Note: I make no claim that the material in these lectures is original. In fact, the bulk of what is contained in the three lectures can be found in [3], and the reader is encouraged to study these papers of Mumford for the many deeper results contained therein. For the basic theory of abelian varieties an excellent reference is [2].

LECTURE 1.

1. Abelian schemes

1.1. A group scheme over a base scheme $S$ is an $S$-scheme

$$G 	o S$$

together with maps

$$m : G \times_S G \to G \quad \text{(multiplication)},$$
$$e : S \to G \quad \text{(identity section)},$$
$$\iota : G \to G \quad \text{(inverse)}$$

such that the following diagrams commute:

$$
\begin{array}{ccc}
G \times_S G \times_S G & \xrightarrow{id \times m} & G \times_S G \\
\downarrow m \times \text{id} & & \downarrow m \\
G \times_S G & \xrightarrow{m} & G,
\end{array}
$$

$$
\begin{array}{ccc}
G \times_S G & \xrightarrow{m} & G \\
\uparrow e \times \text{id} & & \uparrow
\end{array}
$$

$$
\begin{array}{ccc}
G \times_S G & \xrightarrow{\iota} & G \\
\downarrow \text{id} \times e & & \downarrow
\end{array}
$$

$$
\begin{array}{ccc}
G & \xrightarrow{\Delta} & G \times_S G \\
\downarrow e & & \downarrow m \\
S & \xrightarrow{\iota} & G,
\end{array}
$$
We usually suppress the maps $m$, $e$, and $\iota$ from the notation and write simply $G/S$ for a group scheme.

A group scheme $G/S$ is called *abelian* if in addition the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\Delta} & G \times_S G \\
\downarrow & & \downarrow \\
S & \xrightarrow{e} & G
\end{array}
\]

commutes.

**Remark 1.2.** Morphisms of group schemes are morphisms of schemes, which respect the group scheme structure.

**Remark 1.3.** By Yoneda’s lemma, giving a scheme $G/S$ the structure of a group scheme is equivalent to giving a factorization of its functor of points

\[
\begin{array}{ccc}
\text{(Groups)} & \xrightarrow{h_G} & \text{(Set)} \\
\downarrow & & \downarrow \text{forget} \\
(S\text{-schemes})^{\text{op}} & \xrightarrow{b_G} & (\text{abelian groups})
\end{array}
\]

Therefore an equivalent definition of an (abelian) group scheme is a contravariant functor from $S$-schemes to (abelian) groups such that the induced functor to sets is representable. This will usually be the preferred way of describing group schemes.

**Example 1.4.** *(The additive group)* This is the functor

\[
\mathbb{G}_a : (S\text{-schemes})^{\text{op}} \to (\text{abelian groups})
\]

sending $T/S$ to $\Gamma(T, \mathcal{O}_T)$. Note that the underlying scheme of this group scheme is the affine line $\mathbb{A}_S^1$.

**Example 1.5.** *(The multiplicative group)* This is the functor

\[
\mathbb{G}_m : (S\text{-schemes})^{\text{op}} \to (\text{abelian groups})
\]

sending $T/S$ to $\Gamma(T, \mathcal{O}_T^*)$. The underlying scheme of this group scheme is

$S \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Z}[X^\pm])$.

**Example 1.6.** *(The general linear group)* This is the functor (where $n \geq 1$ is a fixed integer)

\[
GL_n : (S\text{-schemes})^{\text{op}} \to (\text{Groups})
\]

sending $T/S$ to the group of $n \times n$ invertible matrices with coefficients in $\Gamma(T, \mathcal{O}_T)$. The underlying scheme of this group scheme is

$S \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Z}[X_{ij}]_{i,j=1}^n[1/\text{det}])$, where $\text{det}$ denotes the polynomial given by the determinant of the $n \times n$-matrix with entries $X_{ij}$.
Example 1.7. (The Jacobian of a curve) Let $S = \text{Spec}(k)$ be the spectrum of a field, and let $C/k$ be a smooth, proper, geometrically connected curve, and fix a point $x \in C(k)$.

If $T/k$ is a $k$-scheme, let $C_T$ denote the base change $C \times_{\text{Spec}(k)} T$. A rigidified line bundle on $C_T$ is a pair $(L, \sigma)$, where $L$ is a line bundle on $C_T$ and

$$\sigma : x^*L \to \mathcal{O}_T$$

is an isomorphism of line bundles on $T$. The set of rigidified line bundles on $C_T$ form an abelian group with addition given by

$$(L, \sigma) + (M, \eta) := (L \otimes M, \sigma \otimes \eta),$$

where $\sigma \otimes \eta$ denotes the isomorphism

$$x^*(L \otimes M) \xrightarrow{\sim} x^*L \otimes x^*M \xrightarrow{\sigma \otimes \eta} \mathcal{O}_T \otimes \mathcal{O}_T, \mathcal{O}_T \cong \mathcal{O}_T.$$

Define

$$P : (k\text{-schemes})^{\text{op}} \to (\text{abelian groups})$$

to be the functor sending $T/k$ to the group of rigidified line bundles on $C_T$. Then one can show the following:

1. $P$ is an abelian group scheme. Moreover, one can define $P$ without the choice of the base point $x$.
2. Taking the degree of a line bundle defines a surjection of group schemes

$$P \to \mathbb{Z},$$

whose kernel $J$ is a smooth, proper, geometrically connected group scheme over $k$ of dimension equal to the genus $g$ of $C$. The group scheme $J$ is called the Jacobian of $C$.

Remark 1.8. Note that the definition of rigidified line bundle makes sense on an arbitrary $S$-scheme $X/S$ with section $x \in X(S)$.

Definition 1.9. An abelian scheme over a base scheme $S$ is a group scheme $A/S$ such that $A \to S$ is of finite presentation, smooth, and all fibers are geometrically connected.

Remark 1.10. Note the distinction between “abelian scheme” and “abelian group scheme”.

Proposition 1.11. (i) Abelian schemes are abelian group schemes.

(ii) The group structure on an abelian scheme $A/S$ is determined by the identity section.

(iii) Let $A$ and $B$ be abelian schemes over a base scheme $S$, and let $f : A \to B$ be a morphism of the underlying schemes. Then there exists a section $b \in B(S)$ such that $f = t_b \circ h$, where $h : A \to B$ is a morphism of group schemes, and $t_b$ denotes translation by the point $b$.

Proof. Note that in (iii) the section $b \in B(S)$ must be $f(e_A)$ (where $e_A$ denotes the identity section of $A$).

(iii) $\implies$ (ii). This is immediate (consider the identity map on $A$).

(iii) $\implies$ (i). Note that a group scheme $A/S$ is an abelian group scheme if and only if the inverse map $\iota : A \to A$ is a homomorphism.
So it suffices to prove (iii). Furthermore, replacing \( f \) by \( t \cdot f(e_A) \circ f \) it suffices to show that any morphism of schemes \( f: A \to B \) with \( f(e_A) = e_B \) is a morphism of group schemes, which is the statement we now prove.

Let
\[
\rho : A \times_S A \to B
\]
be the map given by (here described as a map on functors of points)
\[
(a, a') \mapsto f(a + a') - (f(a) + f(a')).
\]
We need to show that \( \rho \) is the constant map given by \( e_B \in B(S) \).

Consider first the case when \( S = \text{Spec}(k) \) is the spectrum of an algebraically closed field \( k \). We have
\[
\rho(e_A \times A) = \{e_B\} = \rho(A \times \{e_A\}).
\]
Let \( U \subset B \) be an affine open neighborhood of \( e_B \), and let \( Z \subset B \) be its complement. Then
\[
T := \text{pr}_2(\rho^{-1}(Z)) \subset A
\]
is a closed set, since
\[
\text{pr}_2 : A \times A \to A
\]
is a closed map. By definition a point \( \alpha \in A(k) \) lies outside of \( T \) if and only if
\[
\rho(A \times \{\alpha\}) \subset U.
\]
Since \( A \) is proper this is equivalent to the condition that
\[
\rho(A \times \{\alpha\}) = \{e_B\}.
\]
Let \( W \subset A \) denote the complement of \( T \). Then \( W \) is an open subset such that
\[
\rho(A \times W) = \{e_B\}.
\]
Since \( e_A \in W \) then open subset \( W \) is nonempty, whence dense. We therefore see that the restriction of \( \rho \) to the dense open subset
\[
A \times W \subset A \times A
\]
is the constant map \( e_B \), which then implies that \( \rho \) is also the constant map.

The case of a general base scheme \( S \) can be deduced from the case of an algebraically closed field. We leave this as an exercise. \( \square \)

**Exercise 1.12.** Let \( C \) be a smooth, connected, proper curve over an algebraically closed field \( k \). Then \( C \) admits the structure of an abelian variety if and only if the genus of \( C \) is 1.

### 2. The theorem of the cube

2.1. Let \( S \) be a scheme and \( A/S \) an abelian scheme. For a subset \( I \subset \{1, 2, 3\} \), let
\[
m_I : A \times_S A \times_S A \to A
\]
be the map given on functors of points by
\[
(x_1, x_2, x_3) \mapsto \sum_{i \in I} x_i.
\]
Theorem 2.2 (Theorem of the cube). Let \((L, \sigma)\) be a rigidified line bundle on \(A\). Then the line bundle
\[
\bigotimes_{i \in \{1, 2, 3\}} m_i^* L^{(i-1)/i}
\]
is trivial.

Before starting the proof of the theorem, let us note the following corollary. For a point \(a \in A(S)\), define a rigidified line bundle
\[
\Lambda_L(a) := t_a^* L \otimes L^{-1} \otimes L(a + b)^{-1}.
\]

Corollary 2.3. Let \(a, b \in A(S)\) be points. Then there is a unique isomorphism of rigidified line bundles
\[
\Lambda_L(a + b) \cong \Lambda_L(a) \otimes \Lambda_L(b).
\]

The proof of 2.2 occupies the remainder of this section. By a standard limit argument, we may assume that \(S\) is of finite type over an excellent Dedekind ring. Consider first the following general results.

Lemma 2.4. Let \(X/k\) be a connected, integral, and proper scheme over an algebraically closed field \(k\), and let \(L\) be a line bundle on \(X\). Then \(L\) is trivial if and only if both \(H^0(X \hookrightarrow L)\) and \(H^0(X \hookrightarrow L_1)\) are nonzero.

Proof. The ‘only if’ direction is immediate.

For the ‘if’ direction, let \(\alpha \in H^0(X, L)\) and \(\beta \in H^0(X, L^{-1})\) be nonzero sections. We view these sections as maps
\[
\alpha : \mathcal{O}_X \to L, \quad \beta : \mathcal{O}_X \to L^{-1},
\]
and write
\[
\alpha^\wedge : L^{-1} \to \mathcal{O}_X, \quad \beta^\wedge : L \to \mathcal{O}_X
\]
for the induced maps on duals.

The composite map
\[
\mathcal{O}_X \xrightarrow{\alpha} L \xrightarrow{\beta^\wedge} \mathcal{O}_X
\]
is then a nonzero map, and therefore is an isomorphism. In particular, the map \(\beta^\wedge\) is surjective. Tensoring \(\beta^\wedge\) with \(L^{-1}\) we get that \(\beta\) is surjective, whence an isomorphism. \(\square\)

Lemma 2.5. Let \(X\) and \(Y\) be smooth, proper \(S\)-schemes with geometrically connected fibers, and let \(Z\) be any finite type connected \(S\)-scheme. Assume given points
\[
x \in X(S), \quad y \in Y(S), \quad z \in Z(S)
\]
and a line bundle \(L\) on \(X \times_S Y \times_S Z\) such that
\[
L|_{X \times Y \times \{z\}}, \quad L|_{X \times \{y\} \times Z}, \quad L|_{\{x\} \times Y \times Z}
\]
are all trivial. Let
\[
p : X \times_S Y \times_S Z \to Z
\]
be the third projection. Then \(p_\ast L\) is a line bundle on \(Z\) and the adjunction map
\[
p^\ast p_\ast L \to L
\]
is an isomorphism.
LECTURE 2.

3. THE THETA GROUP

3.1. Let $A/S$ be an abelian scheme over some base scheme $S$, and let $L$ be a line bundle on $A$. Define the theta group of $(A, L)$, denoted $\mathcal{G}_{(A, L)}$ to be the functor

$$(S\text{-schemes})^{\text{op}} \to \text{(groups)}$$

sending $T/S$ to the group

$$\{(x, \sigma) | x \in A(T), \sigma : t_x^* L \to L \}.$$

The group structure is given by defining the product of two elements $(x, \sigma)$ and $(y, \eta)$ to be $x + y \in A(T)$ with the isomorphism

$$t^*_{x+y} L \xrightarrow{\cong} t^*_x(t^*_y L) \xrightarrow{t^*_x \eta} t^*_x L \xrightarrow{\sigma} L.$$

Remark 3.2. As we will see, the theta group $\mathcal{G}_{(A, L)}$ is usually not commutative.

Theorem 3.3. The theta group $\mathcal{G}_{(A, L)}$ is a group scheme. Moreover, there is a natural exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}_{(A, L)} \longrightarrow K_{(A, L)} \longrightarrow 1,$$

where $\mathbb{G}_m$ is central in $\mathcal{G}_{(A, L)}$ and $K_{(A, L)}$ is commutative and proper over $S$.

Proof. Note that $\mathcal{G}_{(A, L)}$ is a sheaf with respect to the fppf topology.

The inclusion $\alpha$ is obtained by the map sending $u \in \mathbb{G}_m(T)$ to the pair $(e, u) \in \mathcal{G}_{(A, L)}(T)$. The quotient of $\mathcal{G}_{(A, L)}$ by $\mathbb{G}_m$ (quotient taken in the category of sheaves with respect to the fppf topology) is the functor sending $T/S$ to the set of elements $a \in A(T)$ such that the two line bundles $t_a^* L_T$ and $L_T$ on $A_T$ are fppf-locally on $T$ isomorphic. Let $K_{(A, L)}$ denote this functor. It suffices to show that $K_{(A, L)}$ is a proper $S$-scheme.

For this let

$$P : (S\text{-schemes})^{\text{op}} \to \text{(Groups)}$$

denote the functor sending $T/S$ to the group of isomorphism classes of rigidified line bundles on $A_T$.

Fact: $P$ is a proper algebraic space locally of finite presentation over $S$.

There is a map

$$\lambda_L : A \rightarrow P, \ a \mapsto t_a^* L \otimes L^{-1} \otimes L(e) \otimes L(a)^{-1}.$$

Note that this is a group homomorphism by the theorem of the cube, discussed last lecture.

Now $K_{(A, L)}$ is equal to $\lambda_L^{-1}(\mathcal{O}_A, \text{can})$, and hence is a closed subscheme of $A$. \hfill \Box

Proposition 3.4. If $L$ is relatively ample on $A/S$, then $K_{(A, L)}$ is finite over $S$. 

Proof. Since $K_{(A,L)}$ is proper over $S$, it suffices to show that it is quasi-finite over $S$. We may therefore assume that $S = \text{Spec}(k)$, with $k$ an algebraically closed field.

Let $Y \subset P$ denote the connected component of the identity, with the reduced structure. Then $Y$ is an abelian variety, and we have an ample line bundle $L_Y := L|_Y$ such that for every $y \in Y$ we have

$$t_y^*L_Y \simeq L_Y.$$ 

Consider

$$\Lambda(L) := m^*L \otimes \text{pr}_1^*L^{-1} \otimes \text{pr}_2^*L^{-1}$$

on $Y \times Y$. Then for every $y \in Y$ the restriction of $\Lambda(L)$ to $\{y\} \times Y$ and $Y \times \{y\}$ is trivial. By the argument of lecture 1 we get that $\Lambda(L)$ is trivial. On the other hand, consider the map

$$\text{id} \times \iota : Y \to Y \times Y$$

The pullback of $\Lambda(L)$ along this map is the line bundle

$$(L \otimes \iota^*L)^{-1}.$$ 

It follows that the ample line bundle $L \otimes \iota^*L$ is trivial, which implies that $Y$ must be zero-dimensional.

Example 3.5. Let $S = \text{Spec}(k)$ be the spectrum of an algebraically closed field, and let $E/k$ be an elliptic curve. Let $L = \mathcal{O}_E(ne)$, for some integer $n \geq 1$. Then for a point $a \in E(k)$ we have

$$t_a^*L \otimes L^{-1} \simeq \mathcal{O}_E(n(-a) - n(e))$$

which is trivial if and only if $na = e$. We conclude that $K_{(E,L)}$ is isomorphic to the $n$-torsion group scheme

$$E[n] := \text{Ker}(n : E \to E).$$

Note also that if $a \in E[n]$ then an isomorphism

$$t_a^*L \to L$$

is given by a function $f \in k(E)$ (where $k(E)$ denotes the function field of $E$ such that $\text{div}(f) = n(-a) - n(e)$). The theta group $\mathcal{G}_{(E,L)}(k)$ can therefore be described as the set of pairs $(a, f)$ where $a \in E[n]$ and $f \in k(E)$ is a function such that $\text{div}(f) = n(-a) - n(e)$.

To proceed we will need the following facts:

Facts 3.6. Let $A/k$ be an abelian variety of dimension $g$ over an algebraically closed field $k$, and let $L$ be an ample line bundle on $A$.

(i) There exists an integer $d \geq 1$ such that for all $n \geq 1$ we have

$$h^0(A, L^\otimes n) = dn^g$$

and

$$h^i(A, L^\otimes n) = 0$$

for all $i > 0$. We refer to this integer $d$ as the degree of $L$.

(ii) The group scheme $K_{(A,L)}$ has rank $d^2$. 
Lemma 3.7. Let \( f : A \to S \) be an abelian scheme, and let \( L \) be a relatively ample line bundle on \( A/S \). Then \( f_*L \) is a locally free sheaf on \( S \), whose formation commutes with arbitrary base change on \( S \). In particular, the function sending \( s \in S \) to the degree of \( L|_A \) is a locally constant function on \( S \).

Proof. This follows from fact (i) and cohomology and base change. \( \square \)

Remark 3.8. In particular if \( L \) is a relatively ample line bundle on an abelian scheme \( A/S \) then it makes sense to talk about the degree of \( L \), which is a locally constant function on \( S \).

Proposition 3.9. Let \( A/S \) be an abelian scheme and \( L \) a relatively ample line bundle on \( A \) of degree \( d \). Then \( K_{(A,L)} \) is a finite flat group scheme over \( S \) of rank \( d^2 \).

Proof. We already know that \( K_{(A,L)} \) is finite over \( S \), so it suffices to show that if \( K_{(A,L)} \) is flat over \( S \). If \( S \) is reduced, this can be done as follows. We may work locally on \( S \), so it suffices to consider the case when \( S = \text{Spec}(R) \), where \( R \) is a local ring with maximal ideal \( \mathfrak{m} \subset R \). Let \( M \) denote the coordinate ring of \( K_{(A,L)} \), viewed as a finitely generated \( R \)-module. Since
\[
\dim_{R/\mathfrak{m}}(M/\mathfrak{m}M) = d^2,
\]
we can (by Nakayama’s lemma) choose a surjection
\[
R^{d^2} \xrightarrow{\pi} M,
\]
reducing to an isomorphism modulo \( \mathfrak{m} \). Since the dimension of every fiber of \( M \) is \( d^2 \), the map \( \pi \) induces an isomorphism modulo every prime ideal of \( R \). This implies in particular that at each generic point of \( R \) the map \( \pi \) is an isomorphism. Since \( R \) is reduced, this implies that \( \pi \) is also injective, whence an isomorphism.

For the case of a general base, one needs to use something more sophisticated. In this case we need that the map
\[
\lambda_L : A \to P
\]
is flat, which follows from the theory of the dual abelian variety, which implies that \( \lambda_L \) is surjective onto an open subspace of \( P \) which is smooth over \( S \) and of the same dimension as \( A \). \( \square \)

Summary 3.10. Let \( A/S \) be an abelian scheme, and let \( L \) be a relatively ample invertible sheaf on \( A \) of degree \( d \). Then \( \mathcal{G}_{(A,L)} \) is a flat group scheme over \( S \), which sits in a central extension
\[
1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}_{(A,M)} \longrightarrow K_{(A,L)} \longrightarrow 1,
\]
where \( K_{(A,L)} \) is finite and flat of degree \( d^2 \) over \( S \).

3.11. In the setting of 3.10, note that we get a natural skew-symmetric pairing
\[
e : K_{(A,L)} \times K_{(A,L)} \to \mathbb{G}_m
\]
defined on (scheme-valued) points by sending
\[
(x, y) \mapsto \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1},
\]
where \( \tilde{x}, \tilde{y} \in \mathcal{G}_{(A,L)} \) are liftings of \( x \) and \( y \) respectively (we leave it as an exercise to verify that this is well-defined). This pairing is called the Weil pairing.
4. Level subgroups

4.1. Throughout this section we work over an algebraically closed field \( k \), \( A/k \) denotes an abelian variety, and \( L \) is an ample line bundle on \( A \) of degree \( d \), which we assume invertible in \( k \).

**Definition 4.2.** A level subgroup of \( G(A,L) \) is a subgroup \( \tilde{H} \subset G(A,L)(k) \) such that \( \tilde{H} \cap k^* = \{1\} \).

**Remark 4.3.** Note that if \( \tilde{H} \) is a level subgroup, then the projection \( \tilde{H} \rightarrow K(A,L) \) is injective, whence \( \tilde{H} \) is finite and commutative. This implies that we can also view \( \tilde{H} \) as a subgroup scheme of \( G(A,L) \).

**Exercise 4.4.** Show that if \( x \in K(A,L) \) is an element, then there exists a level subgroup \( \tilde{H} \subset G(A,L) \) whose image in \( K(A,L) \) contains \( x \).

4.5. Let \( \tilde{H} \subset G(A,L) \) be a level subgroup, and let \( H \subset K(A,L) \) denote the image (so the projection \( \tilde{H} \rightarrow H \) is an isomorphism). Then \( H \) is a subgroup scheme of \( A \), and we can form the quotient

\[
\pi : A \rightarrow B := A/H.
\]

Note that if \( M \) is a line bundle on \( B \), then there is a natural embedding \( H \hookrightarrow G(A,\pi^*M) \). Indeed if \( y \in H \subset A \) then there are canonical isomorphisms

\[
t^*_{\pi(y)} \pi^*M \cong \pi^*t^*_\pi M \cong \pi^*M.
\]

Conversely, descent theory implies that the choice of the level subgroup \( \tilde{H} \) lifting \( H \) is precisely equivalent to specifying a line bundle \( M \) on \( B \) and an isomorphism \( \pi^*M \cong L \). In other words, we have a bijection between level subgroups of \( G(A,L) \) and the set of triples \( (\pi : A \rightarrow B, M, \sigma) \), where \( \pi : A \rightarrow B \) is a surjection of abelian varieties, \( M \) is a line bundle on \( B \), and \( \sigma : \pi^*M \cong L \) is an isomorphism of line bundles on \( A \).

**Exercise 4.6.** Let \( \tilde{H} \subset G(A,L) \) be a level subgroup, and let \( (\pi : A \rightarrow B, M, \sigma) \) be the corresponding collection of data.

(i) Show that if \( r \) denotes the order of \( \tilde{H} \), then

\[
h^0(B,M) \cdot r = h^0(A, L).
\]

(ii) Let

\[
\Sigma := \{ z \in G(A,L) \mid z \text{ centralizes } \tilde{H} \}.
\]

Show that there is a natural isomorphism

\[
G(B,M) \cong \Sigma/\tilde{H}.
\]

**Corollary 4.7.** With notation as in 4.6, assume that \( \tilde{H} \) is a maximal level subgroup. Then the order of \( \tilde{H} \) is \( d \), and the degree of \( M \) is 1.

**Proof.** By the formula in (4.6 (i)), the two statements of the corollary are equivalent. Also since \( \tilde{H} \) is maximal, we have

\[
\Sigma = \mathbb{G}_m \cdot \tilde{H}.
\]

It follows that \( G(B,M) \cong \mathbb{G}_m \) which implies that the degree of \( M \) is 1. \( \Box \)
Corollary 4.8. The Weil pairing $e$ is non-degenerate.

Proof. Let $x \in K_{(A,L)}$ be an element, and let $\tilde{H} \subset \mathcal{G}_{(A,L)}$ be a maximal level subgroup whose image $H \subset K_{(A,L)}$ contains $x$. Then since $\tilde{H}$ is maximal, the map

$$K_{(A,L)}/H \to \text{Hom}(H, \mathbb{G}_m), \quad [y] \mapsto e(y, -)$$

is injective. Since this is a map of finite groups of the same order we conclude that it is an isomorphism. Therefore there exists an element $y \in K_{(A,L)}$ such that $e(y, x) \neq 1$. \hfill $\Box$

Exercise 4.9. Let $\tilde{H} \subset \mathcal{G}_{(A,L)}$ be a maximal level subgroup, and let $H \subset K_{(A,L)}$ be its image. Let

$$H^\wedge := \text{Hom}(H, \mathbb{G}_m),$$

denote the Cartier dual of $H$. Show that there exists an isomorphism of schemes

$$\mathcal{G}_{(A,L)} \simeq \mathbb{G}_m \times H \times H^\wedge$$

such that the group law is given by

$$(u, x, \chi) \cdot (v, y, \eta) = (uv\eta(x), x + y, \chi + \eta).$$

Remark 4.10. Since $H$ is a finite abelian group, there exists an integer $s$ and integers $d_1, \ldots, d_s$ such that

$$d_1 d_2 \cdots d_s = d,$$

and

$$H \simeq \oplus_{i=1}^s \mathbb{Z}/(d_i).$$

In this case

$$H^\wedge \simeq \oplus_{i=1}^s \mu_{d_i}.$$

Note that the integers $s$ and $d_1, \ldots, d_s$ are independent of the choice of the level subgroup.

LECTURE 3.

5. Representations of the Heisenberg group

5.1. Fix a collection of positive integers

$$\delta = (d_1, \ldots, d_s),$$

and let $d$ denote $d_1 \cdots d_s$.

Let $H$ denote the group

$$H := \oplus_{i=1}^s \mathbb{Z}/(d_i),$$

and let

$$H^\wedge = \prod_{i=1}^s \mu_{d_i}$$

denote the Cartier dual of $H$. Both $H$ and $H^\wedge$ are group schemes over $\mathbb{Z}$. Let

$$\mathcal{G}(\delta)$$
denote the group scheme whose underlying scheme is
\[ \mathbb{G}_m \times H \times H^\wedge \]
and group law given by the formula (on scheme-valued points)
\[ (u, x, \chi) \cdot (v, y, \eta) = (uv\eta(x), x + y, \chi + \eta). \]
Then \( \mathcal{G}(\delta) \) is a group scheme over \( \mathbb{Z} \) which is a central extension
\[ 1 \to \mathbb{G}_m \to \mathcal{G}(\delta) \to H \times H^\wedge \to 1. \]

**Remark 5.2.** Note that for any \( \eta \in H^\wedge \) and \( x \in H \) we have \( \eta(x) \in \mu_d \). We can therefore also define a group scheme \( \mathcal{G}(\delta)' \) which as a scheme is
\[ \mu_d \times H \times H^\wedge \]
and group given by the formula (5.1.1). Then we have a commutative diagram
\[
\begin{array}{c c c c c c}
1 & \longrightarrow & \mu_d & \longrightarrow & \mathcal{G}(\delta)' & \longrightarrow & H \times H^\wedge & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathcal{G}(\delta) & \longrightarrow & H \times H^\wedge & \longrightarrow & 1,
\end{array}
\]
which identifies \( \mathcal{G}(\delta) \) with the pushout of the diagram
\[ (5.2.1) \quad \mu_d \longrightarrow \mathcal{G}(\delta)' \longrightarrow \mathbb{G}_m. \]

**5.3.** Let \( V(\delta) \) denote the free \( \mathbb{Z} \)-module on the set \( H^\delta \) (so \( V(\delta) \) is the set of functions \( H^\wedge \to \mathbb{Z} \)). For \( \chi \in H^\delta \) let \( f_\chi \in V(\delta) \) denote the corresponding basis element. Then \( V(\delta) \) is naturally a representation \( \rho \) of \( \mathcal{G}(\delta) \). For (scheme-valued points) \( u \in \mathbb{G}_m, x \in H, \) and \( \eta \in H^\wedge \) we have
\[
\rho_u(f_\chi) = u \cdot f_\chi, \quad \rho_x(f_\chi) = \chi(x) \cdot f_\chi, \quad \rho_\eta(f_\chi) = f_{\chi + \eta}.
\]

**Exercise 5.4.** Show that this really defines a representation of \( \mathcal{G}(\delta) \).

**Theorem 5.5.** Let \( R \) be a \( \mathbb{Z}[1/d] \)-algebra, and let \( P \) be a projective \( R \)-module. Let
\[ \rho : \mathcal{G}(\delta) \to GL(P) \]
be a representation of \( \mathcal{G}(\delta) \) over \( R \) such that \( \rho|_{\mathbb{G}_m} \) is the standard multiplication action of \( \mathbb{G}_m \). Then after possibly replacing \( P \) by a flat extension, there exists an isomorphism of \( \mathcal{G}(\delta) \)-representations over \( R \)
\[ P \simeq V(\delta)^r \otimes R \]
for some integer \( r \).

**Proof.** Let us first consider the case when \( R = k \) is an algebraically closed field, and to ease notation write abusively also \( \mathcal{G}(\delta) \) (resp. \( V(\delta) \)) for the base change of \( \mathcal{G}(\delta) \) (resp. \( V(\delta) \)) to \( k \).

Suppose
\[ \gamma : \mathcal{G}(\delta) \to GL(V) \]
is an irreducible representation over $k$ such that $\mathbb{G}_m \subset \mathcal{G}(\delta)$ acts by multiplication. We then show as follows that $V \simeq V(\delta)$. Let $\tilde{H} \subset \mathcal{G}(\delta)$ denote the maximal level subgroup lifting $H$ given by the presentation $\mathcal{G}(\delta) \simeq \mathbb{G}_m \times H \times H^\wedge$. Then since $\tilde{H}$ is a finite commutative group of order invertible in $k$, we have a canonical decomposition

$$V \simeq \oplus_{x \in H^\wedge} V_x,$$

where $x \in \tilde{H}$ acts on $V_x$ by multiplication by $\chi(x)$. Note that for $\eta \in H^\wedge \subset \mathcal{G}(\delta)$, we have for $v \in V_x$ and $x \in \tilde{H}$

$$\gamma_x(\gamma_\eta(v)) = \eta(x)\gamma_\eta(v) = \eta(x)\chi(x)\gamma_\eta(v).$$

It follows that $\gamma_\eta$ maps $V_x$ to $V_{x+\eta}$.

Fix any nonzero vector $v_0 \in V_0$, and let $W \subset V$ be the span of elements $\gamma_\eta(v_0)$ for $\eta \in H^\wedge$. Then $W$ is a subrepresentation of $V$, and since $V$ is irreducible we must have $W = V$. We conclude that each $V_x$ is 1-dimensional. Fix a nonzero element $v_0 \in V_0$, and let $v_x \in V_x$ denote $\gamma_x(v_0)$. We then obtain a surjection

$$V(\delta) \twoheadrightarrow V, \quad f_x \mapsto v_x,$$

which must be an isomorphism since both are irreducible.

To see that any representation $V$ of $\mathcal{G}(\delta)$ on which $\mathbb{G}_m$ acts by multiplication is a direct sum of copies of $V(\delta)$, note that since (5.2.1) is a pushout diagram the restriction functor

(5.5.1) \hspace{2cm} (\text{rep's of } \mathcal{G}(\delta) \text{ on which } \mathbb{G}_m \text{ acts by multiplication}) \twoheadrightarrow (\text{rep's of } \mathcal{G}(\delta)' \text{ on which } \mu_d \text{ acts by multiplication})

is an equivalence of categories. Since $\mathcal{G}(\delta)'$ is a finite group of order invertible in $k$, its category of representations is semisimple, and therefore any representation is a direct sum of irreducible representations.

This proves the proposition when $R$ is an algebraically closed field. We leave it as an exercise to extend the argument to the case of an arbitrary $\mathbb{Z}[1/2d]$-algebra.

\[\square\]

6. Polarizations

6.1. Let $k$ be an algebraically closed field, and let $A/k$ be an abelian variety. Denote by $P_{A/k}$ the Picard scheme representing the functor of rigidified line bundles on $A$. The dual abelian variety of $A$, denoted $A^\vee$, is the connected component of the identity in $P_{A/k}$.

Facts 6.2. (i) $A^\vee$ is an abelian variety of the same dimension as $A$.

(ii) If $M$ is a rigidified line bundle on $A$, then the corresponding point $[M] \in P_{A/k}(k)$ is in $A^\vee$ if and only if for every $a \in A(k)$ we have $t_a^*M \simeq M$, and (by the theorem of the cube) this is in turn equivalent to the statement that the line bundle

$$m^*M \otimes \text{pr}_1^*L^{-1} \otimes \text{pr}_2^*L^{-1}$$

on $A \times A$ is trivial.
6.3. If \( L \) is a line bundle, then we have the homomorphism
\[
\lambda_L : A \to P_{A/k}, \quad a \mapsto [t_a^*L \otimes L^{-1} \otimes L(a)^{-1} \otimes L(e)],
\]
and since \( A \) is connected this in fact has image in \( A' \). If \( L \) is ample then \( \lambda_L : A \to A' \) is surjective since it has finite kernel and \( A \) and \( A' \) have the same dimension.

**Exercise 6.4.** Show directly that if \( a \hookrightarrow b \in A(k) \) are two points, then
\[
t_b^* (t_a^*L \otimes L^{-1}) \simeq t_a^*L \otimes L^{-1}.
\]

**Definition 6.5.** A polarization of degree \( d \) on \( A \) is a map \( \lambda : A \to A' \) isomorphic to \( \lambda_L \) for some ample \( L \).

**Remark 6.6.** If \( L \) is ample and \( M \) is a line bundle on \( A \) corresponding to a point of \( A' \), then \( L \otimes M \) is also ample. Indeed write \( M = t_a^*L \otimes L^{-1} \) for some \( a \in A(k) \). Then
\[
L \otimes M \simeq t_a^*L
\]
and \( t_a^*L \) is ample on \( A \) since \( t_a : A \to A \) is an isomorphism.

Conversely if \( L \) and \( L' \) are two line bundles such that \( \lambda_L = \lambda_{L'} \), then for every \( a \in A \) we have
\[
t_a^*L \otimes L^{-1} \simeq t_a^*L \otimes L'^{-1},
\]
or equivalently if \( M \) denotes \( L \otimes L^{-1} \) then
\[
t_a^*M \simeq M
\]
for all \( a \in A(k) \). Therefore \( L' = L \otimes M \) for some \([M] \in A'(k)\).

It follows that given a polarization \( \lambda : A \to A' \), the set of line bundles \( L \) with \( \lambda_L = \lambda \) is a torsor under \( A' \).

6.7. The preceding discussion can be generalized to families. Let \( A/S \) be an abelian scheme over a base scheme \( S \), and let \( P_{A/S} \) be the algebraic space representing the functor on \( S \)-schemes
\[
T \mapsto \{ \text{group of rigidified line bundles on } A_T \}.
\]
Then on \( A \times_S P_{A/S} \) there is a tautological line bundle \( \mathcal{L} \), and so we get a line bundle
\[
\mathcal{M} := m^*\mathcal{L} \otimes \text{pr}_1^*\mathcal{L}^{-1} \otimes \text{pr}_2^*\mathcal{L}^{-1}
\]
on \( (A \times_S A) \times_S P_{A/S} \) which is rigidified along the zero section
\[
P_{A/S} \to (A \times_S A) \times_S P_{A/S}.
\]
This line bundle corresponds to a morphism
\[
\tau : P_{A/S} \to P_{(A \times_S A) \times_S P_{A/S}}/P_{A/S},
\]
and we define \( A' \) to be the inverse image under this map of the zero section of \( P_{(A \times_S A) \times_S P_{A/S}}/P_{A/S} \).

It is known that \( A' \) is an abelian scheme over \( S \), whose fiber over any point \( s \in S \) is the dual abelian variety of \( A_s \).

**Definition 6.8.** Let \( A/S \) be an abelian scheme over a base scheme \( S \). A polarization on \( A \) is a map \( \lambda : A \to A' \) étale locally on \( S \) is of the form \( \lambda_L : A \to A' \) for some relatively ample line bundle \( L \) on \( A \).
Remark 6.9. Note that $\lambda$ is a homomorphism as this can be verified étale locally on $S$, and $\lambda_L$ is a homomorphism.

Remark 6.10. Just as in the case of a field, given a polarization $\lambda : A \to A'$, the functor on $S$-schemes

$$T \mapsto \{\text{line bundles } L \text{ on } A_T \text{ such that } \lambda_L = \lambda\}$$

is a torsor under $A'$.

Remark 6.11. If $A/S$ is an abelian scheme and $\lambda : A \to A'$ is a polarization, then $\text{Ker}(\lambda)$ is a finite flat group scheme over $S$. Indeed this can be verified étale locally so we may assume that $\lambda = \lambda_L$ for some relatively ample line bundle on $A$. In this case $\text{Ker}(\lambda_L) = K_{(A,L)}$ so the result follows from 3.9. In particular, it makes sense to talk about the degree of $\lambda$, which is the locally constant function $d$ on $S$ for which $\text{Ker}(\lambda)$ has rank $d^2$.

6.12. Fix now positive integers $g$ and $d$, and define

$$\mathcal{A}_{g,d}$$

to be the stack over $\mathbb{Z}[1/2d]$ whose fiber over a $\mathbb{Z}[1/2d]$-scheme $S$ is the groupoid of pairs $(A,\lambda)$, where $A/S$ is an abelian scheme and $\lambda : A \to A'$ is a polarization of degree $d$.

Remark 6.13. Since we restrict to $\mathbb{Z}[1/2d]$-scheme, if $(A,\lambda) \in \mathcal{A}_{g,d}(S)$, the group scheme $\text{Ker}(\lambda)$ is a finite étale group scheme over $S$. It is therefore étale locally on $S$ constant, and we can talk about the type of $\lambda$, which is the unordered collection of integers $\delta = (d_1, \ldots, d_g)$ for which étale locally on $S$ we have

$$\text{Ker}(\lambda) \simeq (\oplus_{i=1}^g \mathbb{Z}/(d_i)) \times \prod_{i=1}^g \mu_{d_i}.$$  

Since the type of a polarization is locally constant on the base, we have an isomorphism of stacks

$$\mathcal{A}_{g,d} \cong \prod_{\delta} \mathcal{A}_{g,\delta},$$

where for a type $\delta = (d_1, \ldots, d_g)$ with $d = d_1 \cdots d_g$ we write $\mathcal{A}_{g,\delta}$ for the substack of $\mathcal{A}_{g,d}$ classifying pairs $(A,\lambda)$ whose type is $\delta$.

6.14. If $A/S$ is an abelian scheme and $\lambda : A \to A'$ is a polarization, then for $n \geq 1$ let $n\lambda$ denote the composite morphism

$$A \overline{\longrightarrow} A' \overset{n}{\longrightarrow} A'.$$

Then $n\lambda$ is also a polarization. Indeed it suffices to verify this étale locally on $S$, so we may assume that $\lambda = \lambda_L$ with $L$ relatively ample on $A$. In this case $n\lambda = \lambda_{L^n}$. Note also that $n\lambda$ is also equal to the composite map

$$A \overset{\times n}{\longrightarrow} A \overset{\lambda}{\longrightarrow} A',$$

since $\lambda$ is a homomorphism.

Exercise 6.15. If the type of $\lambda$ is equal to $\delta = (d_1, \ldots, d_g)$, show that the type of $n\lambda$ is equal to

$$n\delta := (nd_1, \ldots, nd_g).$$
Proposition 6.16. The induced morphism of stacks over $\mathbb{Z}[1/2dn]$

$$\mathcal{A}_{g,\delta} \to \mathcal{A}_{g,n\delta}, \quad (A, \lambda) \mapsto (A, n\lambda)$$

is an isomorphism.

Proof. For the full faithfulness, it suffices to show that if $(A, \lambda)$ and $(B, \eta)$ are objects of $\mathcal{A}_{g,\delta}(S)$ for some scheme $S$, and if $f : A \to B$ is a morphism of abelian schemes such that the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \downarrow n\lambda & \approx & \quad \quad \quad \quad \quad \quad \quad \quad \downarrow n\eta \\
A^t & \xleftarrow{f^*} & B^t \\
\end{array}
$$

commutes, then the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \lambda & \approx & \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \eta \\
A^t & \xleftarrow{f^*} & B^t \\
\end{array}
$$

also commutes. This follows from noting that the map

$$\lambda - f^* \circ \eta \circ f : A \to A^t$$

has image in $A[n]$, and since $A$ has geometrically connected fibers any morphism $A \to A[n]$ is constant.

For the essential surjectivity, let $(A, \eta) \in \mathcal{A}_{g,n\delta}(S)$ be an object over some scheme $S$. We must show that after possibly replacing $S$ by an fpqc covering, we have $\eta = n\lambda$ for some polarization $\lambda$ of type $\delta$. There is no choice for what the map $\lambda$ must be. Indeed note that multiplication by $n$ on $A$ is surjective and induces an isomorphism

$$A/A[n] \to A.$$ 

Since the type of $\eta$ is $n\delta$, we have $A[n] \in \text{Ker}(\eta)$, so we get a factorization of $\eta$ as

$$
\begin{array}{ccc}
A & \xrightarrow{n} & A \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \lambda & \approx & \quad \quad \quad \quad \quad \quad \quad \quad \downarrow A^t, \\
A^t & \xleftarrow{\lambda} & A^t \\
\end{array}
$$

for some map $\lambda : A \to A^t$. We must show that $\lambda$ is a polarization.

For this we may work étale locally on $S$ so we may assume that $\eta = \lambda_M$ for some relatively ample line bundle $M$ on $A$. In this case we can fpqc locally on $S$ write $M \simeq L^\otimes n$ for some line bundle $L$ on $A$, by [2, p. 231]. Then the map

$$\lambda_L - \lambda : A \to A^t$$

has image in $A^t[n]$, and so as above we conclude that $\lambda = \lambda_L$. \qed

7. Interlude on symmetric line bundles

Definition 7.1. Let $A/S$ be an abelian scheme. A line bundle $L$ on $A$ is called symmetric if there exists an isomorphism

$$\iota^* L \to L,$$

where $\iota : A \to A$ is the map sending $x$ to $-x$. 
Remark 7.2. If \( L \) is a symmetric line bundle on \( A \), then there is a canonical choice of isomorphism \( \iota^* L \rightarrow L \). Namely, if \( e : S \rightarrow A \) denotes the identity section, then we have
\[
e\iota^* L \simeq e^* L
\]
since \( e \) is fixed by \( \iota \). Since any two choices of isomorphism \( \iota^* L \rightarrow L \) differ by multiplication by an element of \( \mathcal{O}_S^\times \), there is a unique choice of isomorphism \( \iota^* L \rightarrow L \) such that the induced map \( e^* L \rightarrow e^* L \) is the identity. This implies in particular, that the condition that a line bundle on \( A/S \) is symmetric is local in the fppf topology on \( S \).

Exercise 7.3. Show that the map
\[
\iota^* : A^t \rightarrow A^t
\]
given by pullback along \( \iota \), is equal to the map given by
\[
[M] \mapsto [M^{-1}].
\]

Lemma 7.4. Let \( M \) be a rigidified line bundle on \( A/S \) which is symmetric and such that the corresponding point of \( P_{A/S} \) lies in \( A^t \). Then in fact the point \([M] \in A^t(S)\) lies in the 2-torsion subgroup \( A^t[2] \subset A^t \).

Proof. This is immediate from 7.3. For to say that \( M \) is symmetric is equivalent to saying that
\[
\iota^*[M] = [M^{-1}] \in A^t(S)
\]
is equal to \([M] \in A^t(S)\). Therefore the point \([M^2] \in A^t(S)\) is zero which implies that \([M] \in A^t[2] \).

Lemma 7.5. Let \( L \) and \( L' \) be two relatively ample symmetric invertible sheaves on an abelian scheme \( A/S \). Then \( \lambda_L = \lambda_{L'} \) if and only if there exists an invertible sheaf \( M \) on \( A \) such that \( L' \simeq L \otimes M \) and such that \([M] \in A^t[2] \).

Proof. By 6.6, \( \lambda_L = \lambda_{L'} \) if and only if \( L' = L \otimes M \) for some line bundle with \([M] \in A^t \). Since \( L' \) and \( L \) are symmetric this implies that \( M \) is also symmetric and therefore we must have \([M] \in A^t[2] \).

Lemma 7.6. Let \( A/S \) be an abelian scheme, and let \( \lambda : A \rightarrow A^t \) be a polarization. Then fppf locally on \( S \) there exists an ample, symmetric line bundle \( L \) on \( A \) such that \( \lambda = \lambda_L \).

Proof. After replacing \( S \) by an étale covering, we may assume that \( \lambda = \lambda_L \) for some relatively ample line bundle \( L \). We must show that after possible replacing \( S \) by an fppf covering there exists line bundle \( \bar{R} \) with \([\bar{R}] \in A^t \) such that \( L \otimes \bar{R} \) is symmetric.

Consider the line bundle
\[
M := \iota^* L \otimes L^{-1}.
\]
For any scheme-valued point \( \alpha \in A \), we have
\[
t^\alpha (\iota^* L \otimes L^{-1}) \simeq \iota^*(t^\alpha L \otimes L^{-1}).
\]
Now the line bundle \( t^\alpha L \otimes L^{-1} \) defines a point of \( A^t \) so
\[
\iota^* (t^\alpha L \otimes L^{-1}) \simeq t^\alpha L^{-1} \otimes L.
\]
We conclude that (after possibly making a base change on \( S \))
\[
t^\alpha (\iota^* L \otimes L^{-1}) \simeq \iota^* L \otimes t^\alpha L \otimes t^\alpha L^{-1} \otimes \iota^* L \otimes L^{-1} \otimes (t^\alpha L \otimes L^{-1}) \otimes (t^\alpha L \otimes L^{-1}) \simeq \iota^* L \otimes L^{-1},
\]
where the last isomorphism is using the fact that $\lambda_L$ is a homomorphism.

It follows that $M$ defines a point $[M] \in A^t(S)$. Since the map $\cdot 2 : A^t \to A^t$ is flat and surjective, after replacing $M$ by a finite flat covering we can find a line bundle $R$ such that $R^{\otimes 2} \simeq M$. Then

$$t^*(L \otimes R) \simeq t^* L \otimes R^{-1} \simeq L \otimes R^{\otimes 2} \otimes R \simeq L \otimes R.$$ \qedhere

7.7. If $A/S$ is an abelian scheme and $L$ is a symmetric line bundle on $A$, then the theta group $G(A,L)$ has an additional structure reflecting the fact that $L$ is symmetric. Fix an isomorphism $\gamma : t^* L \to L$ (for example the canonical one). For a scheme-valued point

$$(a, \sigma : t_a^* L \to L) \in G(A,L)$$

define $\tau(a, \sigma)$ to be the point $-a \in A$ together with the isomorphism

$$(7.7.1) \quad t_a^* L \xrightarrow{\gamma^{-1}} t_a^* t^* L \xrightarrow{\gamma} t^* L \xrightarrow{\sigma} L.$$

Exercise 7.8. Show that the isomorphism (7.7.1) is independent of the choice of $\gamma$, and that $\tau$ is a homomorphism.

7.9. Fix $\delta = (d_1, \ldots, d_g)$ and consider the group scheme $G(\delta)$. Then $G(\delta)$ also has an involution

$$\tau_0 : G(\delta) \to G(\delta).$$

To describe this recall that as a scheme

$$G(\delta) \simeq \mathbb{G}_m \times H \times H^\vee.$$  

The map $\tau_0$ sends $(u, x, \chi)$ to $(u, -x, \chi^{-1})$.

Definition 7.10. Let $A/S$ be an abelian scheme and let $L$ be a symmetric relatively ample invertible sheaf on $A$. A symmetric theta level structure of type $\delta$ (or just symmetric theta level structure if the reference to $\delta$ is clear) is an isomorphism of group schemes

$$\theta : G(\delta) \to G(A,L)$$

whose restriction to $\mathbb{G}_m$ is the identity and such that the diagram

$$\begin{array}{ccc}
G(\delta) & \xrightarrow{\theta} & G(A,L) \\
\downarrow{\tau_0} & & \downarrow{\tau} \\
G(\delta) & \xrightarrow{\theta} & G(A,L)
\end{array}$$

commutes.

Exercise 7.11. Fix a type $\delta = (d_1, \ldots, d_g)$, and let $d = \prod d_i$. Let $S$ be a $\mathbb{Z}[1/2d]$-scheme, let $A/S$ be an abelian scheme, and let $L$ be a symmetric relatively ample invertible sheaf on $A$ such that $\lambda_L$ has type $\delta$. Show that étale locally on $S$ there exists a symmetric theta level-structure

$$\theta : G(\delta) \to G(A,L).$$
8. The main construction

8.1. Fix $g$ and $\delta = (d_1, \ldots, d_g)$, and let $\mathcal{M}_{g, \delta}$ denote the stack over $\mathbb{Z}[1/2d]$ whose fiber over a $\mathbb{Z}[1/2d]$-scheme $S$ is the group of triples $(A, L, \theta)$, where $A/S$ is an abelian scheme of dimension $g$, $L$ is a rigidified symmetric relatively ample line bundle on $A$, and $\theta$ is a symmetric theta level structure.

We usually write $\epsilon : e^*L \to \mathcal{O}_S$ for the rigidification on $L$.

**Theorem 8.2.** Assume $4|d_i$ for all $i$. Then $\mathcal{M}_{g, \delta}$ is representable by a quasi-projective scheme over $\mathbb{Z}[1/2d]$.

**Remark 8.3.** The condition that $4|d_i$ for all $i$, implies that the objects of $\mathcal{M}_{g, \delta}$ admit no nontrivial automorphisms. It suffices to consider the case when $S$ is the spectrum of a strictly henselian local ring $R$. The space of global section $\Gamma(A, L)$ is then a representation of $G(A, L)$ over $R$, and hence via $\sigma$ we can view this as a representation of $G(\delta)$ over $R$. If

\[
\chi : A \to A, \quad \chi^b : \chi^*L \to L
\]

is an automorphism of the pair $(A, L)$, then $(\chi, \chi^b)$ induces an automorphism $\alpha$ of the $G(\delta)$-representation $\Gamma(A, L)$ such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\chi} & A \\
\downarrow{j} & & \downarrow{j} \\
\mathbb{P}\Gamma(A, L) & \xrightarrow{\mathbb{P}\alpha} & \mathbb{P}\Gamma(A, L)
\end{array}
\]

commutes, where $j$ is the canonical closed immersion (recall that $L$ is relatively very ample). On the other hand, any automorphism of the $G(\delta)$-representation $\Gamma(A, L) \simeq V(\delta)$ is multiplication by a scalar, and therefore the map $\mathbb{P}\alpha$ is the identity. We conclude that $\chi : A \to A$ is the identity map. This implies that $\chi^b : L \to L$ is multiplication by a scalar $u \in R^*$ such that the diagram

\[
\begin{array}{ccc}
e^*L & \xrightarrow{\epsilon} & e^*L \\
\downarrow{\epsilon} & & \downarrow{\epsilon} \\
R & & R
\end{array}
\]

commutes. It follows that $u = 1$.

In what follows we therefore also write $\mathcal{M}_{g, \delta}$ for the functor obtained by passing to isomorphism classes.

**Remark 8.4.** Note that $\mathcal{M}_{g, \delta}$ is a finite étale covering of $A_{g, \delta}$.

8.5. We will not give a proof of 8.2 (though the diligent reader should be able to fill in the details). Let us just indicate the key point.

Let $\mathcal{H}$ denote the Hilbert scheme classifying subschemes of $\mathbb{P}(V(\delta))$ with Hilbert polynomial $dn^g$, and let

\[
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{\epsilon} & \mathbb{P}(V(\delta)) \\
\downarrow & & \downarrow \\
\mathcal{H} & & \mathcal{H}
\end{array}
\]
be the universal family. There is a natural transformation of functors

\[ \gamma : \mathcal{M}_{g,\delta} \rightarrow \mathcal{Z} \]

defined as follows. Note that \( \mathcal{Z} \) represents the functor which to any scheme \( S \) associates the set of pairs \((Z, \sigma)\), where \( Z \) is a flat closed subscheme of \( \mathbb{P}(V(\delta))_S \) with Hilbert polynomial \( d n^g \), and \( \sigma : S \rightarrow Z \) is a section.

We get such a pair by noting that if \((A, L, \theta) \in \mathcal{M}_{g,\delta}(S)\) is an element and \( f : A \rightarrow S \) is the structure morphism, then there exists a canonical isomorphism

\[ \mathbb{P}(f_*L) \simeq \mathbb{P}(V(\delta))_S \]

since \( f_*L \) is a representation of \( \mathcal{G}(\delta) \) (using \( \theta \)) of rank \( d \). From the triple \((A, L, \theta)\) we therefore get an embedding

\[ A \hookrightarrow \mathbb{P}(V(\delta))_S, \]

and this closed subscheme also comes with a point (the identity section). This defines the transformation \( \gamma \).

We leave it as an exercise to show that this map \( \gamma \) is an immersion.

**8.6.** Finally let us mention another result of Mumford. If \( L \) is a symmetric invertible sheaf on an abelian scheme, and if \( \sigma : \iota^*L \rightarrow L \) is the canonical isomorphism which is the identity over the zero section, then since the 2-torsion \( A[2] \) is invariant under \( \iota \) we obtain an automorphism

\[ L|_{A[2]} \rightarrow L|_{A[2]} \]

We say that \( L \) is *totally symmetric* if this automorphism is the identity. One can show that this is an open and closed condition. Let

\[ \mathcal{M}_{g,\delta}^{\text{tot}} \subset \mathcal{M}_{g,\delta} \]

denote the open and closed subfunctor classifying totally symmetric line bundles.

What Mumford shows among other things is the following [3]:

**Theorem 8.7.** The composite morphism

\[ \mathcal{M}_{g,\delta}^{\text{tot}} \xrightarrow{\gamma} \mathcal{Z} \subset \mathbb{P}(V(\delta))_{\mathcal{Z}} \xrightarrow{\text{proj}} \mathbb{P}(V(\delta))_{Z[1/2d]} \]

is an immersion if \( 8|d_i \) for all \( i \).

**References**

