HANGZHOU – WORKSHOP LECTURES ON ABELIAN VARIETIES

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Note: I make no claim that the material in these lectures is original. In fact, the bulk of what is contained in the three lectures can be found in [3], and the reader is encouraged to study these papers of Mumford for the many deeper results contained therein. For the basic theory of abelian varieties an excellent reference is [2].

LECTURE 1.

1. Abelian schemes

1.1. A *group scheme* over a base scheme S is an S-scheme

$$G \to S$$

together with maps

$$m : G \times_S G \to G \quad \text{(multiplication)},$$
$$e : S \to G \quad \text{(identity section)},$$
$$\iota : G \to G \quad \text{(inverse)}$$

such that the following diagrams commute:

$$\begin{tikzcd}
G \times_S G \times_S G \arrow[r, "\mathrm{id}\times m"] \arrow[d, "m\times\mathrm{id}"] & G \times_S G \arrow[d, "m"] \\
G \times_S G \arrow[r, "m"] & G,
\end{tikzcd}$$

$$\begin{tikzcd}
G \times_S G \arrow[r, "m"] & G \\
G, \arrow[u, "e\times\mathrm{id}"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
G \times_S G \arrow[r, "m"] & G \\
G, \arrow[u, "\mathrm{id}\times e"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "\Delta"] \arrow[d] & G \times_S G \arrow[r, "\mathrm{id}\times\iota"] & G \times_S G \arrow[d, "m"] \\
S \arrow[rr, "e"] & & G,
\end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d] & G \times_S G \arrow[r, "\iota \times \mathrm{id}"] & G \times_S G \arrow[d, "m"] \\ S \arrow[rr, "e"] & & G. \end{tikzcd}$$

We usually suppress the maps m, e, and ι from the notation and write simply G/S for a group scheme.

A group scheme G/S is called *abelian* if in addition the diagram

$$\begin{tikzcd} G \times_S G \arrow[rr, "\mathrm{flip}"] \arrow[dr, "m"'] & & G \times_S G \arrow[dl, "m"] \\ & G & \end{tikzcd}$$

commutes.

Remark 1.2. Morphisms of group schemes are morphisms of schemes, which respect the group scheme structure.

Remark 1.3. By Yoneda's lemma, giving a scheme G/S the structure of a group scheme is equivalent to giving a factorization of its functor of points

$$\begin{tikzcd} & (\text{Groups}) \arrow[d, "\text{forget}"] \\ (S\text{-schemes})^{\mathrm{op}} \arrow[ur, dashed] \arrow[r, "h_G"] & (\text{Set}). \end{tikzcd}$$

Therefore an equivalent definition of an (abelian) group scheme is a contravariant functor from S-schemes to (abelian) groups such that the induced functor to sets is representable. This will usually be the preferred way of describing group schemes.

Example 1.4. (*The additive group*) This is the functor

$$\mathbb{G}_a : (S\text{-schemes})^{\mathrm{op}} \to (\text{abelian groups})$$

sending T/S to $\Gamma(T, \mathscr{O}_T)$. Note that the underlying scheme of this group scheme is the affine line $\mathbb{A}^1_S$.

Example 1.5. (*The multiplicative group*) This is the functor

$$\mathbb{G}_m : (S\text{-schemes})^{\mathrm{op}} \to (\text{abelian groups})$$

sending T/S to $\Gamma(T, \mathscr{O}_T^*)$. The underlying scheme of this group scheme is

$$S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}[X^{\pm}]).$$

Example 1.6. (*The general linear group*) This is the functor (where $n \geq 1$ is a fixed integer)

$$GL_n : (S\text{-schemes})^{\mathrm{op}} \to (\text{Groups})$$

sending T/S to the group of $n \times n$ invertible matrices with coefficients in $\Gamma(T, \mathscr{O}_T)$. The underlying scheme of this group scheme is

$$S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}[X_{ij}]_{i,j=1}^n[1/\det]),$$

where det denotes the polynomial given by the determinant of the $n \times n$-matrix with entries X_{ij}.

Example 1.7. (*The Jacobian of a curve*) Let $S = \mathrm{Spec}(k)$ be the spectrum of a field, and let C/k be a smooth, proper, geometrically connected curve, and fix a point $x \in C(k)$.

If T/k is a k-scheme, let C_T denote the base change $C \times_{\mathrm{Spec}(k)} T$. A *rigidified line bundle* on C_T is a pair (L, σ), where L is a line bundle on C_T and

$$\sigma : x^* L \to \mathscr{O}_T$$

is an isomorphism of line bundles on T. The set of rigidified line bundles on C_T form an abelian group with addition given by

$$(L, \sigma) + (M, \eta) := (L \otimes M, \sigma \otimes \eta),$$

where $\sigma \otimes \eta$ denotes the isomorphism

$$x^*(L \otimes M) \xrightarrow{\simeq} x^* L \otimes x^* M \xrightarrow{\sigma \otimes \eta} \mathscr{O}_T \otimes_{\mathscr{O}_T} \mathscr{O}_T \simeq \mathscr{O}_T.$$

Define

$$P : (k\text{-schemes})^{\mathrm{op}} \to (\text{abelian groups})$$

to be the functor sending T/k to the group of rigidifed line bundles on C_T. Then one can show the following:

(1) P is an abelian group scheme. Moreover, one can define P without the choice of the base point x.
(2) Taking the degree of a line bundle defines a surjection of group schemes

$$P \to \mathbb{Z},$$

whose kernel J is a smooth, proper, geometrically connected group scheme over k of dimension equal to the genus g of C. The group scheme J is called the *Jacobian of* C.

Remark 1.8. Note that the definition of rigidified line bundle makes sense on an arbitrary S-scheme X/S with section $x \in X(S)$.

Definition 1.9. An *abelian scheme* over a base scheme S is a group scheme A/S such that $A \to S$ is of finite presentation, proper smooth, and all fibers are geometrically connected.

Remark 1.10. Note the distinction between "abelian scheme" and "abelian group scheme".

Proposition 1.11. *(i) Abelian schemes are abelian group schemes.*

(ii) The group structure on an abelian scheme A/S is determined by the identity section.

(iii) Let A and B be abelian schemes over a base scheme S, and let $f : A \to B$ be a morphism of the underlying schemes. Then there exists a section $b \in B(S)$ such that $f = t_b \circ h$, where $h : A \to B$ is a morphism of group schemes, and t_b denotes translation by the point b.

Proof. Note that in (iii) the section $b \in B(S)$ must be $f(e_A)$ (where e_A denotes the identity section of A).

$(iii) \implies (ii)$. This is immediate (consider the identity map on A).

$(iii) \implies (i)$. Note that a group scheme A/S is an abelian group scheme if and only if the inverse map $\iota : A \to A$ is a homomorphism.

So it suffices to prove (iii). Furthermore, replacing f by $t_{-f(e_A)} \circ f$ it suffices to show that any morphism of schemes $f : A \to B$ with $f(e_A) = e_B$ is a morphism of group schemes, which is the statement we now prove.

Let

$$\rho : A \times_S A \to B$$

be the map given by (here described as a map on functors of points)

$$(a, a') \mapsto f(a + a') - (f(a) + f(a')).$$

We need to show that ρ is the constant map given by $e_B \in B(S)$.

Consider first the case when $S = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field k. We have

$$\rho(\{e_A\} \times A) = \{e_B\} = \rho(A \times \{e_A\}).$$

Let $U \subset B$ be an affine open neighborhood of e_B, and let $Z \subset B$ be its completement. Then

$$T := \mathrm{pr}_2(\rho^{-1}(Z)) \subset A$$

is a closed set, since

$$\mathrm{pr}_2 : A \times A \to A$$

is a closed map. By definition a point $\alpha \in A(k)$ lies outside of T if and only if

$$\rho(A \times \{\alpha\}) \subset U.$$

Since A is proper this is equivalent to the condition that

$$\rho(A \times \{\alpha\}) = \{e_B\}.$$

Let $W \subset A$ denote the complement of T. Then W is an open subset such that

$$\rho(A \times W) = \{e_B\}.$$

Since $e_A \in W$ then open subset W is nonempty, whence dense. We therefore see that the restriction of ρ to the dense open subset

$$A \times W \subset A \times A$$

is the constant map e_B, which then implies that ρ is also the constant map.

The case of a general base scheme S can be deduced from the case of an algebraically closed field. We leave this as an exercise. □

Exercise 1.12. Let C be a smooth, connected, proper curve over an algebraically closed field k. Then C admits the structure of an abelian variety if and only if the genus of C is 1.

2. The theorem of the cube

2.1. Let S be a scheme and A/S an abelian scheme. For a subset $I \subset \{1, 2, 3\}$, let

$$m_I : A \times_S A \times_S A \to A$$

be the map given on functors of points by

$$(x_1, x_2, x_3) \mapsto \sum_{i \in I} x_i.$$

Theorem 2.2 (Theorem of the cube)**.** *Let (L, σ) be a rigidified line bundle on A. Then the line bundle*

$$\bigotimes_{I \subset \{1,2,3\}} m_I^* L^{(-1)^{|I|}}$$

is trivial.

Before starting the proof of the theorem, let us note the following corollary. For a point $a \in A(S)$, define a rigidified line bundle

$$\Lambda_L(a) := t_a^* L \otimes L^{-1} \otimes L(a+b)^{-1}.$$

Corollary 2.3. *Let $a, b \in A(S)$ be points. Then there is a unique isomorphism of rigidified line bundles*

$$\Lambda_L(a+b) \simeq \Lambda_L(a) \otimes \Lambda_L(b).$$

The proof of 2.2 occupies the remainder of this section. By a standard limit argument, we may assume that S is of finite type over an excellent Dedekind ring. Consider first the following general results.

Lemma 2.4. *Let X/k be a connected, integral, and proper scheme over an algebraically closed field k, and let L be a line bundle on X. Then L is trivial if and only if both $H^0(X, L)$ and $H^0(X, L^{-1})$ are nonzero.*

Proof. The 'only if' direction is immediate.

For the 'if' direction, let $\alpha \in H^0(X, L)$ and $\beta \in H^0(X, L^{-1})$ be nonzero sections. We view these sections as maps

$$\alpha : \mathscr{O}_X \to L, \quad \beta : \mathscr{O}_X \to L^{-1},$$

and write

$$\alpha^\wedge : L^{-1} \to \mathscr{O}_X, \quad \beta^\wedge : L \to \mathscr{O}_X$$

for the induced maps on duals.

The composite map

$$\mathscr{O}_X \xrightarrow{\alpha} L \xrightarrow{\beta^\wedge} \mathscr{O}_X$$

is then a nonzero map, and therefore is an isomorphism. In particular, the map $\beta^\wedge$ is surjective. Tensoring $\beta^\wedge$ with L^{-1} we get that β is surjective, whence an isomorphism. □

Lemma 2.5. *Let X and Y be smooth, proper S-schemes with geometrically connected fibers, and let Z be any finite type connected S-scheme. Assume given points*

$$x \in X(S), \quad y \in Y(S), \quad z \in Z(S)$$

and a line bundle L on $X \times_S Y \times_S Z$ such that

$$L|_{X \times Y \times \{z\}}, \quad L|_{X \times \{y\} \times Z}, \quad L|_{\{x\} \times Y \times Z}$$

are all trivial. Let

$$p : X \times_S Y \times_S Z \to Z$$

be the third projection. Then $p_* L$ is a line bundle on Z and the adjunction map

$$p^* p_* L \to L$$

is an isomorphism.

LECTURE 2.

3. The theta group

3.1. Let A/S be an abelian scheme over some base scheme S, and let L be a line bundle on A. Define the *theta group* of (A, L), denoted $\mathscr{G}_{(A,L)}$ to be the functor

$$(S\text{-schemes})^{\mathrm{op}} \to (\text{groups})$$

sending T/S to the group

$$\{(x, \sigma) | x \in A(T), \quad \sigma : t_x^* L \to L\}.$$

The group structure is given by defining the product of two elements (x, σ) and (y, η) to be $x + y \in A(T)$ with the isomorphism

$$t_{x+y}^* L \xrightarrow{\simeq} t_x^*(t_y^* L) \xrightarrow{t_x^* \eta} t_x^* L \xrightarrow{\sigma} L.$$

Remark 3.2. As we will see, the theta group $\mathscr{G}_{(A,L)}$ is usually not commutative.

Theorem 3.3. *The theta group $\mathscr{G}_{(A,L)}$ is a group scheme. Moreover, there is a natural exact sequence*

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\alpha} \mathscr{G}_{(A,L)} \longrightarrow K_{(A,L)} \longrightarrow 1,$$

where $\mathbb{G}_m$ is central in $\mathscr{G}_{(A,L)}$ and $K_{(A,L)}$ is commutative and proper over S.

Proof. Note that $\mathscr{G}_{(A,L)}$ is a sheaf with respect to the fppf topology.

The inclusion α is obtained by the map sending $u \in \mathbb{G}_m(T)$ to the pair $(e, u) \in \mathscr{G}_{(A,L)}(T)$. The quotient of $\mathscr{G}_{(A,L)}$ by $\mathbb{G}_m$ (quotient taken in the category of sheaves with respect to the fppf topology) is the functor sending T/S to the set of elements $a \in A(T)$ such that the two line bundles $t_a^* L_T$ and L_T on A_T are fppf-locally on T isomorphic. Let $K_{(A,L)}$ denote this functor. It suffices to show that $K_{(A,L)}$ is a proper S-scheme.

For this let

$$P : (S\text{-schemes})^{\mathrm{op}} \to (\text{Groups})$$

denote the functor sending T/S to the group of isomorphism classes of rigidified line bundles on A_T.

Fact: P is a proper algebraic space locally of finite presentation over S.

There is a map

$$\lambda_L : A \to P, \quad a \mapsto t_a^* L \otimes L^{-1} \otimes L(e) \otimes L(a)^{-1}.$$

Note that this is a group homomorphism by the theorem of the cube, discussed last lecture.

Now $K_{(A,L)}$ is equal to $\lambda_L^{-1}(\mathscr{O}_A, \text{can})$, and hence is a closed subscheme of A. □

Proposition 3.4. *If L is relatively ample on A/S, then $K_{(A,L)}$ is finite over S.*

Proof. Since $K_{(A,L)}$ is proper over S, it suffices to show that it is quasi-finite over S. We may therefore assume that $S = \mathrm{Spec}(k)$, with k an algebraically closed field.

Let $Y \subset P$ denote the connected component of the identity, with the reduced structure. Then Y is an abelian variety, and we have an ample line bundle $L_Y := L|_Y$ such that for every $y \in Y$ we have

$$t_y^* L_Y \simeq L_Y.$$

Consider

$$\Lambda(L) := m^*L \otimes \mathrm{pr}_1^* L^{-1} \otimes \mathrm{pr}_2^* L^{-1}$$

on $Y \times Y$. Then for every $y \in Y$ the restriction of $\Lambda(L)$ to $\{y\} \times Y$ and $Y \times \{y\}$ is trivial. By the argument of lecture 1 we get that $\Lambda(L)$ is trivial. On the other hand, consider the map

$$\mathrm{id} \times \iota : Y \to Y \times Y$$

The pullback of $\Lambda(L)$ along this map is the line bundle

$$(L \otimes \iota^* L)^{-1}.$$

It follows that the ample line bundle $L \otimes \iota^* L$ is trivial, which implies that Y must be zero-dimensional. □

Example 3.5. Let $S = \mathrm{Spec}(k)$ be the spectrum of an algebraically closed field, and let E/k be an elliptic curve. Let $L = \mathscr{O}_E(ne)$, for some integer $n \geq 1$. Then for a point $a \in E(k)$ we have

$$t_a^* L \otimes L^{-1} \simeq \mathscr{O}_E(n(-a) - n(e))$$

which is trivial if and only if $na = e$. We conclude that $K_{(E,L)}$ is isomorphic to the n-torsion group scheme

$$E[n] := \mathrm{Ker}(\times n : E \to E).$$

Note also that if $a \in E[n]$ then an isomorphism

$$t_a^* L \to L$$

is given by a function $f \in k(E)$ (where $k(E)$ denotes the function field of E such that $\mathrm{div}(f) = n(-a) - n(e)$). The theta group $\mathscr{G}_{(E,L)}(k)$ can therefore be described as the set of pairs (a, f) where $a \in E[n]$ and $f \in k(E)$ is a function such that $\mathrm{div}(f) = n(-a) - n(e)$.

To proceed we will need the following facts:

Facts 3.6. Let A/k be an abelian variety of dimension g over an algebraically closed field k, and let L be an ample line bundle on A.

(i) There exists an integer $d \geq 1$ such that for all $n \geq 1$ we have

$$h^0(A, L^{\otimes n}) = dn^g$$

and

$$h^i(A, L^{\otimes n}) = 0$$

for all $i > 0$. We refer to this integer d as the *degree* of L.

(ii) The group scheme $K_{(A,L)}$ has rank d^2.

Lemma 3.7. *Let $f : A \to S$ be an abelian scheme, and let L be a relatively ample line bundle on A/S. Then f_*L is a locally free sheaf on S, whose formation commutes with arbitrary base change on S. In particular, the function sending $s \in S$ to the degree of $L|_{A_s}$ is a locally constant function on S.*

Proof. This follows from fact (i) and cohomology and base change. □

Remark 3.8. In particular if L is a relatively ample line bundle on an abelian scheme A/S then it makes sense to talk about the degree of L, which is a locally constant function on S.

Proposition 3.9. *Let A/S be an abelian scheme and L a relatively ample line bundle on A of degree d. Then $K_{(A,L)}$ is a finite flat group scheme over S of rank d^2.*

Proof. We already know that $K_{(A,L)}$ is finite over S, so it suffices to show that if $K_{(A,L)}$ is flat over S. If S is reduced, this can be done as follows. We may work locally on S, so it suffices to consider the case when $S = \mathrm{Spec}(R)$, where R is a local ring with maximal ideal $\mathfrak{m} \subset R$. Let M denote the coordinate ring of $K_{(A,L)}$, viewed as a finitely generated R-module. Since

$$\dim_{R/\mathfrak{m}}(M/\mathfrak{m}M) = d^2,$$

we can (by Nakayama's lemma) choose a surjection

$$R^{d^2} \xrightarrow{\pi} M,$$

reducing to an isomorphism modulo $\mathfrak{m}$. Since the dimension of every fiber of M is d^2, the map π induces an isomorphism modulo every prime ideal of R. This implies in particular that at each generic point of R the map π is an isomorphism. Since R is reduced, this implies that π is also injective, whence an isomorphism.

For the case of a general base, one needs to use something more sophisticated. In this case we need that the map

$$\lambda_L : A \to P$$

is flat, which follows from the theory of the dual abelian variety, which implies that λ_L is surjective onto an open subspace of P which is smooth over S and of the same dimension as A. □

Summary 3.10. Let A/S be an abelian scheme, and let L be a relatively ample invertible sheaf on A of degree d. Then $\mathscr{G}_{(A,L)}$ is a flat group scheme over S, which sits in a central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathscr{G}_{(A,M)} \longrightarrow K_{(A,L)} \longrightarrow 1,$$

where $K_{(A,L)}$ is finite and flat of degree d^2 over S.

3.11. In the setting of 3.10, note that we get a natural skew-symmetric pairing

$$e : K_{(A,L)} \times K_{(A,L)} \to \mathbb{G}_m$$

defined on (scheme-valued) points by sending

$$(x, y) \mapsto \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1},$$

where $\tilde{x}, \tilde{y} \in \mathscr{G}_{(A,L)}$ are liftings of x and y respectively (we leave it as an exercise to verify that this is well-defined). This pairing is called the *Weil pairing.*

4. Level subgroups

4.1. Throughout this section we work over an algebraically closed field k, A/k denotes an abelian variety, and L is an ample line bundle on A of degree d, which we assume invertible in k.

Definition 4.2. A *level subgroup* of $\mathscr{G}_{(A,L)}$ is a subgroup $\tilde{H} \subset \mathscr{G}_{(A,L)}(k)$ such that $\tilde{H} \cap k^* = \{1\}$.

Remark 4.3. Note that if $\tilde{H}$ is a level subgroup, then the projection $\tilde{H} \to K_{(A,L)}$ is injective, whence $\tilde{H}$ is finite and commutative. This implies that we can also view $\tilde{H}$ as a subgroup scheme of $\mathscr{G}_{(A,L)}$.

Exercise 4.4. Show that if $x \in K_{(A,L)}$ is an element, then there exists a level subgroup $\tilde{H} \subset \mathscr{G}_{(A,L)}$ whose image in $K_{(A,L)}$ contains x.

4.5. Let $\tilde{H} \subset \mathscr{G}_{(A,L)}$ be a level subgroup, and let $H \subset K_{(A,L)}$ denote the image (so the projection $\tilde{H} \to H$ is an isomorphism). Then H is a subgroup scheme of A, and we can form the quotient

$$\pi : A \to B := A/H.$$

Note that if M is a line bundle on B, then there is a natural embedding $H \hookrightarrow \mathscr{G}_{(A,\pi^*M)}$. Indeed if $y \in H \subset A$ then there are canonical isomorphisms

$$t_y^*\pi^*M \simeq \pi^*t_{\pi(y)}^*M \simeq \pi^*M.$$

Conversely, descent theory implies that the choice of the level subgroup $\tilde{H}$ lifting H is precisely equivalent to specifying a line bundle M on B and an isomorphism $\pi^*M \simeq L$. In other words, we have a bijection between level subgroups of $\mathscr{G}_{(A,L)}$ and the set of triples $(\pi : A \to B, M, \sigma)$, where $\pi : A \to B$ is a surjection of abelian varieties, M is a line bundle on B, and $\sigma : \pi^*M \simeq L$ is an isomorphism of line bundles on A.

Exercise 4.6. Let $\tilde{H} \subset \mathscr{G}_{(A,L)}$ be a level subgroup, and let $(\pi : A \to B, M, \sigma)$ be the corresponding collection of data.

(i) Show that if r denotes the order of $\tilde{H}$, then

$$h^0(B, M) \cdot r = h^0(A, L).$$

(ii) Let

$$\Sigma := \{z \in \mathscr{G}_{(A,L)} | z \text{ centralizes } \tilde{H}\}.$$

Show that there is a natural isomorphism

$$\mathscr{G}_{(B,M)} \simeq \Sigma/\tilde{H}.$$

Corollary 4.7. *With notation as in 4.6, assume that $\tilde{H}$ is a maximal level subgroup. Then the order of $\tilde{H}$ is d, and the degree of M is* 1.

Proof. By the formula in (4.6 (i)), the two statements of the corollary are equivalent. Also since $\tilde{H}$ is maximal, we have

$$\Sigma = \mathbb{G}_m \cdot \tilde{H}.$$

It follows that $\mathscr{G}_{(B,M)} \simeq \mathbb{G}_m$ which implies that the degree of M is 1. $\square$

Corollary 4.8. *The Weil pairing e is non-degenerate.*

Proof. Let $x \in K_{(A,L)}$ be an element, and let $\tilde{H} \subset \mathscr{G}_{(A,L)}$ be a maximal level subgroup whose image $H \subset K_{(A,L)}$ contains x. Then since $\tilde{H}$ is maximal, the map

$$K_{(A,L)}/H \to \mathrm{Hom}(H, \mathbb{G}_m), \quad [y] \mapsto e(y, -)$$

is injective. Since this is a map of finite groups of the same order we conclude that it is an isomorphism. Therefore there exists an element $y \in K_{(A,L)}$ such that $e(y, x) \neq 1$. □

Exercise 4.9. Let $\tilde{H} \subset \mathscr{G}_{(A,L)}$ be a maximal level subgroup, and let $H \subset K_{(A,L)}$ be its image. Let

$$H^\wedge := \mathrm{Hom}(H, \mathbb{G}_m),$$

denote the Cartier dual of H. Show that there exists an isomorphism of schemes

$$\mathscr{G}_{(A,L)} \simeq \mathbb{G}_m \times H \times H^\wedge$$

such that the group law is given by

$$(u, x, \chi) \cdot (v, y, \eta) = (uv\eta(x), x + y, \chi + \eta).$$

Remark 4.10. Since H is a finite abelian group, there exists an integer s and integers $d_1, \ldots, d_s$ such that

$$d_1 d_2 \cdots d_s = d,$$

and

$$H \simeq \oplus_{i=1}^s \mathbb{Z}/(d_i).$$

In this case

$$H^\wedge \simeq \oplus_{i=1}^s \mu_{d_i}.$$

Note that the integers s and $d_1, \ldots, d_s$ are independent of the choice of the level subgroup.

LECTURE 3.

5. Representations of the Heisenberg group

5.1. Fix a collection of positive integers

$$\delta = (d_1, \ldots, d_s),$$

and let d denote $d_1 \cdots d_s$.

Let H denote the group

$$H := \oplus_{i=1}^s \mathbb{Z}/(d_i),$$

and let

$$H^\wedge = \prod_{i=1}^s \mu_{d_i}$$

denote the Cartier dual of H. Both H and $H^\wedge$ are group schemes over $\mathbb{Z}$. Let

$$\mathscr{G}(\delta)$$

denote the group scheme whose underlying scheme is

$$\mathbb{G}_m \times H \times H^\wedge$$

and group law given by the formula (on scheme-valued points)

$$(u, x, \chi) \cdot (v, y, \eta) = (uv\eta(x), x + y, \chi + \eta). \tag{5.1.1}$$

Then $\mathscr{G}(\delta)$ is a group scheme over $\mathbb{Z}$ which is a central extension

$$1 \to \mathbb{G}_m \to \mathscr{G}(\delta) \to H \times H^\wedge \to 1.$$

Remark 5.2. Note that for any $\eta \in H^\wedge$ and $x \in H$ we have $\eta(x) \in \mu_d$. We can therefore also define a group scheme $\mathscr{G}(\delta)'$ which as a scheme is

$$\mu_d \times H \times H^\wedge$$

and group given by the formula (5.1.1). Then we have a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mu_d \arrow[r] \arrow[d, hook] & \mathscr{G}(\delta)' \arrow[r] \arrow[d, hook] & H \times H^\wedge \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathbb{G}_m \arrow[r] & \mathscr{G}(\delta) \arrow[r] & H \times H^\wedge \arrow[r] & 1, \end{tikzcd}$$

which identifies $\mathscr{G}(\delta)$ with the pushout of the diagram

$$\begin{tikzcd} \mu_d \arrow[r] \arrow[d] & \mathscr{G}(\delta)' \\ \mathbb{G}_m. & \end{tikzcd} \tag{5.2.1}$$

5.3. Let $V(\delta)$ denote the free $\mathbb{Z}$-module on the set H^δ (so $V(\delta)$ is the set of functions $H^\wedge \to \mathbb{Z}$). For $\chi \in H^\delta$ let $f_\chi \in V(\delta)$ denote the corresponding basis element. Then $V(\delta)$ is naturally a representation ρ of $\mathscr{G}(\delta)$. For (scheme-valued points) $u \in \mathbb{G}_m$, $x \in H$, and $\eta \in H^\wedge$ we have

$$\rho_u(f_\chi) = u \cdot f_\chi, \quad \rho_x(f_\chi) = \chi(x) \cdot f_\chi, \quad \rho_\eta(f_\chi) = f_{\chi+\eta}.$$

Exercise 5.4. Show that this really defines a representation of $\mathscr{G}(\delta)$.

Theorem 5.5. *Let R be a $\mathbb{Z}[1/d]$-algebra, and let P be a projective R-module. Let*

$$\rho : \mathscr{G}(\delta) \to GL(P)$$

be a representation of $\mathscr{G}(\delta)$ over R such that $\rho|_{\mathbb{G}_m}$ is the standard multiplication action of $\mathbb{G}_m$. Then after possibly replacing P by a flat extension, there exists an isomorphism of $\mathscr{G}(\delta)$-representations over R

$$P \simeq V(\delta)^r \otimes R$$

for some integer r.

Proof. Let us first consider the case when $R = k$ is an algebraically closed field, and to ease notation write abusively also $\mathscr{G}(\delta)$ (resp. $V(\delta)$) for the base change of $\mathscr{G}(\delta)$ (resp. $V(\delta)$) to k.

Suppose

$$\gamma : \mathscr{G}(\delta) \to GL(V)$$

is an irreducible representation over k such that $\mathbb{G}_m \subset \mathscr{G}(\delta)$ acts by multiplication. We then show as follows that $V \simeq V(\delta)$. Let $\tilde{H} \subset \mathscr{G}(\delta)$ denote the maximal level subgroup lifting H given by the presentation $\mathscr{G}(\delta) \simeq \mathbb{G}_m \times H \times H^\wedge$. Then since $\tilde{H}$ is a finite commutative group of order invertible in k, we have a canonical decomposition

$$V \simeq \oplus_{\chi \in H^\wedge} V_\chi,$$

where $x \in \tilde{H}$ acts on V_χ by multiplication by $\chi(x)$. Note that for $\eta \in H^\wedge \subset \mathscr{G}(\delta)$, we have for $v \in V_\chi$ and $x \in \tilde{H}$

$$\gamma_x(\gamma_\eta(v)) = \eta(x)\gamma_\eta\gamma_x(v) = \eta(x)\chi(x)\gamma_\eta(v).$$

It follows that γ_η maps V_χ to $V_{\chi+\eta}$.

Fix any nonzero vector $v_0 \in V_0$, and let $W \subset V$ be the span of elements $\gamma_\eta(v_0)$ for $\eta \in H^\wedge$. Then W is a subrepresentation of V, and since V is irreducible we must have $W = V$. We conclude that each V_χ is 1-dimensional. Fix a nonzero element $v_0 \in V_0$, and let $v_\chi \in V_\chi$ denote $\gamma_\chi(v_0)$. We then obtain a surjection

$$V(\delta) \to V, \quad f_\chi \mapsto v_\chi,$$

which must be an isomorphism since both are irreducible.

To see that any representation V of $\mathscr{G}(\delta)$ on which $\mathbb{G}_m$ acts by multiplication is a direct sum of copies of $V(\delta)$, note that since (5.2.1) is a pushout diagram the restriction functor

$$\begin{array}{c}\text{(rep's of } \mathscr{G}(\delta) \text{ on which } \mathbb{G}_m \text{ acts by multiplication)} \\ \downarrow \\ \text{(rep's of } \mathscr{G}(\delta)' \text{ on which } \mu_d \text{ acts by multiplication)}\end{array} \tag{5.5.1}$$

is an equivalence of categories. Since $\mathscr{G}(\delta)'$ is a finite group of order invertible in k, its category of representations is semisimple, and therefore any representation is a direct sum of irreducible representations.

This proves the proposition when R is an algebraically closed field. We leave it as an exercise to extend the argument to the case of an arbitrary $\mathbb{Z}[1/2d]$-algebra. □

6. Polarizations

6.1. Let k be an algebraically closed field, and let A/k be an abelian variety. Denote by $P_{A/k}$ the Picard scheme representing the functor of rigidified line bundles on A. The *dual abelian variety* of A, denoted A^t, is the connected component of the identity in $P_{A/k}$.

Facts 6.2. (i) A^t is an abelian variety of the same dimension as A.

(ii) If M is a rigidified line bundle on A, then the corresponding point $[M] \in P_{A/k}(k)$ is in A^t if and only if for every $a \in A(k)$ we have $t_a^*M \simeq M$, and (by the theorem of the cube) this is in turn equivalent to the statement that the line bundle

$$m^*M \otimes \mathrm{pr}_1^*L^{-1} \otimes \mathrm{pr}_2^*L^{-1}$$

on $A \times A$ is trivial.

6.3. If L is a line bundle, then we have the homomorphism

$$\lambda_L : A \to P_{A/k}, \quad a \mapsto [t_a^*L \otimes L^{-1} \otimes L(a)^{-1} \otimes L(e)],$$

and since A is connected this in fact has image in A^t. If L is ample then $\lambda_L : A \to A^t$ is surjective since it has finite kernel and A and A^t have the same dimension.

Exercise 6.4. Show directly that if $a, b \in A(k)$ are two points, then

$$t_b^*(t_a^*L \otimes L^{-1}) \simeq t_a^*L \otimes L^{-1}.$$

Definition 6.5. A *polarization of degree d* on A is a map $\lambda : A \to A^t$ isomorphic to λ_L for some ample L.

Remark 6.6. If L is ample and M is a line bundle on A corresponding to a point of A^t, then $L \otimes M$ is also ample. Indeed write $M = t_a^*L \otimes L^{-1}$ for some $a \in A(k)$. Then

$$L \otimes M \simeq t_a^*L$$

and t_a^*L is ample on A since $t_a : A \to A$ is an isomorphism.

Conversely if L and L' are two line bundles such that $\lambda_L = \lambda_{L'}$ then for every $a \in A$ we have

$$t_a^*L \otimes L^{-1} \simeq t_a^*L' \otimes L'^{-1},$$

or equivalently if M denotes $L' \otimes L^{-1}$ then

$$t_a^*M \simeq M$$

for all $a \in A(k)$. Therefore $L' = L \otimes M$ for some $[M] \in A^t(k)$.

It follows that given a polarization $\lambda : A \to A^t$, the set of line bundles L with $\lambda_L = \lambda$ is a torsor under A^t.

6.7. The preceding discussion can be generalized to families. Let A/S be an abelian scheme over a base scheme S, and let $P_{A/S}$ be the algebraic space representing the functor on S-schemes

$$T \mapsto \{\text{group of rigidified line bundles on } A_T\}.$$

Then on $A \times_S P_{A/S}$ there is a tautological line bundle $\mathscr{L}$, and so we get a line bundle

$$\mathscr{M} := m^*\mathscr{L} \otimes \mathrm{pr}_1^*\mathscr{L}^{-1} \otimes \mathrm{pr}_2^*\mathscr{L}^{-1}$$

on $(A \times_S A) \times_S P_{A/S}$ which is rigidified along the zero section

$$P_{A/S} \to (A \times_S A) \times_S P_{A/S}.$$

This line bundle corresponds to a morphism

$$\tau : P_{A/S} \to P_{(A\times_S A)\times P_{A/S}/P_{A/S}},$$

and we define A^t to be the inverse image under this map of the zero section of $P_{(A\times_S A)\times P_{A/S}/P_{A/S}}$.

It is known that A^t is an abelian scheme over S, whose fiber over any point $s \in S$ is the dual abelian variety of A_s.

Definition 6.8. Let A/S be an abelian scheme over a base scheme S. A *polarization* on A is a map $\lambda : A \to A^t$ which étale locally on S is of the form $\lambda_L : A \to A^t$ for some relatively ample line bundle L on A.

Remark 6.9. Note that λ is a homomorphism as this can be verified étale locally on S, and λ_L is a homomorphism.

Remark 6.10. Just as in the case of a field, given a polarization $\lambda : A \to A^t$, the functor on S-schemes

$$T \mapsto \{\text{line bundles } L \text{ on } A_T \text{ such that } \lambda_L = \lambda\}$$

is a torsor under A^t.

Remark 6.11. If A/S is an abelian scheme and $\lambda : A \to A^t$ is a polarization, then $\mathrm{Ker}(\lambda)$ is a finite flat group scheme over S. Indeed this can be verified étale locally so we may assume that $\lambda = \lambda_L$ for some relatively ample line bundle on A. In this case $\mathrm{Ker}(\lambda_L) = K_{(A,L)}$ so the result follows from 3.9. In particular, it makes sense to talk about the *degree of* λ, which is the locally constant function d on S for which $\mathrm{Ker}(\lambda)$ has rank d^2.

6.12. Fix now positive integers g and d, and define

$$\mathscr{A}_{g,d}$$

to be the stack over $\mathbb{Z}[1/2d]$ whose fiber over a $\mathbb{Z}[1/2d]$-scheme S is the groupoid of pairs (A, λ), where A/S is an abelian scheme and $\lambda : A \to A^t$ is a polarization of degree d.

Remark 6.13. Since we restrict to $\mathbb{Z}[1/2d]$-scheme, if $(A, \lambda) \in \mathscr{A}_{g,d}(S)$, the group scheme $\mathrm{Ker}(\lambda)$ is a finite étale group scheme over S. It is therefore étale locally on S constant, and we can talk about the *type* of λ, which is the unordered collection of integers $\delta = (d_1, \ldots, d_g)$ for which étale locally on S we have

$$\mathrm{Ker}(\lambda) \simeq (\oplus_{i=1}^g \mathbb{Z}/(d_i)) \times \prod_{i=1}^g \mu_{d_i}.$$

Since the type of a polarization is locally constant on the base, we have an isomorphism of stacks

$$\mathscr{A}_{g,d} = \coprod_\delta \mathscr{A}_{g,\delta},$$

where for a type $\delta = (d_1, \ldots, d_g)$ with $d = d_1 \cdots d_g$ we write $\mathscr{A}_{g,\delta}$ for the substack of $\mathscr{A}_{g,d}$ classifying pairs (A, λ) whose type is δ.

6.14. If A/S is an abelian scheme and $\lambda : A \to A^t$ is a polarization, then for $n \geq 1$ let $n\lambda$ denote the composite morphism

$$A \xrightarrow{\lambda} A^t \xrightarrow{\cdot n} A^t.$$

Then $n\lambda$ is also a polarization. Indeed it suffices to verify this étale locally on S, so we may assume that $\lambda = \lambda_L$ with L relatively ample on A. In this case $n\lambda = \lambda_{L^n}$. Note also that $n\lambda$ is also equal to the composite map

$$A \xrightarrow{\times n} A \xrightarrow{\lambda} A^t,$$

since λ is a homomorphism.

Exercise 6.15. If the type of λ is equal to $\delta = (d_1, \ldots, d_g)$, show that the type of $n\lambda$ is equal to

$$n\delta := (nd_1, \ldots, nd_g).$$

Proposition 6.16. *The induced morphism of stacks over* $\mathbb{Z}[1/2dn]$

$$\mathscr{A}_{g,\delta} \to \mathscr{A}_{g,n\delta}, \quad (A,\lambda) \mapsto (A, n\lambda)$$

is an isomorphism.

Proof. For the full faithfulness, it suffices to show that if (A,λ) and (B,η) are objects of $\mathscr{A}_{g,\delta}(S)$ for some scheme S, and if $f : A \to B$ is a morphism of abelian schemes such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "n\lambda"] & B \arrow[d, "n\eta"] \\ A^t & B^t \arrow[l, "f^*"] \end{tikzcd}$$

commutes, then the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\lambda"] & B \arrow[d, "\eta"] \\ A^t & B^t \arrow[l, "f^*"] \end{tikzcd}$$

also commutes. This follows from noting that the map

$$\lambda - f^* \circ \eta \circ f : A \to A^t$$

has image in $A[n]$, and since A has geometrically connected fibers any morphism $A \to A[n]$ is constant.

For the essential surjectivity, let $(A,\eta) \in \mathscr{A}_{g,n\delta}(S)$ be an object over some scheme S. We must show that after possibly replacing S by an fppf covering, we have $\eta = n\lambda$ for some polarization λ of type δ. There is no choice for what the map λ must be. Indeed note that multiplication by n on A is surjective and induces an isomorphism

$$A/A[n] \to A.$$

Since the type of η is $n\delta$, we have $A[n] \in \mathrm{Ker}(\eta)$, so we get a factorization of η as

$$A \xrightarrow{\cdot n} A \xrightarrow{\lambda} A^t,$$

for some map $\lambda : A \to A^t$. We must show that λ is a polarization.

For this we may work étale locally on S so we may assume that $\eta = \lambda_M$ for some relatively ample line bundle M on A. In this case we can fppf locally on S write $M \simeq L^{\otimes n}$ for some line bundle L on A, by [2, p. 231]. Then the map

$$\lambda_L - \lambda : A \to A^t$$

has image in $A^t[n]$, and so as above we conclude that $\lambda = \lambda_L$. $\square$

7. Interlude on symmetric line bundles

Definition 7.1. Let A/S be an abelian scheme. A line bundle L on A is called *symmetric* if there exists an isomorphism

$$\iota^* L \to L,$$

where $\iota : A \to A$ is the map sending x to $-x$.

Remark 7.2. If L is a symmetric line bundle on A, then there is a canonical choice of isomorphism $\iota^*L \to L$. Namely, if $e : S \to A$ denotes the identity section, then we have

$$e^*\iota^*L \simeq e^*L$$

since e is fixed by ι. Since any two choices of isomorphism $\iota^*L \to L$ differ by multiplication by an element of $\mathscr{O}_S^*$, there is a unique choice of isomorphism $\iota^*L \to L$ such that the induced map $e^*L \to e^*L$ is the identity. This implies in particular, that the condition that a line bundle on A/S is symmetric is local in the fppf topology on S.

Exercise 7.3. Show that the map

$$\iota^* : A^t \to A^t$$

given by pullback along ι, is equal to the map given by

$$[M] \mapsto [M^{-1}].$$

Lemma 7.4. *Let M be a rigidified line bundle on A/S which is symmetric and such that the corresponding point of $P_{A/S}$ lies in A^t. Then in fact the point $[M] \in A^t(S)$ lies in the 2-torsion subgroup $A^t[2] \subset A^t$.*

Proof. This is immediate from 7.3. For to say that M is symmetric is equivalent to saying that

$$\iota^*[M] = [M^{-1}] \in A^t(S)$$

is equal to $[M] \in A^t(S)$. Therefore the point $[M^2] \in A^t(S)$ is zero which implies that $[M] \in A^t[2]$. □

Lemma 7.5. *Let L and L' be two relatively ample symmetric invertible sheaves on an abelian scheme A/S. Then $\lambda_L = \lambda_{L'}$ if and only if there exists an invertible sheaf M on A such that $L' \simeq L \otimes M$ and such that $[M] \in A^t[2]$.*

Proof. By 6.6, $\lambda_L = \lambda_{L'}$ if and only if $L' = L \otimes M$ for some line bundle with $[M] \in A^t$. Since L' and L are symmetric this implies that M is also symmetric and therefore we must have $[M] \in A^t[2]$. □

Lemma 7.6. *Let A/S be an abelian scheme, and let $\lambda : A \to A^t$ be a polarization. Then fppf locally on S there exists an ample, symmetric line bundle L on A such that $\lambda = \lambda_L$.*

Proof. After replacing S by an étale covering, we may assume that $\lambda = \lambda_L$ for some relatively ample line bundle L. We must show that after possible replacing S by an fppf covering there exists line bundle R with $[R] \in A^t$ such that $L \otimes R$ is symmetric.

Consider the line bundle

$$M := \iota^*L \otimes L^{-1}.$$

For any scheme-valued point $\alpha \in A$, we have

$$t_\alpha^*\iota^*L \otimes \iota^*L^{-1} \simeq \iota^*(t_{-\alpha}^*L \otimes L^{-1}).$$

Now the line bundle $t_{-\alpha}^*L \otimes L^{-1}$ defines a point of A^t so

$$\iota^*(t_{-\alpha}^*L \otimes L^{-1}) \simeq t_{-\alpha}^*L^{-1} \otimes L.$$

We conclude that (after possibly making a base change on S)

$$t_\alpha^*(\iota^*L \otimes L^{-1}) \simeq \iota^*L \otimes t_\alpha^*L \otimes t_{-\alpha}^*L^{-1} \otimes L \simeq \iota^*L \otimes L^{-1} \otimes (t_\alpha^*L \otimes L^{-1}) \otimes (t_{-\alpha}^*L \otimes L^{-1}) \simeq \iota^*L \otimes L^{-1},$$

where the last isomorphism is using the fact that λ_L is a homomorphism.

It follows that M defines a point $[M] \in A^t(S)$. Since the map $\cdot 2 : A^t \to A^t$ is flat and surjective, after replacing M by a finite flat covering we can find a line bundle R such that $R^{\otimes 2} \simeq M$. Then

$$\iota^*(L \otimes R) \simeq \iota^* L \otimes R^{-1} \simeq L \otimes R^{\otimes 2} \otimes R \simeq L \otimes R.$$

□

7.7. If A/S is an abelian scheme and L is a symmetric line bundle on A, then the theta group $\mathscr{G}_{(A,L)}$ has an additional structure reflecting the fact that L is symmetric. Fix an isomorphism $\gamma : \iota^* L \to L$ (for example the canonical one). For a scheme-valued point

$$(a, \sigma : t_a^* L \to L) \in \mathscr{G}_{(A,L)}$$

define $\tau(a, \sigma)$ to be the point $-a \in A$ together with the isomorphism

$$t_{-a}^* L \xrightarrow{\gamma^{-1}} t_{-a}^* \iota^* L \xrightarrow{\simeq} \iota^* t_a^* L \xrightarrow{\sigma} \iota^* L \xrightarrow{\gamma} L. \tag{7.7.1}$$

Exercise 7.8. Show that the isomorphism (7.7.1) is independent of the choice of γ, and that τ is a homomorphism.

7.9. Fix $\delta = (d_1, \ldots, d_g)$ and consider the group scheme $\mathscr{G}(\delta)$. Then $\mathscr{G}(\delta)$ also has an involution

$$\tau_0 : \mathscr{G}(\delta) \to \mathscr{G}(\delta).$$

To describe this recall that as a scheme

$$\mathscr{G}(\delta) \simeq \mathbb{G}_m \times H \times H^\wedge.$$

The map τ_0 sends (u, x, χ) to $(u, -x, \chi^{-1})$.

Definition 7.10. Let A/S be an abelian scheme and let L be a symmetric relatively ample invertible sheaf on A. A *symmetric theta level structure of type* δ (or just *symmetric theta level structure* if the reference to δ is clear) is an isomorphism of group schemes

$$\theta : \mathscr{G}(\delta) \to \mathscr{G}_{(A,L)}$$

whose restriction to $\mathbb{G}_m$ is the identity and such that the diagram

$$\begin{tikzcd} \mathscr{G}(\delta) \arrow[r, "\theta"] \arrow[d, "\tau_0"] & \mathscr{G}_{(A,L)} \arrow[d, "\tau"] \\ \mathscr{G}(\delta) \arrow[r, "\theta"] & \mathscr{G}_{(A,L)} \end{tikzcd}$$

commutes.

Exercise 7.11. Fix a type $\delta = (d_1, \ldots, d_g)$, and let $d = \prod_i d_i$. Let S be a $\mathbb{Z}[1/2d]$-scheme, let A/S be an abelian scheme, and let L be a symmetric relatively ample invertible sheaf on A such that λ_L has type δ. Show that étale locally on S there exists a symmetric theta level-structure

$$\theta : \mathscr{G}(\delta) \to \mathscr{G}_{(A,L)}.$$

8. The main construction

8.1. Fix g and $\delta = (d_1, \dots, d_g)$, and let $\mathscr{M}_{g,\delta}$ denote the stack over $\mathbb{Z}[1/2d]$ whose fiber over a $\mathbb{Z}[1/2d]$-scheme S is the group of triples (A, L, θ), where A/S is an abelian scheme of dimension g, L is a rigidifed symmetric relatively ample line bundle on A, and θ is a symmetric theta level structure.

We usually write $\epsilon : e^*L \to \mathscr{O}_S$ for the rigidification on L.

Theorem 8.2. *Assume $4|d_i$ for all i. Then $\mathscr{M}_{g,\delta}$ is representable by a quasi-projective scheme over $\mathbb{Z}[1/2d]$.*

Remark 8.3. The condition that $4|d_i$ for all i, implies that the objects of $\mathscr{M}_{g,\delta}$ admit no nontrivial automorphisms. It suffices to consider the case when S is the spectrum of a strictly henselian local ring R. The space of global section $\Gamma(A, L)$ is then a representaton of $\mathscr{G}_{(A,L)}$ over R, and hence via σ we can view this as a representation of $\mathscr{G}(\delta)$ over R. If

$$\chi : A \to A, \quad \chi^b : \chi^*L \to L \tag{8.3.1}$$

is an automorphism of the pair (A, L), then (χ, χ^b) induces an automorphism α of the $\mathscr{G}(\delta)$-representation $\Gamma(A, L)$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\chi"] \arrow[d, hook, "j"] & A \arrow[d, hook, "j"] \\ \mathbb{P}\Gamma(A, L) \arrow[r, "\mathbb{P}\alpha"] & \mathbb{P}\Gamma(A, L) \end{tikzcd} \tag{8.3.2}$$

commutes, where j is the canonical closed immersion (recall that L is relatively very ample). On the other hand, any automorphism of the $\mathscr{G}(\delta)$-representation $\Gamma(A, L) \simeq V(\delta)$ is multiplication by a scalar, and therefore the map $\mathbb{P}\alpha$ is the identity. We conclude that $\chi : A \to A$ is the identity map. This implies that $\chi^b : L \to L$ is multiplication by a scalar $u \in R^*$ such that the diagram

$$\begin{tikzcd} e^*L \arrow[rr, "\cdot u"] \arrow[dr, "\epsilon"'] & & e^*L \arrow[dl, "\epsilon"] \\ & R & \end{tikzcd} \tag{8.3.3}$$

commutes. It follows that $u = 1$.

In what follows we therefore also write $\mathscr{M}_{g,\delta}$ for the functor obtained by passing to isomorphism classes.

Remark 8.4. Note that $\mathscr{M}_{g,\delta}$ is a finite étale covering of $\mathscr{A}_{g,\delta}$.

8.5. We will not give a proof of 8.2 (though the diligent reader should be able to fill in the details). Let us just indicate the key point.

Let $\mathscr{H}$ denote the Hilbert scheme classifying subschemes of $\mathbb{P}(V(\delta))$ with Hilbert polynomial dn^g, and let

$$\begin{tikzcd} \mathscr{Z} \arrow[r, hook] \arrow[d] & \mathbb{P}(V(\delta))_{\mathscr{H}} \arrow[dl] \\ \mathscr{H} & \end{tikzcd}$$

be the universal family. There is a natural transformation of functors

$$\gamma : \mathscr{M}_{g,\delta} \to \mathscr{X}$$

defined as follows. Note that $\mathscr{X}$ represents the functor which to any scheme S associates the set of pairs (Z, σ), where Z is a flat closed subscheme of $\mathbb{P}(V(\delta))_S$ with Hilbert polynomial dn^g, and $\sigma : S \to Z$ is a section.

We get such a pair by noting that if $(A, L, \theta) \in \mathscr{M}_{g,\delta}(S)$ is an element and $f : A \to S$ is the structure morphism, then there exists a canonical isomorphism

$$\mathbb{P}(f_*L) \simeq \mathbb{P}(V(\delta))_S$$

since f_*L is a representation of $\mathscr{G}(\delta)$ (using θ) of rank d. From the triple (A, L, θ) we therefore get an embedding

$$A \hookrightarrow \mathbb{P}(V(\delta))_S,$$

and this closed subscheme also comes with a point (the identity section). This defines the transformation γ.

We leave it as an exercise to show that this map γ is an immersion.

8.6. Finally let us mention another result of Mumford. If L is a symmetric invertible sheaf on an abelian scheme, and if $\sigma : \iota^*L \to L$ is the canonical isomorphism which is the identity over the zero section, then since the 2-torsion $A[2]$ is invariant under ι we obtain an automorphism

$$L|_{A[2]} \to L|_{A[2]}.$$

We say that L is *totally symmetric* if this automorphism is the identity. One can show that this is an open and closed condition. Let

$$\mathscr{M}_{g,\delta}^{\mathrm{tot}} \subset \mathscr{M}_{g,\delta}$$

denote the open and closed subfunctor classifying totally symmetric line bundles.

What Mumford shows among other things is the following [3]:

Theorem 8.7. *The composite morphism*

$$\mathscr{M}_{g,\delta}^{\mathrm{tot}} \xrightarrow{\gamma} \mathscr{X} \subset \mathbb{P}(V(\delta))_{\mathscr{H}} \xrightarrow{\mathrm{proj}} \mathbb{P}(V(\delta))_{\mathbb{Z}[1/2d]}$$

is an immersion if $8|d_i$ *for all* i.

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