}_{\bar{b}^{\log}}$. Choose a lifting $\overline{M}_{B, \bar{b}, \mathbb{Z}(p)} \rightarrow M_{\bar{b}'^{\log}}$ of this map, and let $M_{\bar{b}'^{\log}}$ denote the associated log structure on \bar{b}' . Setting $\bar{b}'^{\log} := (\bar{b}', M_{\bar{b}'^{\log}})$ we get the desired factorization (8.1.1).

In particular, by this discussion and 1.7 it suffices to prove 1.9 in the case when $\bar{b}^{\log} \rightarrow (B, M_B)$ is quasi-strict.

8.2. Let $\bar{s}^{\log} \rightarrow (B, M_B)$ and $\bar{b}^{\log} \rightarrow (B, M_B)$ be two log geometric points of (B, M_B) . A *specialization* $\bar{s}^{\log} \rightsquigarrow \bar{b}^{\log}$ is a commutative diagram

$$\begin{array}{ccc} \bar{b}^{\log} & \xrightarrow{\alpha} & (E, M_E) \xleftarrow{\beta} \bar{s}^{\log} \\ & \searrow & \downarrow \swarrow \\ & & (B, M_B), \end{array}$$

where α and β are strict and (E, M_E) is log strictly local in the sense of [N, 2.8 (6)] with residue field given by \bar{b} .

Given such a specialization we get an induced cospecialization functor

$$(8.2.1) \quad \text{cosp} : \text{Fet}((X, M_X)_{(\bar{b}^{\log})}) \rightarrow \text{Fet}((X, M_X)_{(\bar{s}^{\log})})$$

defined as follows.

Write $\bar{b}^{\log} = (\bar{b}, M_{\bar{b}^{\log}})$, and $M_{\bar{b}^{\log}} = \text{colim}_{\lambda} M_{\bar{b}, \lambda}$, where $M_{\bar{b}, \lambda}$ is a fs log structure contained in $M_{\bar{b}^{\log}}$ and containing the image of $M_{B, \bar{b}}$. Let \bar{b}_{λ}^{\log} denote $(\bar{b}, M_{\bar{b}, \lambda})$ so $\bar{b}^{\log} = \lim_{\lambda} \bar{b}_{\lambda}^{\log}$. Let $M_{E, \lambda} \subset M_E$ be the sublog structure induced by the submonoid $M_{E, \bar{b}} \times_{\overline{M}_{E, \bar{b}}} \overline{M}_{\bar{b}, \lambda}$, so $M_E = \text{colim}_{\lambda} M_{E, \lambda}$, and let $M_{\bar{s}, \lambda} \subset M_{\bar{s}^{\log}}$ denote the log structure defined by the image of

$M_{E,\lambda}$. Setting $\bar{s}_\lambda^{\log} := (\bar{s}, M_{\bar{s},\lambda})$ we then have a commutative diagram for all λ

$$\begin{array}{ccccc} \bar{b}^{\log} & \hookrightarrow & (E, M_E) & \longleftarrow & \bar{s}^{\log} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{b}_\lambda & \hookrightarrow & (E, M_{E,\lambda}) & \longleftarrow & \bar{s}_\lambda^{\log} \end{array}$$

We then get restriction functors

$$\mathrm{Fet}((X, M_X)_{\bar{b}_\lambda^{\log}}) \xleftarrow{u} \mathrm{Fet}((X, M_X)_{(E, M_{E,\lambda})}) \xrightarrow{v} \mathrm{Fet}((X, M_X)_{\bar{s}_\lambda^{\log}}),$$

where u is an equivalence by 5.1 and 1.2. The functor $v \circ u^{-1}$ then is a functor

$$\mathrm{Fet}((X, M_X)_{\bar{b}_\lambda^{\log}}) \rightarrow \mathrm{Fet}((X, M_X)_{\bar{s}_\lambda^{\log}}).$$

Passing to the limit over λ we get the functor (8.2.1).

Lemma 8.3. *Let $\bar{x} \rightarrow B$ be a geometric point. After replacing (B, M_B) by a strict étale neighborhood of \bar{x} there exists a morphism of log schemes $(\tilde{B}, M_{\tilde{B}}) \rightarrow (B, M_B)$ with the following properties:*

- (i) \bar{x} lifts to a geometric point of \tilde{B} .
- (ii) \tilde{B} is integral and the morphism $\tilde{B} \rightarrow B$ is dominant.
- (iii) There exists an fs monoid M with a face $F \subset M$ and a strict étale morphism

$$(\tilde{B}, M_{\tilde{B}}) \rightarrow (A_F, M_{A_F}),$$

where (A_F, M_{A_F}) is as in setup 1 or setup 2 in 7.2 and 7.3 (for some discrete valuation ring V).

Proof. We may assume that we have a chart $P \rightarrow \mathcal{O}_B$ for M_B , with P an fs sharp monoid. Let $I \subset P$ be the face of elements whose image in B is nonzero, and let $Z \subset B$ be the closed subscheme defined by I .

We now apply de Jong's alteration results.

In the case when the base ring is a field k , we apply [dJ, 4.1] to find a dominant morphism $Y \rightarrow B$ such that Y is smooth over k and the preimage of Z is a divisor with simply normal crossings. Let B_1 denote an open subset of a point of Y lying over the image of \bar{x} such that the restriction of the divisor to B_1 is given by a morphism $\mathbf{N}^r \rightarrow \mathcal{O}_{B_1}$. Let M_{int} (resp. M) denote the pushout in the category of fine (resp. fs) monoids of the diagram

$$\begin{array}{ccc} I & \longrightarrow & P \\ \downarrow & & \\ \mathbf{N}^r & & \end{array}$$

By [O, Chapter I, 2.2.1] there exists a morphism $h : P \rightarrow \mathbf{N}$ such that $h^{-1}(0) = I$. Let $\tilde{h} : M \rightarrow \mathbf{N}$ be the unique extension which is 0 on \mathbf{N}^r , and let $F \subset M$ denote $h^{-1}(0)$. If $m = p + f \in M_{\mathrm{int}} \cap F$, where $p \in P$ and $f \in \mathbf{N}^r$, then we must have $p \in I$ and therefore $m \in \mathbf{N}^r$.

It follows that F is the saturation of \mathbf{N}^r in M , and that \mathbf{N}^r is a face in M_{int} . The given map $\mathbf{N}^r \rightarrow \mathcal{O}_{B_1}$ then extends to a map $M_{\text{int}} \rightarrow \mathcal{O}_{B_1}$. Let \tilde{B} denote the fiber product

$$B_1 \times_{\text{Spec}(k[\mathbf{N}^r])} \text{Spec}(k[F]),$$

and let $M_{\tilde{B}}$ be the log structure induced by the map $M \rightarrow k[F]$. Then this satisfies the assumptions of the lemma.

In the case when the base is a discrete valuation ring, we proceed in a very similar manner. By [dJ, 6.5] we can find $B_1 \rightarrow B$ and an étale map

$$B_1 \rightarrow \text{Spec}(V[\mathbf{N}^r]/(\pi - f))$$

for some $f \in \mathbf{N}^r$ and a discrete valuation ring V . Now proceeding with the pushout of monoids as in the previous case we obtain the desired $(\tilde{B}, M_{\tilde{B}})$. \square

8.4. Using this lemma we are then further reduced to the case when (B, M_B) admits a strict étale morphism

$$(B, M_B) \rightarrow (A_F, M_{A_F}),$$

and to showing that for a specialization of quasi-strict log geometric points $\bar{\eta}^{\text{log}} \rightsquigarrow \bar{b}^{\text{log}}$, with $\bar{\eta}^{\text{log}}$ lying over the generic point of B and \bar{b}^{log} mapping to the vertex of A_F , the cospecialization functor is an equivalence of categories.

8.5. Let S denote the strict henselization of B at \bar{b} , and for an integer $n \geq 1$ prime to p let S_n denote the fiber product of the diagram

$$\begin{array}{ccc} & & A_{F,n} \\ & & \downarrow \times n \\ \text{Spec}(S) & \longrightarrow & A_F, \end{array}$$

where $\times n : A_{F,n} \rightarrow A_F$ is as in the proof of 7.5, and let M_{S_n} denote the log structure on $\text{Spec}(S_n)$ induced by that on $A_{F,n}$. Let (X_n, M_{X_n}) denote the base change of (X, M_X) to $(\text{Spec}(S_n), M_{S_n})$. Let

$$(b_n, M_{b_n}) \hookrightarrow (\text{Spec}(S_n), M_{S_n})$$

be the closed point with the induced log structure, and let $(\text{Spec}(K_n), M_{K_n})$ be the generic fiber over $(\text{Spec}(S_n), M_{S_n})$.

Finally we may assume that

$$\bar{b}^{\text{log}} = \lim_n (b_n, M_{b_n})$$

and that $\bar{\eta}^{\text{log}}$ admits a strict morphism

$$\bar{\eta}^{\text{log}} \rightarrow \lim_n (\text{Spec}(K_n), M_{K_n})$$

whose underlying morphism of schemes is given by a separable closure of $\text{colim}_n K_n$.

8.6. We then get a diagram

$$\begin{array}{ccc} \text{colim}_n \text{Fet}^{(p)}((X, M_X) \times_{(b, M_b)} (b_n, M_{b_n})) & \xleftarrow{u} & \text{colim}_n \text{Fet}^{(p)}(X_n, M_{X_n}) & \xrightarrow{v} & \text{Fet}^{(p)}((X, M_X)_{(\bar{\eta}^{\text{log}})}) \\ \downarrow \simeq & & & & \\ \text{Fet}^{(p)}((X, M_X)_{(\bar{b}^{\text{log}})}) & & & & \end{array}$$

By 1.2 the functor u is an equivalence.

To show that v is an equivalence it suffices to show that for any morphism

$$\rho_{\bar{\eta}^{\log}} : (U_{\bar{\eta}}, M_{U_{\bar{\eta}}}) \rightarrow (V_{\bar{\eta}}, M_{V_{\bar{\eta}}})$$

in $\text{Fet}^{(p)}((X, M_X)_{\bar{\eta}^{\log}})$ there exists an integer n prime to p and a morphism

$$\rho_n : (U_n, M_{U_n}) \rightarrow (V_n, M_{V_n})$$

in $\text{Fet}^{(p)}(X_n, M_{X_n})$ inducing $\rho_{\bar{\eta}^{\log}}$. Note that such a morphism is necessarily unique by the purity theorem 7.9.

By the definition of $\text{Fet}^{(p)}((X, M_X)_{\bar{\eta}^{\log}})$ as a direct limit, there exists an integer m and a finite extension L/K_m such that if M_L is the log structure on $\text{Spec}(L)$ obtained by pullback from M_{K_m} then there exists a morphism

$$\rho_L : (U_L, M_{U_L}) \rightarrow (V_L, M_{V_L})$$

in $\text{Fet}^{(p)}((X, M_X) \times_{(B, M_B)} (\text{Spec}(L), M_L))$ inducing $\rho_{\bar{\eta}^{\log}}$.

Let T denote the normalization of S_m in L so we get a finite strict morphism

$$(\text{Spec}(T), M_T) \rightarrow (\text{Spec}(S_m), M_{S_m})$$

inducing the map $(\text{Spec}(L), M_L) \rightarrow (\text{Spec}(K_m), M_{K_m})$. Applying 8.3 to a finite type approximation of $(\text{Spec}(T), M_T)$ we then obtain a commutative diagram

$$\begin{array}{ccccc}
 \bar{b}'^{\log} & & & & \bar{\eta}'^{\log} \\
 \downarrow & \searrow & & & \downarrow & \searrow \\
 & (\text{Spec}(S'), M_{S'}) & \longleftarrow & (\text{Spec}(K'), M_{K'}) & & \bar{\eta}^{\log} \\
 & \downarrow & & \downarrow & & \downarrow \\
 \bar{b}^{\log} & & & (\text{Spec}(T), M_T) & \longleftarrow & (\text{Spec}(L), M_L) \\
 & \downarrow & & \downarrow & & \downarrow \\
 & (\text{Spec}(S_m), M_{S_m}) & \longleftarrow & (\text{Spec}(K_m), M_{K_m}) & &
 \end{array}$$

where $(\text{Spec}(S'), M_{S'})$ admits a strict étale morphism to $(A_{F'}, M_{A_{F'}})$ for suitable fs monoid M' with face F' (and in setup 2 a discrete valuation ring V'). By the purity theorem we then get an extension of ρ_L to a morphism in $\text{Fet}^{(p)}((X, M_X) \times_{(B, M_B)} (\text{Spec}(S'), M_{S'}))$. By 1.7 the image of this morphism in $\text{Fet}^{(p)}((X, M_X)_{(\bar{b}'^{\log})})$ is induced by a morphism in $\text{Fet}^{(p)}((X, M_X)_{(\bar{b}^{\log})})$. Using the fact that the reduction functor u is an equivalence we therefore obtain a morphism ρ_n in $\text{Fet}^{(p)}(X_n, M_{X_n})$, for some n , whose image in $\text{Fet}^{(p)}((X, M_X)_{(\bar{\eta}'^{\log})})$ agrees with the image of $\rho_{\bar{\eta}^{\log}}$. By 1.7 we then conclude that $v(\rho_n) = \rho_{\bar{\eta}^{\log}}$ as desired.

This completes the proof of 1.9 in the case of an integral morphism. \square

Remark 8.7. It is in the application of the log purity theorem [M, 3.3] that the prime-to- p assumption is crucial.

9. INVARIANCE UNDER LOG BLOWUPS

To prove 1.7 and 1.9 in general we will use results of Fujiwara and Kato on invariance under log blowups of the category of finite Kummer étale covers. Since these results are not published we provide a proof.

9.1. Let (X, M_X) be an fs log scheme and let $\mathcal{I} \subset M_X$ be a coherent sheaf of ideals (see [O, Chapter II, 2.6.1]), all of whose stalks are nonempty. We can then consider the log blowup

$$\pi : (X', M_{X'}) \rightarrow (X, M_X)$$

defined as in [O, Chapter III, 2.6.2].

Theorem 9.2 (Fujiwara-Kato [I, 6.10 and references therein]). *Assume that X is locally of finite type over an excellent Dedekind ring A , and let \mathbf{L} denote the set of residue characteristics of X . Then the pullback functor*

$$(9.2.1) \quad \text{Fet}^{\mathbf{L}}(X, M_X) \rightarrow \text{Fet}^{\mathbf{L}}(X', M_{X'})$$

is an equivalence of categories.

Proof. By [I, 3.13] the categories of Kummer étale covers of (X, M_X) and $(X', M_{X'})$ form stacks for the Kummer étale topology. It therefore suffices to prove the theorem after replacing X by an étale cover. We may therefore assume that $X = \text{Spec}(R)$ is affine and that we have a chart $P \rightarrow R$ such that $(X', M_{X'})$ is defined by blowing up a coherent sheaf of ideals $I \subset P$, where P is an fs sharp monoid.

Observe that a log blowup has connected fibers. Indeed, for this it suffices to consider the case when $X = \text{Spec}(\mathbf{Z}[P])$. In this case $\pi_* \mathcal{O}_{X'}$ is a coherent sheaf of algebras corresponding to a finite $\mathbf{Z}[P]$ -algebra C which is an integral domain and with $\mathbf{Z}[P] \rightarrow C$ an isomorphism over $\mathbf{Z}[P^{\text{gp}}]$. Since P is saturated, which implies that $\mathbf{Z}[P]$ is normal, this implies that $C = \mathbf{Z}[P]$; that is, $\mathcal{O}_X = \pi_* \mathcal{O}_{X'}$. From this and [EGA, III, 4.3.2] we conclude that the fibers are connected. This, in turn, implies that the functor (9.2.1) is fully faithful.

For an integer $n \geq 1$ not divisible by the primes in \mathbf{L} let $(X_n, M_{X_n}) \rightarrow (X, M_X)$ be the Kummer étale cover given by the fiber product

$$(9.2.2) \quad X_n := X \times_{\text{Spec}(\mathbf{Z}[P]), \times n} \text{Spec}(\mathbf{Z}[P]),$$

where $\times n : \text{Spec}(\mathbf{Z}[P]) \rightarrow \text{Spec}(\mathbf{Z}[P])$ is induced by multiplication by n on P , and M_{X_n} is induced by the natural log structure on the second factor. Let $\mathcal{I}_n \subset M_{X_n}$ be the coherent sheaf of ideals given by $I \subset P$ (again on the second factor in (9.2.2)). Let $(X'_n, M_{X'_n})$ denote the log blowup of (X_n, M_{X_n}) along \mathcal{I}_n . Note that the ideal \mathcal{I} generates an invertible sheaf of ideals in $M_{X'_n}$ so we obtain a commutative diagram

$$(9.2.3) \quad \begin{array}{ccc} (X'_n, M_{X'_n}) & \xrightarrow{q_n} & (X', M_{X'}) \\ \downarrow & & \downarrow \\ (X_n, M_{X_n}) & \longrightarrow & (X, M_X) \end{array}$$

We can describe this diagram explicitly locally on X' . Let $a_1, \dots, a_r \in I$ be generators for I , and for $j = 1, \dots, r$ define $P_j \subset P^{\text{gp}}$ denote the monoid generated by P and the elements

$$z_i := a_i - a_j.$$

Then X' is covered by the affine schemes

$$\mathrm{Spec}(R \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P_j]).$$

Now observe that the diagram of monoids

$$\begin{array}{ccc} P & \xrightarrow{\times n} & P \\ \downarrow & & \downarrow \\ P_j & \xrightarrow{\times n} & P_j \end{array}$$

is a pushout diagram in the category of fs monoids. From this it follows that the diagram (9.2.3) is cartesian, in the category of fs log schemes, and that for any object $(U, M_U) \in \mathrm{Fet}^{\mathbf{L}}(X', M_{X'})$ there exists an integer n not divisible by primes in \mathbf{L} such that the pullback of (U, M_U) to $(X'_n, M_{X'_n})$ is given by a finite étale morphism $U_n \rightarrow X'_n$ with the pullback log structure.

Making a base change $(X_n, M_{X_n}) \rightarrow (X, M_X)$, we are therefore reducing to showing that any finite étale covering of schemes $U' \rightarrow X'$ descends to X . By a standard limit argument, and an application of Artin approximation we are then reduced to the variant statement that if R is a complete local ring with separably closed residue field, which is the completion of the strict henselization of a finite type algebra over a field or discrete valuation ring at a point, then any object of $\mathrm{Fet}^{\mathbf{L}}(X')$ is trivialized by a Kummer étale extension of (X, M_X) . Let R_0 be a finite type algebra over our base field or discrete valuation ring such that R is the completion of the local ring $\mathcal{O}_{\mathrm{Spec}(R_0), \bar{x}}$ at a geometric point, then choose an epimorphism $S_0 := A[X_1, \dots, X_t] \rightarrow R_0$ for some t and consider the resulting epimorphism

$$S \rightarrow R,$$

where S is the completion of the strict henselization of $A[X_1, \dots, X_t][P]$ at the point given by \bar{x} . Now observe that by 1.2 the restriction functor

$$\mathrm{Fet}^{\mathbf{L}}(X'_S, M_{X'_S}) \rightarrow \mathrm{Fet}^{\mathbf{L}}(X', M_{X'})$$

is an equivalence of categories, where $(X'_S, M_{X'_S})$ denotes the log blowup of I over S . In this way we are further reduced to the case when the complete local base $(X, M_X) = (\mathrm{Spec}(R), M_R)$ is log regular. Let $X_{\mathrm{triv}} \subset X$ be the maximal open subset over which M_X is trivial. By the log purity theorem [M, 3.3] both the restriction functors

$$\mathrm{Fet}^{\mathbf{L}}(X, M_X) \rightarrow \mathrm{Fet}^{\mathbf{L}}(X_{\mathrm{triv}}), \quad \mathrm{Fet}^{\mathbf{L}}(X', M_{X'}) \rightarrow \mathrm{Fet}^{\mathbf{L}}(X_{\mathrm{triv}})$$

are equivalences of categories, from which it follows that U' is obtained from a Kummer étale covering $u : (U, M_U) \rightarrow (X, M_X)$. This covering must, in fact, be strict since the map $u^{-1}\overline{M}_X^{\mathrm{gp}} \rightarrow \overline{M}_U^{\mathrm{gp}}$ is an isomorphism, since this can be verified over X' . Therefore $U \rightarrow X$ is a finite étale covering and consequently trivial, since R is strictly local. \square

10. LIFTING ÉTALE COVERS OF FIBERS

10.1. In order to prove theorems 1.7 and 1.9 it will be useful to have some results about lifting étale covers from a geometric fiber to the total space of a fibration. In the classical setting of topology, the problem of finding such liftings can be understood in terms of the

higher homotopy groups of the base, using the long exact sequence of homotopy groups of a fibration. We use this idea to obtain results in the logarithmic setting.

10.2. Let

$$(f, f^b) : (X, M_X) \rightarrow (B, M_B)$$

be a log smooth integral morphism of fs log schemes, with B connected.

Theorem 10.3. *Assume that for any locally constant sheaf of finite abelian groups A on $(B, M_B)_{\text{ket}}$ and class $\alpha \in H^2((B, M_B)_{\text{ket}}, A)$ there exists a finite Kummer étale covering $(B_\alpha, M_{B_\alpha}) \rightarrow (B, M_B)$ such that the class α maps to 0 in $H^2((B_\alpha, M_{B_\alpha})_{\text{ket}}, A)$.*

Let $\bar{b}^{\log} \rightarrow (B, M_B)$ be a log geometric point and let $(U_{\bar{b}^{\log}}, M_{U_{\bar{b}^{\log}}}) \in \text{Fet}((X, M_X)_{(\bar{b}^{\log})})$ be an object. Then, after possibly replacing (B, M_B) by a covering in $\text{Fet}(B, M_B)$ there exists an object $(U, M_U) \in \text{Fet}(X, M_X)$ inducing $(U_{\bar{b}^{\log}}, M_{U_{\bar{b}^{\log}}})$. Moreover, any two such objects become isomorphic after a finite Kummer étale extension of (B, M_B) .

Proof. By [I, 9.9] for any prime ℓ invertible in k the sheaf (pushforward for Kummer étale topoi)

$$f_{\text{két}*}(\mathbf{Z}/(\ell)),$$

is locally constant constructible on (B, M_B) and its formation commutes with base change. Replacing (B, M_B) by a Kummer étale covering over which this locally constant sheaf is trivial we are then reduced to the case when the geometric fibers of f are connected, which we assume for the rest of the proof.

Let G be a finite group of order invertible in k . To prove the theorem it suffices to show the variant statement that if $(U_{\bar{b}^{\log}}, M_{U_{\bar{b}^{\log}}})$ is a covering of the geometric fiber which is Galois with group G , then after replacing (B, M_B) by a covering as in the theorem we can find a G -covering (U, M_U) of (X, M_X) inducing the given G -cover in the fiber.

Let $R^1 f_{\text{két}*} G$ be the sheaf associated to the presheaf on the Kummer étale site $\text{Ket}(B, M_B)$ which to any $(U, M_U) \rightarrow (B, M_B)$ associates the pointed set of isomorphism classes of G -torsors on $(X, M_X) \times_{(B, M_B)} (U, M_U)$.

Lemma 10.4. *The natural map*

$$(10.4.1) \quad (R^1 f_{\text{két}*} G)_{\bar{b}^{\log}} \rightarrow |\text{Fet}^G((X, M_X)_{(\bar{b}^{\log})})|$$

is an isomorphism, where $|\text{Fet}^G((X, M_X)_{(\bar{b}^{\log})})|$ denotes the set of isomorphism classes in the category of G -coverings $\text{Fet}^G((X, M_X)_{(\bar{b}^{\log})})$, defined as in [I, 3.1].

Proof. This is an immediate consequence of (5.1) □

Let $\mathcal{F}_{\bar{b}^{\log}} \subset (R^1 f_{\text{két}*} G)_{\bar{b}^{\log}}$ be the subset obtained from the $\pi_1((b, M_b), \bar{b}^{\log})$ -orbit of the class of $(U_{\bar{b}^{\log}}, M_{U_{\bar{b}^{\log}}})$ and the isomorphism (10.4.1).

Lemma 10.5. *There exists a unique locally constant subsheaf $\mathcal{F} \subset R^1 f_{\text{két}*} G$ whose stalk at \bar{b}^{\log} is $\mathcal{F}_{\bar{b}^{\log}}$. Furthermore, for any other log geometric point $\bar{b}'^{\log} \rightarrow (B, M_B)$ with image point $(b', M_{b'})$ the stalk $\mathcal{F}_{\bar{b}'^{\log}} \subset (R^1 f_{\text{két}*} G)_{\bar{b}'^{\log}}$ is an orbit for the $\pi_1((b', M_{b'}), \bar{b}'^{\log})$ -action.*

Proof. By the uniqueness, it suffices to consider the case when B is integral and \bar{b}^{\log} maps to the generic point of B .

Consider the sheaf \mathcal{S} on $(B, M_B)_{\text{két}}$ which to any $(V, M_V) \rightarrow (B, M_B)$ associates

$$\prod_{\rho: \bar{b}^{\log} \rightarrow (V, M_V)} |\text{Fet}^G((X, M_X)_{(\bar{b}^{\log})})|,$$

where the product is taken over lifts to (V, M_V) of the morphism $\bar{b}^{\log} \rightarrow (B, M_B)$. Let $T \subset |\text{Fet}^G((X, M_X)_{(\bar{b}^{\log})})|$ denote the orbit of the class of $(U_{\bar{b}^{\log}}, M_{U_{\bar{b}^{\log}}})$ under the action of the automorphism group $\pi_1((b, M_b), \bar{b}^{\log})$ of \bar{b}^{\log} over (b, M_b) . Since $(U_{\bar{b}^{\log}}, M_{U_{\bar{b}^{\log}}})$ is defined over some finite extension of (b, M_b) the set T is finite. Let \mathcal{T} denote the subsheaf of \mathcal{S} which to any $(V, M_V) \rightarrow (B, M_B)$ associates

$$\prod_{\rho: \bar{b}^{\log} \rightarrow (V, M_V)} T,$$

and let $\mathcal{F} \subset R^1 f_{\text{két}*} G$ denote the preimage of \mathcal{T} under the natural inclusion

$$R^1 f_{\text{két}*} G \rightarrow \mathcal{S}.$$

It then follows from inspection of the stalks that \mathcal{F} is a locally constant constructible subsheaf of $R^1 f_{\text{két}*} G$ whose restriction to \bar{b}^{\log} is the set T . \square

Returning to the proof of 10.3, after replacing (B, M_B) by a finite Kummer étale cover, we may assume that \mathcal{F} is a constant sheaf. We therefore have a distinguished section $\gamma \in H^0((B, M_B)_{\text{két}}, R^1 f_{\text{két}*} G)$ whose image in $|\text{Fet}^G((X, M_X)_{(\bar{b}^{\log})})|$ is the class of $(U_{\bar{b}^{\log}}, M_{U_{\bar{b}^{\log}}})$.

Let \mathcal{G}_γ be the fibered category of the Kummer étale site of (B, M_B) which to any $(V, M_V) \rightarrow (B, M_B)$ associates the category of G -torsors $(P, M_P) \rightarrow (X, M_X) \times_{(B, M_B)} (V, M_V)$ whose induced class in $H^0((V, M_V), R^1 f_{\text{két}*} G)$ is the image of the class of γ . Then \mathcal{G}_γ is a gerbe with associated band, in the sense of [G, Chapitre IV, §1 and 2], equal to the band L_G associated to G . Let $Z \subset G$ be the center. Then by [G, Chapitre IV, 3.3.3] the natural action of $H^2((B, M_B)_{\text{két}}, Z)$ on $H^2((B, M_B)_{\text{két}}, L_G)$ is simply transitive. This combined with our assumption that every class in $H^2((B, M_B)_{\text{két}}, Z)$ can be killed by a finite Kummer étale cover implies that there exists such a cover of (B, M_B) over which \mathcal{G}_γ is trivial. Making such cover we obtain the desired global torsor.

Finally to see that any two such torsors

$$(U_i, M_{U_i}) \rightarrow (X, M_X), \quad i = 1, 2$$

become isomorphic after a Kummer étale finite extension of (B, M_B) , note that it follows from the above discussion that the sheaf on the Kummer étale site of (B, M_B) of isomorphisms between these two covers is a locally constant nonempty sheaf.

This completes the proof of 10.3. \square

11. PROOFS OF 1.7 AND 1.9 IN GENERAL

11.1. Let $f : (X, M_X) \rightarrow (B, M_B)$ be a log smooth proper morphism of finite type. Combining [O, Chapter III, 2.6.7] and [IKN, A.3.4], we know that étale locally on B there exists a

sequence of morphisms

$$(B_1, M_{B_1}) \xrightarrow{q} (B_2, M_{B_2}) \xrightarrow{b} (B, M_B),$$

where b is a log blowup and q is a Kummer étale covering, such that the base change in the category of fs log schemes

$$(X_1, M_{X_1}) \rightarrow (B_1, M_{B_1})$$

is integral.

In fact we can also reverse the order of the Kummer étale cover and the log blowup. To see this we may assume that $B = \text{Spec}(R)$ is affine, that we have a chart $P \rightarrow R$ for M_B , and that b is defined by blowing up a finitely generated ideal

$$K = \langle f_1, \dots, f_r \rangle \subset P.$$

For an integer $n \geq 1$ invertible on B let

$$(B_2^{1/n}, M_{B_2^{1/n}}) \rightarrow (B_2, M_{B_2}), \quad (B^{1/n}, M_{B^{1/n}}) \rightarrow (B, M_B)$$

be the Kummer étale coverings induced by multiplication by n on P . Then $M_{B^{1/n}}$ again admits a chart $\beta_n : P \rightarrow M_{B^{1/n}}$ and the natural map

$$(B_2^{1/n}, M_{B_2^{1/n}}) \rightarrow (B^{1/n}, M_{B^{1/n}})$$

is the log blowup of the target with respect to $\beta_n(K)$.

Since the assertions of 1.7 and 1.9 are local for the Kummer étale topology on (B, M_B) , we see that it suffices to prove both results under the further assumption that there exists a log blowup

$$(B', M_{B'}) \rightarrow (B, M_B)$$

such that the base change

$$(X', M_{X'}) \rightarrow (B', M_{B'})$$

is integral.

Note furthermore that $B' \rightarrow B$ has geometrically connected fibers, as discussed in the proof of 9.2. In particular, if B is connected then so is B' .

By the case of an integral morphism, discussed in section 8, we know that 1.9 holds for $(X', M_{X'}) \rightarrow (B', M_{B'})$, and therefore to prove 1.9 in general it suffices to prove that for any commutative diagram

$$\begin{array}{ccc} \bar{b}'^{\log} & \longrightarrow & (B', M_{B'}) \\ \downarrow & & \downarrow \\ \bar{b}^{\log} & \longrightarrow & (B, M_B), \end{array}$$

where the horizontal maps are log geometric points, the induced functor

$$\text{Fet}((X, M_X)_{(\bar{b}^{\log})}) \rightarrow \text{Fet}((X', M_{X'})_{(\bar{b}'^{\log})})$$

is an equivalence. In other words, it suffices to prove the general case of 1.7.

To complete the proofs of 1.7 and 1.9 we are thus reduced to proving 1.7 in the case when $(B, M_B) = (\text{Spec}(k), M_k)$ is a log point and $\bar{b}'^{\log} \rightarrow (B, M_B)$ lifts to $(B', M_{B'})$. We may

further assume that we have a chart $P \rightarrow M_k$ inducing an isomorphism $k^* \oplus P \simeq M_B$, and that k is separably closed.

11.2. For an integer $n \geq 1$ invertible in k let

$$\times n : (b, M_b) \rightarrow (b, M_b), \quad \times n : (B', M_{B'}) \rightarrow (B', M_{B'})$$

be the maps induced by multiplication by n on P , and denote by

$$(X_n, M_{X_n}) := (X, M_X) \times_{(b, M_b), \times n} (b, M_b), \quad (X'_n, M_{X'_n}) := (X', M_{X'}) \times_{(B', M_{B'}), \times n} (B', M_{B'})$$

the base changes. By the discussion above, each morphism

$$(X'_n, M_{X'_n}) \rightarrow (X_n, M_{X_n})$$

is a log blowup. From this and 9.2 it follows that the pullback functor

$$\mathrm{Fet}^{(p)}((X, M_X)_{(\bar{b}^{\log})}) \simeq \mathrm{colim}_n \mathrm{Fet}^{(p)}(X_n, M_{X_n}) \rightarrow \mathrm{colim}_n \mathrm{Fet}^{(p)}(X'_n, M_{X'_n})$$

is an equivalence of categories.

This reduces us to showing that the functor

$$\mathrm{colim}_n \mathrm{Fet}^{(p)}(X'_n, M_{X'_n}) \rightarrow \mathrm{Fet}^{(p)}((X', M_{X'})_{(\bar{b}'^{\log})})$$

is an equivalence of categories for any log geometric point $\bar{b}'^{\log} \rightarrow (B', M_{B'})$, which follows from 10.3 and the result that the pullback functor

$$\mathrm{Fet}^{(p)}(B, M_B) \rightarrow \mathrm{Fet}^{(p)}(B', M_{B'})$$

is an equivalence of categories. Note that here the assumptions of 10.3 are satisfied by [I, 6.2].

This completes the proofs of 1.7 and 1.9. □

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