F–ISOCRYSTALS AND HOMOTOPY TYPES

(F–ISOCRISTAUX ET TYPES D'HOMOTOPIE)

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Abstract. We study a positive characteristic analog of the nonabelian Hodge structure constructed by Katzarkov, Pantev, and Toen on the homotopy type of a complex algebraic variety. Given a proper smooth scheme X over a perfect field of characteristic p and a Tannakian category $\mathcal{C}$ of isocrystals on X, we construct an object $X_{\mathcal{C}}$ in a suitable homotopy category of simplicial presheaves whose category of local systems is equivalent to $\mathcal{C}$ in a manner compatible with cohomology. We then study F–isocrystal structure on these simplicial presheaves. As applications of the theory, we prove a p–adic analog of a result of Hain on relative Malcev completions, a generalization to the level of homotopy types of a theorem of Katz relating p–adic étale local systems and F–isocrystals, as well as a p–adic version of the formality theorem in homotopy theory. We have also included a new proof based on reduction modulo p of the formality theorem for complex algebraic varieties.

1. Introduction

1.1. One of the most powerful tools used in arithmetic geometry is the F–crystal structure on the cohomology of a smooth proper scheme over a perfect field k of positive characteristic. This structure is often viewed as an arithmetic analogue of the Hodge structure on the cohomology of a variety over $\mathbb{C}$. Hodge theory, however, extends well beyond the abelian theory of cohomology to the nonabelian theory of homotopy types and homotopy groups, thanks to work of Sullivan ([Su]), Morgan ([Mo]), Hain ([Ha3], [Ha4]), and more recently Simpson ([Si1], [Si2]), Katzarkov, Pantev, and Toen ([KPT]) among others. Our investigation in this paper is an attempt to define and study a crystalline analog of the Hodge structure on the homotopy type of a smooth proper variety over $\mathbb{C}$ ([KPT]).

1.2. Let k be a perfect field of characteristic $p > 0$, W its ring of Witt vectors, and K the field of fractions of W. Let X/k be a smooth proper scheme with a point $x \in X(k)$. It is well known ([Og1], 4.1) that the category $\text{Isoc}(X/K)$ of isocrystals on X is a Tannakian category with fiber functor

$$\omega_x : \text{Isoc}(X/K) \longrightarrow \text{Vec}_K, \quad \mathcal{V} \mapsto x^*\mathcal{V}. \tag{1.2.1}$$

Moreover, for any sub–Tannakian category $\mathcal{C} \subset \text{Isoc}(X/K)$, we can define its Galois group $\pi_1(\mathcal{C}, \omega_x)$ as the automorphism group of the fiber functor $\omega_x|_{\mathcal{C}}$. The first main goal of this paper is to associate to $\mathcal{C}$ a "homotopy type" encoding the group $\pi_1(\mathcal{C}, \omega_x)$ as well as the cohomological theory of isocrystals in $\mathcal{C}$.

1.3. For this we use the theory of simplicial presheaves developed in ([Bl], [H-S], [Ja], [To1]). Let $\text{SPr}(K)$ denote the category of simplicial presheaves on the site Aff_K of affine K–schemes with the fpqc topology (in what follows follows we often abuse notation and denote an object

$\mathrm{Spec}(R) \in \mathrm{Aff}_K$ simply by R). That is, $\mathrm{SPr}(K)$ is the category of contravariant functors $F : \mathrm{Aff}_K \to \mathrm{SSet}$ from Aff_K to the category of simplicial sets.

For an object $F \in \mathrm{SPr}(K)$ with a point $* \to F(R)$ over some $R \in \mathrm{Aff}_K$ and integer $i > 0$ there is an associated sheaf $\pi_i(F, *)$ on Aff_R defined to be the sheaf associated to the presheaf $R' \mapsto \pi_i(|(F(R')|, *)$, where $|F(R')|$ denotes the geometric realization of the simplicial set $F(R')$. There is also a sheaf $\pi_0(F)$ defined to be the sheaf associated to the presheaf $R \mapsto \pi_0(|F(R)|)$.

By a theorem of Jardine (see ([Bl], 1.3)) there is a simplicial closed model category structure on $\mathrm{SPr}(K)$ such that a morphism $F \to F'$ is a cofibration if for every $R \in \mathrm{Aff}_K$ the map $F(R) \to F'(R)$ is a monomorphism (that is, a cofibration in the category of simplicial sets with the usual model category structure ([G-J], I.11.3)) and an equivalence if the map of sheaves $\pi_0(F) \to \pi_0(F')$ is an isomorphism and for every point $* \to F(R)$ the induced map of sheaves $\pi_i(F, *) \to \pi_i(F', *)$ on Aff_R is an isomorphism. Fibrations are defined by the right lifting property with respect to trivial cofibrations.

We denote by $\mathrm{Ho}(\mathrm{SPr}(K))$ the associated homotopy category and call the objects *stacks*. A stack F is called *connected* if $\pi_0(F) \simeq \{*\}$. We will also consider the category, denoted $\mathrm{SPr}_*(K)$, of pointed objects in $\mathrm{SPr}(K)$. The model category structure on $\mathrm{SPr}(K)$ induces a model category structure on $\mathrm{SPr}_*(K)$, and we refer to the objects of $\mathrm{Ho}(\mathrm{SPr}_*(K))$ as *pointed stacks.*

In section 2, we prove the following (see (2.26) for the definition of cohomology groups of a stack):

Theorem 1.4. *For any Tannakian subcategory $\mathcal{C} \subset \mathrm{Isoc}(X/K)$, which is closed under extensions, there is a natural associated pointed stack $X_{\mathcal{C}}$ over K for which:*

(i) *There is a natural isomorphism $\pi_1(X_{\mathcal{C}}) \simeq \pi_1(\mathcal{C}, \omega_x)$.*

(ii) *For any object $\mathcal{V} \in \mathrm{Isoc}(X/K)$ corresponding to a representation $\rho : \pi_1(X_{\mathcal{C}}) \to \mathrm{Aut}(V)$, there is a natural isomorphism*

$$H^*(X_{\mathcal{C}}, (V, \rho)) \simeq H^*_{\mathrm{cris}}(X, \mathcal{V}), \tag{1.4.1}$$

where $H^*(X_{\mathcal{C}}, (V, \rho))$ denotes the cohomology of the local system (V, ρ).

Remark 1.5. It follows from the construction of $X_{\mathcal{C}}$ that it is a schematic homotopy type in the sense of ([To1]). However, since we give direct constructions of all stacks used in this paper we do not use this terminology.

1.6. If pullback by Frobenius induces an auto–equivalence on $\mathcal{C}$ (for example if $\mathcal{C}$ is the category of all isocrystals on X/K), then the pointed stack $X_{\mathcal{C}}$ possesses a natural F–isocrystal structure; that is, an isomorphism $\varphi : X_{\mathcal{C}} \longrightarrow X_{\mathcal{C}} \otimes_{K,\sigma} K$. This action of Frobenius on $X_{\mathcal{C}}$ unifies both the action of Frobenius on $\pi_1(\mathcal{C}, \omega_x)$ and the action on cohomology of F–isocrystals in $\mathcal{C}$ and is the topic of section 3. It is the crystalline analog of the action of $\mathbb{C}^*$ on the pointed stack $(X \otimes \mathbb{C})^{sch}$ associated to a smooth complete complex algebraic variety $X/\mathbb{C}$ in ([KPT]).

The main issue studied in section 3 is to what extent this F–isocrystal action on $X_{\mathcal{C}}$ is "continuous". As has been pointed out to us by Toen the Frobenius structure on $X_{\mathcal{C}}$ is in general only an F–isocrystal structure in a weak sense. This is perhaps best understood by analogy with Hodge theory. The point in the complex situation is that in general the action of $\mathbb{C}^*$ on $(X\otimes\mathbb{C})^{sch}$ is not continuous, and is just an action of $\mathbb{C}^*$ as a discrete group. To give a continuous action of $\mathbb{C}^*$ on a pointed stack F means by definition to give an action of $\mathbb{G}_m$ on F. To give such a pointed stack is in turn equivalent to giving a pointed stack $[F/\mathbb{G}_m]$ over $B\mathbb{G}_m$.

In the case of isocrystals, the Tannakian category of F–isocrystals $\mathrm{FIsoc}(k/K)$ is not neutral, and so the analog of having a continuous action of $\mathbb{C}^*$ on a stack F is that F is a stack over the gerbe of fiber functors for $\mathrm{FIsoc}(k/K)$. We show in section 3 that if $\{(\mathcal{V}_i,\varphi_i)\}$ is a collection of F–isocrystals and $\mathcal{C}$ denotes the smallest Tannakian subcategory of $\mathrm{Isoc}(X/K)$ closed under extensions and containing the $\mathcal{V}_i$, then in fact the stacks $X_{\mathcal{C}}$ are naturally stacks over the gerbe of fiber functors for $\mathrm{FIsoc}(k/K)$.

More precisely, the gerbe $\mathcal{G}$ of fiber functors for the category $\mathrm{FIsoc}(k/K)$ corresponds in a natural way to a stack $B\mathcal{G}\in\mathrm{Ho}(\mathrm{SPr}(\mathbb{Q}_p))$. The statement that $X_{\mathcal{C}}$ admits the structure of a pointed stack over $\mathcal{G}$ is equivalent to saying that there is an object $F^0_{\mathcal{C}}\in\mathrm{Ho}(\mathrm{SPr}(\mathbb{Q}_p)|_{B\mathcal{G}})$ such that $X_{\mathcal{C}}$ with its F–isocrystal structure is obtained as the homotopy fiber product of the diagram

(1.6.1)
$$\begin{array}{ccc} & & F^0_{\mathcal{C}} \\ & & \big\downarrow \\ \mathrm{Spec}(K) & \xrightarrow{\omega_0} & B\mathcal{G}, \end{array}$$

where ω_0 denotes the fiber functor with values in Vec_K which forgets the F–isocrystal structure.

The construction of $F^0_{\mathcal{C}}$ also leads to a description of F–isocrystals in terms of local systems on stacks:

Theorem 1.7 (3.63)**.** *With $\mathcal{C}$ as above, there is a natural equivalence between the category of local systems on $F^0_{\mathcal{C}}$ and the category of F–isocrystals $(\mathcal{V},\varphi)$ on X/K with $\mathcal{V}\in\mathcal{C}$.*

1.8. In section 4 we turn to applications of the general theory.

First we explain how the techniques of this paper give rise to restrictions on fundamental groups. Let k be a finite field of characteristic $p>0$. Denote by W the ring of Witt vectors of k and let K be the field of fractions of W. Suppose X/k is a smooth proper variety with a rational point $x\in X(k)$, and that $(\mathcal{V},\varphi)$ is a ι–pure (in the sense of ([Ke], 5.1)) F–isocrystal on X/K, for some embedding $\iota: K\hookrightarrow\mathbb{C}$. Let G denote the reductive completion of $\pi_1(\langle\mathcal{V}\rangle_\otimes,\omega_x)$, where $\langle\mathcal{V}\rangle_\otimes\subset\mathrm{Isoc}(X/K)$ denotes the tensor category generated by $\mathcal{V}$, and let $\mathcal{C}$ denote the smallest full Tannakian subcategory of $\mathrm{Isoc}(X/K)$ closed under extensions and containing $\mathcal{V}$. Set $\widetilde{G}:=\pi_1(\mathcal{C},\omega_x)$ (the *relative Malcev completion*) and define

(1.8.1)
$$\mathcal{U}:=\mathrm{Ker}(\widetilde{G}\longrightarrow G).$$

The group $\mathcal{U}$ is a pro–unipotent group scheme, and we denote by $\mathfrak{u}$ its Lie algebra. The following is a p–adic version of a result of Hain ([Ha1], 13.4) for variations of Hodge structure.

Theorem 1.9 (4.2)**.** *The Lie algebra $\mathfrak{u}$ admits a quadratic presentation.*

Secondly, we generalize in section 4 the classical equivalence of categories between unit–root F–isocrystals and étale local systems of $\mathbb{Q}_p$–vector spaces ([Cr1], 2.1). Let k be a separably closed field of characteristic $p > 0$ and $(X, x)/k$ a smooth proper pointed scheme. As explained by Toen in ([To1]), one can associate to X a pointed stack X_{et} whose category of local systems is naturally equivalent to the category of $\mathbb{Q}_p$–local systems on X_{et}, and whose cohomology groups compute étale cohomology. In (4.19) we show that if one takes $\mathcal{C}$ to be the smallest Tannakian category closed under extensions and containing the underlying isocrystals of unit–root F–isocrystals (denote the associated pointed stack by X_{ur}), then one can recover X_{et} from X_{ur} by taking the "slope zero part" of X_{ur} in a suitable sense. A consequence of this result is that for every $i \geq 1$ one can recover the group $\pi_i(X_{\text{et}})$ from $\pi_i(X_{\text{ur}})$ with its F–isocrystal structure. For example (see (4.19)–(4.22) for stronger and variant assertions):

Theorem 1.10. *For each $i \geq 1$ there is a natural isomorphism*

$$\text{Lie}(\pi_i(X_{\text{et}})) \simeq \text{Lie}(\pi_i(X_{\text{ur}}))^{\varphi=1}. \tag{1.10.1}$$

Finally we prove a positive characteristic version of the formality theorem ([DGMS], section 6) and its generalization to non–rational homotopy theory ([KPT]). Roughly, our result can be stated as follows. Let k be a finite field, $(X, x)/k$ a smooth proper pointed scheme, and $\mathcal{C}$ a Tannakian category of isocrystals associated to a family of F–isocrystals $\{(\mathcal{V}_i, \varphi_i)\}$ as in (1.6). Assume further that the $(\mathcal{V}_i, \varphi_i)$ are ι–pure in the sense of ([Ke], 5.1) for some embedding $\iota : \text{Frac}(W(k)) \hookrightarrow \mathbb{C}$. Let G be the pro–reductive completion of $\pi_1(\mathcal{C}, \omega_x)$, and let $\mathcal{O}_G$ be its coordinate ring. Right translation defines a left action of G on $\mathcal{O}_G$ which by the Tannakian formalism corresponds to an ind–isocrystal $\mathbb{L}(\mathcal{O}_G)$ on X/K. Furthermore, left translation induces a right action of G on $\mathbb{L}(\mathcal{O}_G)$.

Theorem 1.11 (4.25)**.** *The pointed stack $X_{\mathcal{C}}$ is determined by the ring $H^*(X/K, \mathbb{L}(\mathcal{O}_G))$ with its natural G–action.*

Corollary 1.12. *The group scheme $\pi_1(\mathcal{C}, \omega_x)$ is determined by the ring $H^*(X/K, \mathbb{L}(\mathcal{O}_G))$ with its natural right G–action.*

Theorem (4.25) can also be applied to prove a formality theorem for certain homotopy types attached to smooth $\mathbb{Q}_p$–sheaves (4.34).

We conclude in (4.37) by explaining how one can deduce the formality theorem for rational homotopy types of complex algebraic varieties ([DGMS], §6) from our result using reduction modulo p techniques (4.37) instead of Hodge theory.

We should also note that in the case when $\mathcal{C}$ is generated by the underlying isocrystals of unit–root F–isocrystals one can construct a generalization of the de Rham–Witt complex which provides a model for $X_{\mathcal{C}}$ just as the usual de Rham–Witt complex can be viewed as a model for the rational homotopy type of X ([H-K1]). In the interest of space, however, we have not included the development of this theory in this paper.

Finally we remark that in a sequel to this paper we will use the techniques developed here to study p–adic Hodge theory on the level of homotopy types and in particular for homotopy groups ([Ol]).

1.13 (*Acknowledgements*)**.** Starting with the paper of Deligne ([De1]), a number of people have investigated the crystalline aspects of rational homotopy theory (the theory of unipotent bundles). This includes work of Chiarellotto and Le Stum ([C-LS]), Kim and Hain ([H-K1], [H-K2]), Shiho ([Sh1], [Sh2]), and Vologodsky ([Vo]) among others. Thus the main new

contribution of this paper is to extend some of the above work to a theory with non–unipotent coefficients, though we also obtain some new results about rational homotopy theory. In the complex setting, such a theory was first investigated by Hain ([Ha1]), and subsequently generalized by Katzarkov, Pantev, and Toen in ([KPT]). It is from this latter paper and its authors that we learned many of the technical ideas used here.

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1.14 (*Conventions and prerequisites*)**.** We fix a universe $\mathbb{U}$ containing the natural numbers, and another universe $\mathbb{V}$ with $\mathbb{U} \in \mathbb{V}$. For a ring R, we write Aff/R for the category of affine $\mathbb{U}$–schemes over R. We shall often consider this category as a site $(\mathrm{Aff}/R)_{\mathrm{fpqc}}$ with the faithfully flat and quasi–compact topology. We use the expression $\mathbb{U}$–set (resp. $\mathbb{U}$–group, $\mathbb{U}$–simplicial set, etc.) to mean a set (resp. group, simplicial set, etc.) in $\mathbb{U}$. The only exception is the term $\mathbb{U}$–category by which we mean a category for which the set of maps between any two objects is a $\mathbb{U}$–set. Unless stated otherwise, all schemes, sheaves, etc. are $\mathbb{U}$–schemes, $\mathbb{U}$-sheaves, etc.

We denote by Δ the standard simplicial category ([G-J], I.1). The objects of Δ are the ordered sets $[n] := \{0, \dots, n\}$ and morphisms in Δ are order preserving maps. Note that the category Δ is contained in $\mathbb{U}$.

Our reference for closed model categories is ([Ho]), and we shall assume that all model categories are $\mathbb{V}$–categories. For a model category C, we write $\mathrm{Ho}(C)$ for the associated homotopy category. If $A, B \in \mathrm{Ho}(C)$ are two objects, we write $[A, B]_{\mathrm{Ho}(C)}$ for the set of homotopy classes of maps between A and B.

We shall also need some results about Tannakian categories for which our reference is ([Sa]). For a Tannakian category C with a fiber functor ω, we denote by $\pi_1(C, \omega)$ the group scheme of automorphisms of ω. If $V \in C$ is an object, we write $\langle V \rangle_\otimes$ for the sub–Tannakian category of C generated by V. We write $\mathrm{Ind}(\mathcal{C})$ for the category of Ind–objects in $\mathcal{C}$.

The reader is assumed to be familiar with crystalline cohomology at the level of the book ([B-O]), as well as the definition and basic results about the convergent topos ([Og1]). If X/k is a scheme over a perfect field of characteristic $p > 0$, K the field of fractions of the ring of Witt vectors of k, then we write $\mathrm{Isoc}(X/K)$ for the category of isocrystals on X/K and $\mathrm{FIsoc}(X/K)$ for the category of F–isocrystals.

2. Homotopy types and isocrystals

Review of affine stacks ([To1])

Throughout this section K denotes a field of characteristic 0.

2.1. Let Alg^Δ denote the category of cosimplicial K–algebras belonging to $\mathbb{V}$. That is, the category of covariant functors from Δ to the category of $\mathbb{V}$–K–algebras. For $A \in \mathrm{Alg}^\Delta$, we denote by $N(A)$ the normalized chain complex of the cosimplicial K–module underlying A ([G-J], III.2). By ([To1], 2.1.1), the category Alg^Δ has a cofibrantly generated simplicial closed model category structure for which the fibrations are epimorphisms and equivalences are maps $f : A \to B$ for which the induced morphism

$$N(f) : N(A) \longrightarrow N(B) \tag{2.1.1}$$

is a quasi–isomorphism.

Similarly, one can define a category $\mathrm{Alg}^\Delta_{\mathbb{U}}$ of cosimplicial K–algebras belonging to $\mathbb{U}$, and $\mathrm{Alg}^\Delta_{\mathbb{U}}$ also has a cofibrantly generated simplicial closed model category structure. The model category $\mathrm{Alg}^\Delta_{\mathbb{U}}$ is a sub–model category of Alg^Δ, and the natural functor

$$\mathrm{Ho}(\mathrm{Alg}^\Delta_{\mathbb{U}}) \longrightarrow \mathrm{Ho}(\mathrm{Alg}^\Delta) \tag{2.1.2}$$

is fully faithful by ([KPT], p.11).

2.2. Let SSet denote the category of simplicial $\mathbb{V}$–sets, and let $\mathrm{SPr}(K)$ denote the simplicial model category of simplicial presheaves on Aff_K in $\mathbb{V}$ described in (1.3). There is a natural functor

$$\mathrm{Spec} : (\mathrm{Alg}^\Delta)^{\mathrm{op}} \longrightarrow \mathrm{SPr}(K), \tag{2.2.1}$$

which sends $A \in \mathrm{Alg}^\Delta$ to the functor

$$\mathrm{Spec}(A) : (\mathrm{Aff}/K)^{\mathrm{op}} \longrightarrow \mathrm{SSet}, \quad B \mapsto \underline{\mathrm{Hom}}(A, B), \tag{2.2.2}$$

where $\underline{\mathrm{Hom}}(A, B)$ denotes the simplicial set of morphisms of cosimplicial algebras $A \to B$ (B viewed as a constant cosimplicial algebra). By ([To1], 2.2.2), Spec is a right Quillen functor, and we denote its derived functor by

$$\mathbb{R}\mathrm{Spec} : \mathrm{Ho}(\mathrm{Alg}^\Delta) \longrightarrow \mathrm{Ho}(\mathrm{SPr}(K)). \tag{2.2.3}$$

Definition 2.3 ([To1], 2.2.4)**.** An *affine stack* is a stack F isomorphic in $\mathrm{Ho}(\mathrm{SPr}(K))$ to $\mathbb{R}\mathrm{Spec}(A)$, for some $A \in \mathrm{Alg}^\Delta_{\mathbb{U}}$.

2.4. We can also consider the category Alg^Δ_* of algebras $A \in \mathrm{Alg}^\Delta$ with an augmentation $A \to K$. Just as in ([Ho], 1.1.8) the model category structure on Alg^Δ induces a natural model category structure on Alg^Δ_*. As above, the functor Spec induces a functor

$$\mathbb{R}\mathrm{Spec} : \mathrm{Ho}(\mathrm{Alg}^\Delta_*) \longrightarrow \mathrm{Ho}(\mathrm{SPr}_*(K)). \tag{2.4.1}$$

We will refer to objects of $\mathrm{Ho}(\mathrm{SPr}_*(K))$ isomorphic to $\mathbb{R}\mathrm{Spec}(A)$ for $A \in \mathrm{Alg}^\Delta_{\mathbb{U}}$ as *pointed affine stacks.*

The most important property of affine stacks that we need is the following:

Theorem 2.5 ([To1], 2.4.5)**.** *Let $F \in \mathrm{Ho}(\mathrm{SPr}_*(K))$ be a connected pointed affine stack. Then for all $i \geq 1$ the sheaf $\pi_i(F)$ is representable by a pro–unipotent group scheme.*

Stacks associated to isocrystals

In this section, we review a construction of pointed stacks due to Katzarkov, Pantev, and Toen ([KPT]), and use it to associate pointed stacks to categories of isocrystals.

2.6. Let K be a field of characteristic 0, and let G/K be an affine group scheme. Denote by $\mathrm{Rep}(G)$ the category of quasi–coherent sheaves of $\mathcal{O}$–modules in $\mathbb{V}$ on $(\mathrm{Aff}/K)_{\mathrm{fpqc}}$ with a linear right action of the presheaf of groups defined by G. We will often view the category $\mathrm{Rep}(G)$ as the category of K–vector spaces V in $\mathbb{V}$ with comodule structure ([DMOS], II.2.2).

The category of *cosimplicial G–modules* is by definition the category $\mathrm{Rep}(G)^{\Delta}$ of cosimplicial objects in $\mathrm{Rep}(G)$. By ([KPT], 1.3.1), $\mathrm{Rep}(G)^{\Delta}$ has a natural structure of a simplicial closed model category. The category $\mathrm{Rep}(G)^{\Delta}$ has a natural symmetric monoidal structure given by level–wise tensor product. We denote by $G-\mathrm{Alg}^{\Delta}$ the category of commutative unital monoids in $\mathrm{Rep}(G)$, and refer to $G-\mathrm{Alg}^{\Delta}$ as the category of G–equivariant cosimplicial algebras. The category $G-\mathrm{Alg}^{\Delta}$ is also naturally viewed as the opposite category of the category of simplicial affine schemes with G–action. By ([KPT], 1.3.2), the model category structure on $\mathrm{Rep}(G)^{\Delta}$ induces a simplicial closed model category structure on $G-\mathrm{Alg}^{\Delta}$. This model category structure restricts to a simplicial closed model category structure on the full subcategory of $G-\mathrm{Alg}^{\Delta}$ consisting of cosimplicial algebras belonging to $\mathbb{U}$.

2.7. We can also define a category $G-\mathrm{SPr}(K)$ of G–equivariant simplicial presheaves. This is the category of objects in $\mathrm{SPr}(K)$ equipped with a left action of the presheaf G. It has a natural structure of a simplicial closed model category (see ([KPT], p.16)). For any $A \in G-\mathrm{Alg}^{\Delta}$, the spectrum of A defined in (2.2.2) has a natural left G–action induced by the action on A, and this enables one to define a functor

$$\mathrm{Spec}_G : G-\mathrm{Alg}^{\Delta} \longrightarrow G-\mathrm{SPr}(K), \tag{2.7.1}$$

which by ([KPT], 1.3.6) is right Quillen. We denote by

$$\mathbb{R}\mathrm{Spec}_G : \mathrm{Ho}(G-\mathrm{Alg}^{\Delta}) \longrightarrow \mathrm{Ho}(G-\mathrm{SPr}(K)) \tag{2.7.2}$$

the resulting derived functor.

2.8. For a presheaf of groups G, there is a connected stack $BG \in \mathrm{Ho}(\mathrm{SPr}_*(K))$ characterized by the condition that $\pi_i(BG) = 0$ for $i > 1$ and $\pi_1(BG) = G$ ([To1], 1.3).

If EG is a cofibrant model of $* \in G-\mathrm{SPr}(K)$, so that $BG := EG/G$ is a model for the stack $BG \in \mathrm{Ho}(\mathrm{SPr}_*(K))$. We fix such a model EG for $*$ so we have this concrete model for BG.

Consider the coma category $\mathrm{SPr}(K)/BG$ of simplicial presheaves over BG, with its natural simplicial model category structure. There is a natural functor

$$De : G-\mathrm{SPr}(K) \longrightarrow \mathrm{SPr}(K)/BG, \quad F \longmapsto (EG \times F)/G, \tag{2.8.1}$$

where G acts diagonally on $EG \times F$. By ([KPT], 1.2.1), the functor (2.8.1) is part of a Quillen equivalence (De, Mo) which induces an equivalence of categories

$$\mathbb{L}De : \mathrm{Ho}(G-\mathrm{SPr}(K)) \longrightarrow \mathrm{Ho}(\mathrm{SPr}(K)/BG). \tag{2.8.2}$$

Definition 2.9 ([KPT], 1.2.2)**.** For $F \in \mathrm{Ho}(G-\mathrm{SPr}(K))$, the *quotient stack* $[F/G]$ is the G–equivariant stack $\mathbb{L}De(F) \in \mathrm{Ho}(\mathrm{SPr}(K)/BG)$.

2.10. If G is a pro–algebraic group scheme, and if $A \in G-\mathrm{Alg}^{\Delta}_{\mathbb{U}}$ is an algebra with $H^0(A) = K$ and a K–augmentation $x : A \to K$ (not necessarily compatible with the G–action), then the stack $[\mathbb{R}\mathrm{Spec}_G(A)/G]$ has a natural structure of an object in $\mathrm{Ho}(\mathrm{SPr}_*(K))$. For this, note first that giving the augmentation x is equivalent to giving a G–equivariant map $A \to \mathcal{O}_G$. Thus

$\mathbb{R}\mathrm{Spec}_G(A)$ is naturally an object of $\mathrm{Ho}(G-\mathrm{SPr}(K)_{G/})$ of G–equivariant stacks F under the constant sheaf G (note that $\mathcal{O}_G$ is cofibrant in $G-\mathrm{Alg}^{\Delta}$). The construction of the equivalence (2.8.2) combined with the observation that $[G/G] \simeq *$ then gives an equivalence

$$\mathrm{Ho}(G-\mathrm{SPr}(K)_{G/}) \simeq \mathrm{Ho}(\mathrm{SPr}_*(K)|_{BG}). \tag{2.10.1}$$

For an algebra A as above, we will therefore usually view $[\mathbb{R}\mathrm{Spec}_G(A)/G]$ as an object in $\mathrm{Ho}(\mathrm{SPr}_*(K)|_{BG})$.

The following proposition will be important in what follows:

Proposition 2.11 ([KPT], 1.3.10). *Let $A \in G-\mathrm{Alg}^{\Delta}$ be a G–equivariant cosimplicial algebra in $\mathbb{U}$ with a K–augmentation $x : A \to K$, and suppose that $H^0(N(A)) = K$. Then the homotopy fiber of the map*

$$[\mathbb{R}\mathrm{Spec}_G(A)/G] \longrightarrow BG \tag{2.11.1}$$

is naturally isomorphic to $\mathbb{R}\mathrm{Spec}(A)$.

Corollary 2.12 ([KPT], 1.3.11). *If A is as in (2.11), then for all $i \geq 1$ the sheaf of groups $\pi_i([\mathbb{R}\mathrm{Spec}_G(A)/G])$ is representable by a pro–algebraic group scheme which is pro–unipotent if $i \geq 2$.*

Proof. This follows from (2.11) and the long–exact sequence of homotopy groups associated to a fibration combined with (2.5). □

2.13. Now let k denote a perfect field of characteristic $p > 0$, W its ring of Witt vectors, and K the field of fractions of W. Let (X, x) be a proper smooth pointed k–scheme, and let

$$\omega_x : \mathrm{Isoc}(X/K) \longrightarrow \mathrm{Vec}_K, \quad \mathcal{V} \mapsto x^*\mathcal{V} \tag{2.13.1}$$

be the fiber functor defined by x. Fix a Tannakian subcategory $\mathcal{C} \subset \mathrm{Isoc}(X/K)$, closed under extensions, and let G be the pro–reductive completion of the affine group scheme $\pi_1(\mathcal{C}, \omega_x|_{\mathcal{C}})$. We use the above construction to associate to $\mathcal{C}$ a pointed stack.

2.14. Consider first the situation when we fix an *embedding system* for X. By this we mean an étale hypercovering $U^\bullet \to X$ and an embedding $U^\bullet \hookrightarrow Y^\bullet$ of $U^\bullet$ into a simplicial p–adic formal scheme $Y^\bullet$, formally smooth over W. We further assume that a lifting of Frobenius $F_{Y^\bullet} : Y^\bullet \to Y^\bullet$ is chosen compatible with the canonical lifting to W.

Let $\mathrm{Rep}(G)_{Y^\bullet}$ denote the abelian category of sheaves taking values in the abelian category $\mathrm{Rep}(G)$ in the topos $Y^\bullet_{\mathrm{et}}$ ([L-MB], 12.4), and let $C^+(\mathrm{Rep}(G)_{Y^\bullet})$ denote the category of positively graded complexes in $\mathrm{Rep}(G)_{Y^\bullet}$. The Dold–Kan correspondence ([G-J], III.2.3) furnishes an equivalence of categories

$$D : C^+(\mathrm{Rep}(G)_{Y^\bullet}) \longrightarrow \mathrm{Rep}(G)^{\Delta}_{Y^\bullet}, \quad N : \mathrm{Rep}(G)^{\Delta}_{Y^\bullet} \longrightarrow C^+(\mathrm{Rep}(G)_{Y^\bullet}), \tag{2.14.1}$$

where $\mathrm{Rep}(G)^{\Delta}_{Y^\bullet}$ denotes the category of cosimplicial objects in $\mathrm{Rep}(G)_{Y^\bullet}$. The category of complexes $C^+(\mathrm{Rep}(G)_{Y^\bullet})$ has a natural model category structure ([Ho], 2.3) which, through the above equivalence, defines a model category structure on $\mathrm{Rep}(G)^{\Delta}_{Y^\bullet}$. This model category structure extends naturally to a simplicial model category structure ([G-J], II.5).

The category $C^+(\mathrm{Rep}(G)_{Y^\bullet})$ (resp. $\mathrm{Rep}(G)^{\Delta}_{Y^\bullet}$) has a natural symmetric monoidal structure given by tensor product, and we denote by $G-\mathrm{dga}(Y^\bullet)$ (resp. $G-\mathrm{Alg}^{\Delta}(Y^\bullet)$) the category of commutative monoids in $C^+(\mathrm{Rep}(G)_{Y^\bullet})$ (resp. $\mathrm{Rep}(G)^{\Delta}_{Y^\bullet}$). As in ([KPT], A1), there exists a unique closed model category structure on $G - \mathrm{dga}(Y^\bullet)$ (resp. $G - \mathrm{Alg}^\Delta(Y^\bullet)$) for which a morphism $f : A \to B$ is an equivalence (resp. fibration) if and only if the induced morphism in $C^+(\mathrm{Rep}(G)_{Y^\bullet})$ (resp. $\mathrm{Rep}(G)^\Delta_{Y^\bullet}$) is an equivalence (resp. fibration). By ([KPT], p. 49), the functor D induces an equivalence

$$D : \mathrm{Ho}(G - \mathrm{Alg}^\Delta(Y^\bullet)) \longrightarrow \mathrm{Ho}(G - \mathrm{dga}(Y^\bullet)), \tag{2.14.2}$$

whose inverse is the functor of *Thom–Sullivan cochains* ([H-S], 4.1)

$$Th : \mathrm{Ho}(G - \mathrm{dga}(Y^\bullet)) \longrightarrow \mathrm{Ho}(G - \mathrm{Alg}^\Delta(Y^\bullet)). \tag{2.14.3}$$

The global section functor Γ induces functors

$$\Gamma : G - \mathrm{dga}(Y^\bullet) \longrightarrow G - \mathrm{dga}, \quad \Gamma : G - \mathrm{Alg}^\Delta(Y^\bullet) \longrightarrow G - \mathrm{Alg}^\Delta, \tag{2.14.4}$$

which are right Quillen. Moreover, the induced diagram

$$\begin{array}{ccc} \mathrm{Ho}(G - \mathrm{dga}(Y^\bullet)) & \xrightarrow{D} & \mathrm{Ho}(G - \mathrm{Alg}^\Delta(Y^\bullet)) \\ {\scriptstyle \mathbb{R}\Gamma}\downarrow & & \downarrow{\scriptstyle \mathbb{R}\Gamma} \\ \mathrm{Ho}(G - \mathrm{dga}) & \xrightarrow{D} & \mathrm{Ho}(G - \mathrm{Alg}^\Delta) \end{array} \tag{2.14.5}$$

is naturally 2–commutative.

2.15. For an Ind–object $E = \varinjlim E_i$ in $\mathrm{Isoc}(X/K)$, one can defines its de Rham–complex on $Y^\bullet_{\mathrm{et}}$ as follows. Pulling the E_i back to $U^\bullet$, we obtain an Ind–object $\widetilde{E} = \varinjlim \widetilde{E}_i$ in the category of isocrystals on $U^\bullet$, and we define

$$\widetilde{E} \otimes \Omega^\bullet_{D^\bullet} := \varinjlim \widetilde{E}_i \otimes \Omega^\bullet_{D^\bullet}, \tag{2.15.1}$$

where $\widetilde{E}_i \otimes \Omega^\bullet_{D^\bullet}$ denotes the complex computing the cohomology of $\widetilde{E}_i$ defined in ([B-O], 7.23) and $D^\bullet$ denotes the p–adic completion of the divided power enveloped of $U^\bullet$ in $Y^\bullet$. We will sometimes refer to $\widetilde{E} \otimes \Omega^\bullet_{D^\bullet}$ as the *cohomology complex* of E.

2.16. Let $\mathcal{O}_G$ denote the coordinate ring of G. Right translation defines a left action of G on $\mathcal{O}_G$, and by the definition of G this action defines an Ind–object $\mathbb{L}(\mathcal{O}_G)$ in $\mathrm{Isoc}(X/K)$. Moreover, right translation defines an action of G on $\mathbb{L}(\mathcal{O}_G)$, which makes $\mathbb{L}(\mathcal{O}_G)$ a ring Ind–object with G–action in $\mathrm{Isoc}(X/K)$. Write $C^\bullet(\mathcal{O}_G)$ for the cohomology complex of $\mathbb{L}(\mathcal{O}_G)$. The ring structure on $\mathbb{L}(\mathcal{O}_G)$ makes $C^\bullet(\mathcal{O}_G)$ an object in $G - \mathrm{dga}(Y^\bullet)$.

Lemma 2.17. (i) *There is a natural isomorphism $H^0(Y^\bullet, C^\bullet(\mathcal{O}_G)) \simeq K$.*

(ii) *For any representation (V, ρ) of G corresponding to a module with integrable connection $(\mathcal{V}, \nabla_{\mathcal{V}})$, there is a natural isomorphism of Ind–isocrystals with G–action*

$$(\mathcal{V} \otimes_{\mathcal{O}_X} \mathbb{L}(\mathcal{O}_G), 1 \otimes \gamma) \simeq (V^c \otimes_K \mathbb{L}(\mathcal{O}_G), \rho^c \otimes \gamma), \tag{2.17.1}$$

where γ denotes the right action of G on $\mathbb{L}(\mathcal{O}_G)$ and (V^c, ρ^c) denotes the contragredient representation.

(iii) *The Ind–isocrystal with G–action $\mathbb{L}(\mathcal{O}_G)$ is equal to the direct limit of its coherent sub–isocrystals with right G–action.*

Proof. To prove (i), note that
(2.17.2)
$$H^0(Y^\bullet, C^\bullet(\mathcal{O}_G)) \simeq H^0_{\mathrm{cris}}(X/K, \mathbb{L}(\mathcal{O}_G)) \simeq \mathrm{Hom}(\mathcal{O}_{X/K}, \mathbb{L}(\mathcal{O}_G)) \simeq \mathrm{Hom}_{\mathrm{Rep}(G)}(K, \mathcal{O}_G).$$

Hence it suffices to show that the invariants in $\mathcal{O}_G$ under the left G–action is equal to K which follows from (2.18 (i)) below.

To prove (ii) and (iii), note that (ii) is equivalent to the statement that there is a natural isomorphism of (G, G)–bimodules

(2.17.3) $$(V \otimes \mathcal{O}_G, \rho \otimes \ell_G, 1 \otimes \gamma) \simeq (V \otimes \mathcal{O}_G, 1 \otimes \ell_G, \rho^c \otimes \gamma),$$

where ℓ_G denotes the left action of G on $\mathcal{O}_G$, and (iii) is equivalent to the statement that $\mathcal{O}_G$ is equal to the union over its finite dimensional sub–(G, G)–bimodules. Thus (ii) and (iii) also follow from the following lemma. □

Lemma 2.18. *Let H/K be a pro–algebraic group with coordinate ring $\mathcal{O}_H$.*

(i) *The vector space $\mathcal{O}_H$ viewed as a right H–module by the action coming from left translation is injective in the category of right H–representations and the invariants of $\mathcal{O}_H$ are isomorphic to K.*

(ii) *For any right representation W of H there is a natural isomorphism of (H, H)–bimodules $W^c \otimes \mathcal{O}_H \simeq \mathcal{O}_H \otimes W$. In particular, there is a natural isomorphism of left H–representations $W^c \simeq (\mathcal{O}_H \otimes W)^{r_H}$, where r_H denotes the right action.*

(iii) *The (H, H)–bimodule $\mathcal{O}_H$ is a direct limit of finite–dimensional (H, H)–bimodules.*

Proof. By ([Sa], II.2.3.2.1), for any right representation W of H, the (H, H)–bimodule $\mathcal{O}_H \otimes W$ has the following universal property: For any left representation V of H there is a natural isomorphism of right H–modules

(2.18.1) $$\mathrm{Hom}_{\text{left } H\text{–modules}}(V, \mathcal{O}_H \otimes W) \simeq V^* \otimes W.$$

Let W^c denote W with the contragredient action (a left H–module). Then by the same result there is a natural isomorphism

(2.18.2) $$\mathrm{Hom}_{\text{left } H\text{–modules}}(V, W^c \otimes \mathcal{O}_H) \simeq \mathrm{Hom}_{\text{left } H\text{–modules}}(V \otimes W^*, \mathcal{O}_H) \simeq V^* \otimes W.$$

It follows that for any W there is a natural isomorphism of (H, H)–bimodules $\mathcal{O}_H \otimes W \simeq W^c \otimes \mathcal{O}_H$. In particular, for any right H–representation W

(2.18.3) $$\mathrm{Hom}_{\text{right } H\text{–modules}}(W, \mathcal{O}_H) \simeq ((W^c)^* \otimes \mathcal{O}_H)^H \simeq (\mathcal{O}_H)^H \otimes W^*$$

It follows that for any inclusion $W \hookrightarrow W'$ the induced map

(2.18.4) $$\mathrm{Hom}_{\text{right } H\text{–modules}}(W', \mathcal{O}_H) \to \mathrm{Hom}_{\text{right } H\text{–modules}}(W, \mathcal{O}_H)$$

is surjective and hence $\mathcal{O}_H$ is injective.

To see the second statement in (i), note that a non–trivial map $K \hookrightarrow H$ would give a non–trivial map of schemes $H \to \mathbb{A}^1$ invariant under the action of H by right translation. Since the right action of H is torsorial this is impossible.

Statement (ii) also follows from the preceding discussion.

For (iii), write $H = \varprojlim H_i$, where H_i is an algebraic group over K. Then $\mathcal{O}_H = \varinjlim \mathcal{O}_{H_i}$ so it suffices to show that $\mathcal{O}_{H_i}$ is a direct limit of finite–dimensional (H_i, H_i)–submodules. This reduces the lemma to the case when H is of finite type over K.

In this case choose a faithful representation $H \hookrightarrow GL(V)$ for some finite–dimensional K–space V. Then $\mathcal{O}_H$ is a quotient of $\mathcal{O}_{GL(V)}$ viewed as a (H, H)–bimodule via the inclusion $H \subset GL(V)$. This reduces the lemma to the case $H = GL(V)$ where the result is straightforward.

□

2.19. The point

$$j : x = \mathrm{Spec}(k) \to X, \tag{2.19.1}$$

defines an augmentation $\mathbb{R}\Gamma(C^\bullet(\mathcal{O}_G)) \to K$. It follows from this that $Th(\mathbb{R}\Gamma(C^\bullet(\mathcal{O}_G)))$ is isomorphic in $G - \mathrm{Alg}^\Delta$ to a cosimplicial algebra A belonging to $\mathbb{U}$, with $H^0(A) = K$ and a natural augmentation $A \to K$.

Definition 2.20. The *pointed stack attached to* $\mathcal{C}$, denoted $X_\mathcal{C}$, is the pointed stack

$$[\mathbb{R}\mathrm{Spec}_G(\mathbb{R}\Gamma(Th(C^\bullet(\mathcal{O}_G)))) / G] \tag{2.20.1}$$

Notation 2.21. We will often write simply $\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_G))$ or $\mathbb{R}\Gamma(\mathbb{L}(\mathcal{O}_G))$ for the equivariant algebra $\mathbb{R}\Gamma(Th(C^\bullet(\mathcal{O}_G)))$.

Remark 2.22. It follows from ([KPT], 1.3.11) that $X_\mathcal{C}$ is a schematic homotopy type in the sense of ([To1]).

2.23. Of course, this definition seems to depend on the choice of the embedding system $Y^\bullet$. That it does not is seen as follows. Suppose $(V^\bullet \hookrightarrow Z^\bullet)$ is a second embedding system for X. Then $(V^\bullet \times_X U^\bullet \hookrightarrow Z^\bullet \times_W Y^\bullet)$ is also an embedding system for X, so to prove that $(V^\bullet \hookrightarrow Z^\bullet)$ and $(U^\bullet \hookrightarrow Y^\bullet)$ give rise to the same pointed stack, we may assume that there is a map $g : (V^\bullet \hookrightarrow Z^\bullet) \to (U^\bullet \hookrightarrow Y^\bullet)$ of embedding systems. Let $\widetilde{E}_Z \otimes \Omega^\bullet_{D^\bullet}$ (resp. $\widetilde{E}_Y \otimes \Omega^\bullet_{D^\bullet}$) denote the complex (2.15.1) for $(V^\bullet \hookrightarrow Z^\bullet)$ (resp. $(U^\bullet \hookrightarrow Y^\bullet)$). The natural functor

$$g_* : G - \mathrm{dga}(Z^\bullet) \longrightarrow G - \mathrm{dga}(Y^\bullet) \tag{2.23.1}$$

is right Quillen, and there is a natural map

$$\widetilde{E}_Y \otimes \Omega^\bullet_{D^\bullet} \longrightarrow \mathbb{R}g_*(\widetilde{E}_Z \otimes \Omega^\bullet_{D^\bullet}). \tag{2.23.2}$$

This map is an equivalence, because this can be verified after forgetting the algebra structure and the action of G, in which case it follows from cohomological descent (see for example the proof of ([B-O], 7.8)). We conclude that up to canonical isomorphism $X_\mathcal{C}$ does not depend on the choice of the embedding system.

Remark 2.24. The fact that G is reductive was never used in the construction of $X_\mathcal{C}$. If G is the Tannaka dual of any sub–Tannakian category $\mathcal{C} \subset \mathrm{Isoc}$, the above construction yields a pointed stack

$$[\mathbb{R}\mathrm{Spec}_G(\mathbb{R}\Gamma(Th(C^\bullet(\mathcal{O}_G)))) / G]. \tag{2.24.1}$$

This remark will be useful for technical reasons in what follows.

Local systems and cohomology

2.25. There is a notion of a local system on any pointed stack $F \in \mathrm{Ho}(\mathrm{SPr}_*(K))$ ([To1], 1.3.1). By definition, a local system on F, is a sheaf of abelian groups M on $(\mathrm{Aff}/K)_{\mathrm{fpqc}}$ together with an action of the sheaf of groups $\pi_1(F)$. We shall henceforth assume that $\pi_1(F)$ is represented by a pro–algebraic group scheme. In this case, $\pi_1(F)$ is determined by its category of linear representations (by Tannaka duality), and we refer to the associated local systems as *coherent local systems*. That is, a coherent local system on F is a coherent sheaf M on $(\mathrm{Aff}/K)_{\mathrm{fpqc}}$ with a linear action of the group scheme $\pi_1(F)$.

2.26. For a coherent local system M on F, Toen defines cohomology groups as follows (see ([To1],1.3) for the case of general F). There is a natural map in $\mathrm{Ho}(\mathrm{SPr}_*(K))$

$$F \longrightarrow \tau_{\leq 1} F \simeq B(\pi_1(F)), \tag{2.26.1}$$

and hence we can view F as an object in $\mathrm{Ho}(\mathrm{SPr}(K)/B\pi_1(F))$. Also, the action of $\pi_1(F)$ on M enables us to view $K(M,m)$ as an object in $\mathrm{Ho}(\pi_1(F) - \mathrm{SPr}(K))$ which by (2.8) corresponds to an object $K(\pi_1(F), M, m) \in \mathrm{Ho}(\mathrm{SPr}(K)/B\pi_1(F))$, and by definition

$$H^m(F, M) := \pi_0(\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{Ho}(\mathrm{SPr}(K)/B\pi_1(F))}(F, K(\pi_1(F), M, m))), \tag{2.26.2}$$

where $\underline{\mathrm{Hom}}$ denotes "simplicial hom."

The main result of this section is the following:

Theorem 2.27. *Let X/k be as in* (2.13)*, and let $\mathcal{C} \subset \mathrm{Isoc}(X/K)$ be a Tannakian subcategory closed under extensions.*

(i) *There is a natural isomorphism $\pi_1(X_{\mathcal{C}}) \simeq \pi_1(\mathcal{C}, \omega_x)$.*

(ii) *If (V, ρ) is a representation of $\pi_1(X_{\mathcal{C}})$ giving rise via* (i) *to an isocrystal $\mathcal{V} \in \mathcal{C}$, then there is a natural isomorphism*

$$H^m_{\mathrm{cris}}(X, \mathcal{V}) \simeq H^m(X_{\mathcal{C}}, (V, \rho)). \tag{2.27.1}$$

The proof of the theorem occupies the remainder of this section.

2.28. For technical reasons, it will be useful to consider a second pointed pointed stack, which will be denoted $\widetilde{X}$. This is the stack obtained by replacing the group G by the group $\widetilde{G} := \pi_1(\mathrm{Isoc}(X/K), \omega_x)$ everywhere in the construction (2.15) (see remark (2.24)). It follows from the construction that there is a natural commutative diagram of pointed stacks

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\pi}} & B\widetilde{G} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle r} \\ X_{\mathcal{C}} & \xrightarrow{\pi} & BG. \end{array} \tag{2.28.1}$$

Moreover, by (2.11) the homotopy fibers of $\tilde{\pi}$ and π are affine stacks.

Lemma 2.29. *There is a natural isomorphism $\pi_1(\widetilde{X}) \simeq \widetilde{G}$.*

Proof. Choose an embedding system $U^\bullet \hookrightarrow Y^\bullet$ for X, and let $C^\bullet(\mathcal{O}_{\widetilde{G}})$ be the complex on $Y^\bullet_{\mathrm{et}}$ obtained as in (2.15). By (2.11), the homotopy fiber of $\tilde{\pi}$ is isomorphic to $F := \mathbb{R}\mathrm{Spec}(\mathbb{R}\Gamma(Th(C^\bullet(\mathcal{O}_{\widetilde{G}}))))$, so by the long exact sequence of homotopy groups, it suffices to show that $\pi_1(F) = 0$. By (2.5), the fundamental group of F is a unipotent group scheme, so it suffices to show that

$$\text{Hom}(\pi_1(F), \mathbb{G}_a) \simeq \pi_0(\mathbb{R}\underline{\text{Hom}}_*(F, K(\mathbb{G}_a, 1)) \tag{2.29.1}$$

is zero. By ([To1], 2.2.6), this group is equal to $H^1(\mathbb{R}\Gamma(C^\bullet(\mathcal{O}_{\widetilde{G}})))$, which in turn is equal to

$$\varinjlim H^1_{\text{cris}}(X, \mathcal{E}_i) \simeq \varinjlim \text{Ext}^1(\mathcal{O}_{X/K}, \mathcal{E}_i), \tag{2.29.2}$$

where the direct limit is taken over sub–isocrystals $\mathcal{E}_i \subset \mathbb{L}(\mathcal{O}_{\widetilde{G}})$ corresponding to subrepresentations $E_i \subset \mathcal{O}_G$ of finite type and $\mathcal{O}_{X/K}$ denotes the structure isocrystal. On the other hand, there are natural isomorphisms

$$\varinjlim \text{Ext}^1(\mathcal{O}_{X/K}, \mathcal{E}_i) \simeq \varinjlim \text{Ext}^1_{\text{Rep}(\widetilde{G})}(K, \mathcal{E}_i) \simeq \text{Ext}^1_{\text{Rep}(\widetilde{G})}(K, \mathcal{O}_{\widetilde{G}}). \tag{2.29.3}$$

Hence the vanishing of this cohomology group follows from the fact that $\mathcal{O}_{\widetilde{G}}$ is injective in the category $\text{Rep}(\widetilde{G})$ (2.18 (i)). □

Note that since $\mathcal{C} \subset \text{Isoc}(X/K)$ is a Tannakian subcategory, there is a natural surjection $\widetilde{G} \to \pi_1(\mathcal{C}, \omega_x)$.

Lemma 2.30. (i) *The proreductive completion of $\pi_1(X_{\mathcal{C}})$ is naturally isomorphic to G.*

(ii) *The map $f_* : \widetilde{G} = \pi_1(\widetilde{X}) \to \pi_1(X_{\mathcal{C}})$ factors through $\pi_1(\mathcal{C}, \omega_x)$.*

Proof. By (2.11) and (2.17 (i)), the homotopy fiber of the map π is a connected affine stack. Since the homotopy groups of affine stacks are unipotent group schemes (2.5), the long exact sequence of homotopy groups associated to π shows that there is a natural surjection $\pi_1(X_{\mathcal{C}}) \to G$ whose kernel is a unipotent group scheme. By the universal property of the pro–reductive completion $\pi_1(X_{\mathcal{C}})^{\text{red}}$, we obtain a surjection $\pi_1(X_{\mathcal{C}})^{\text{red}} \to G$ whose kernel K is a quotient of a unipotent group, hence unipotent. Since $\pi_1(X_{\mathcal{C}})^{\text{red}}$ is reductive this implies that $K = 0$ and (i) follows.

To see (ii), note that (i) implies that any $V \in \text{Rep}(\pi_1(X_{\mathcal{C}}))$ admits a finite filtration whose associated graded pieces lie in $\text{Rep}(G) \subset \text{Rep}(\pi_1(X_{\mathcal{C}}))$. Since the category $\mathcal{C}$ is closed under extensions, this implies (ii). □

2.31. In order to understand the cohomology of local systems on $X_{\mathcal{C}}$, we need to understand better the equivariant stacks $K(V, m)$ associated to representations V of G.

Let $K_V[m]$ denote the complex with zero differentials which in degree 0 is equal to K and in degree m is equal to V^*. The left action of G on V induces a right action of G on $K_V[m]$. The Dold–Kan correspondence ([G-J], III.2.3) associates to $K_V[m]$ a simplicial K–module which we denote by S^m_V. Taking the symmetric algebra on S^m_V, we obtain a cosimplicial algebra which we denote by $S_V(m)$. Note that the natural morphism of complexes $K_V[m] \to K$ (where K denotes the complex which is equal to K in degree 0 and 0 elsewhere) induces a morphism of cosimplicial algebras $S_V(m) \to K$. Also, the action of G on V induces a right action of G on $S_V(m)$, and hence $S_V(m)$ is naturally viewed as a connected object in $G-\text{Alg}^\Delta$ with an augmentation to K. Note that for any $A \in G - \text{Alg}^\Delta$ we have

$$[S_V(m), A]_{\text{Ho}(G-\text{Alg}^\Delta)} \simeq [V^*[m], N(A)]_{\text{Ho}(C^+(\text{Rep}(G)))}. \tag{2.31.1}$$

Lemma 2.32. *The quotient stack* $[\mathbb{R}\mathrm{Spec}(S_V(m))/G] \in \mathrm{Ho}(\mathrm{SPr}_*(K)/BG)$ *is isomorphic to* $K(G,V,m)$.

Proof. This follows from the same reasoning as in ([To1], 2.2.5). □

Corollary 2.33. *If* $A \in G-\mathrm{Alg}^\Delta$ *is a* G*–equivariant cosimplicial algebra in* $\mathbb{U}$ *with augmentation* $A \to K$, *and if* (V,ρ) *is a representation of* G *viewed as a representation of* $\pi_1([\mathbb{R}\mathrm{Spec}(A)/G])$, *then there is a natural isomorphism*

$$H^m([\mathbb{R}\mathrm{Spec}(A)/G],(V,\rho)) \simeq H^m(G, V^c \otimes N(A)). \tag{2.33.1}$$

where $N(A)$ *denotes the normalized complex associated to* A, V^c *is the contragredient representation, and the right hand side of* (2.33.1) *denotes group cohomology.*

Proof. By ([KPT], 1.3.8) The functor $[\mathbb{R}\mathrm{Spec}(-)/G]$ is fully faithful when restricted to objects of $\mathrm{Ho}(G-\mathrm{Alg}^\Delta)$ isomorphic to algebras in $\mathbb{U}$. Therefore by the isomorphism (2.31.1) there is a natural isomorphism

$$H^m([\mathbb{R}\mathrm{Spec}(A)/G],(V,\rho)) \simeq [V^*[m], N(A)]_{\mathrm{Ho}(C^+(\mathrm{Rep}(G)))}, \tag{2.33.2}$$

where $C^+(\mathrm{Rep}(G))$ denotes the category of complexes in $\mathrm{Rep}(G)$ with support in positive degrees. The model category structure on $C^+(\mathrm{Rep}(G))$ is that described in ([Ho], 2.3), and hence the homotopy category $\mathrm{Ho}(C^+(\mathrm{Rep}(G))$ is simply the derived category of objects in $\mathrm{Rep}(G)$. The result follows. □

Proposition 2.34. (i) *For any representation* V *of* $\widetilde{G}$ *with associated isocrystal* $\mathcal{V}$, *there is a natural isomorphism*

$$H^*_{\mathrm{cris}}(X,\mathcal{V}) \simeq H^*(\widetilde{X}, V). \tag{2.34.1}$$

(ii) *For any representation* V *of* G, *corresponding to an isocrystal* $\mathcal{V}$ *by Tannaka duality, there is a natural isomorphism*

$$H^*_{\mathrm{cris}}(X,\mathcal{V}) \simeq H^*(X_{\mathcal{C}}, V). \tag{2.34.2}$$

Proof. The proof of (ii) is the same as the proof of (i) (except slightly simpler since G is reductive), so we just prove (i).

Choose an embedding system $Y^\bullet$ as in (2.14), and let $C^\bullet(\mathcal{O}_{\widetilde{G}})$ denote the cohomology complex of $\mathbb{L}(\mathcal{O}_{\widetilde{G}})$ on $Y^\bullet_{\mathrm{et}}$. Combining (2.33) with the construction of $X_{\mathcal{C}}$, we see that the right hand side of (2.34.1) is isomorphic to the cohomology of $\mathbb{R}I(C^\bullet(\mathcal{O}_{\widetilde{G}}) \otimes_K V^c)$, where

$$I : \mathrm{Rep}(\widetilde{G})_{Y^\bullet} \longrightarrow \mathrm{Vec}_K \tag{2.34.3}$$

is the composite of the global section functor Γ with the functor

$$\pi : \mathrm{Rep}(\widetilde{G}) \longrightarrow \mathrm{Vec}_K, \quad V \mapsto V^{\widetilde{G}}. \tag{2.34.4}$$

Therefore the group $H^m(\widetilde{X}, V)$ is isomorphic to

$$\mathbb{R}^m\pi(\mathbb{R}\Gamma(C^*(\mathcal{O}_{\widetilde{G}})) \otimes_K V^c). \tag{2.34.5}$$

Let $C^*(\mathcal{V}\otimes\mathcal{O}_{\widetilde{G}})$ denote the cohomology complex of the Ind–isocrystal $\mathcal{V}\otimes\mathbb{L}(\mathcal{O}_{\widetilde{G}})$. By (2.17 (ii)), there is a natural isomorphism in the derived category of $\mathrm{Rep}(\widetilde{G})$

$$\mathbb{R}\Gamma(C^*(\mathcal{O}_{\widetilde{G}})) \otimes_K V^c \simeq \mathbb{R}\Gamma(C^*(\mathcal{V} \otimes \mathcal{O}_{\widetilde{G}})). \tag{2.34.6}$$

The map $\mathcal{V} \to \mathcal{V} \otimes \mathbb{L}(\mathcal{O}_{\widetilde{G}})$ then induces a map

$$\mathbb{R}\Gamma_{\text{cris}}(\mathcal{V}) \longrightarrow \mathbb{R}\pi\mathbb{R}\Gamma(C^*(\mathcal{O}_{\widetilde{G}}) \otimes V). \tag{2.34.7}$$

To prove (2.34 (i)) it therefore suffices to prove the following lemma: □

Lemma 2.35. *The map* (2.34.7) *is a quasi–isomorphism.*

Proof. Let $\mathcal{S}_{\widetilde{G}}$ denote the abelian category of pairs $(\mathcal{F}, \rho)$, where $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X/K}$–modules in $(X/K)_{\text{conv}}$ and $\rho : \mathcal{F} \to \mathcal{F} \otimes_K \mathcal{O}_{\widetilde{G}}$ is a morphism of sheaves such that for every enlargement T the induced morphism $\mathcal{F}(T) \to \mathcal{F}(T) \otimes_K \mathcal{O}_{\widetilde{G}}$ defines a comodule structure on $\mathcal{F}(T)$. Equivalently $\mathcal{S}_{\widetilde{G}}$ is the category of sheaves of $\mathcal{O}_{X/K}$–modules on $(X/K)_{\text{conv}}$ with algebraic action of the sheaf of groups $\widetilde{G}$.

Denote by $\mathcal{S}_{\mathcal{O}_{X/K}}$ the category of sheaves of $\mathcal{O}_{X/K}$–modules. There is a natural forgetful functor

$$\mathcal{S}_{\widetilde{G}} \longrightarrow \mathcal{S}_{\mathcal{O}_{X/K}}, \quad E \mapsto E^f. \tag{2.35.1}$$

This functor has a right adjoint sending $\mathcal{V}$ to $\mathcal{V} \otimes_K \mathcal{O}_{\widetilde{G}}$ with comodule structure

$$\mathcal{V} \otimes_K \mathcal{O}_{\widetilde{G}} \xrightarrow{1 \otimes \Delta} \mathcal{V} \otimes_K \mathcal{O}_{\widetilde{G}} \otimes_K \mathcal{O}_{\widetilde{G}}, \tag{2.35.2}$$

where Δ is the map giving the multiplication on $\widetilde{G}$. This shows in particular that any object $E \in \mathcal{S}_{\widetilde{G}}$ admits an inclusion $E \hookrightarrow I$ where I is injective in $\mathcal{S}_{\widetilde{G}}$ and I^f is a direct limit of injectives in $\mathcal{S}_{\mathcal{O}_{X/K}}$. Namely, choose an inclusion $E^f \hookrightarrow J$ with J injective in $\mathcal{S}_{\mathcal{O}_{X/K}}$ and set $I = J \otimes_K \mathcal{O}_{\widetilde{G}}$.

There is a commutative diagram of functors between abelian categories with enough injectives

$$\begin{array}{ccc} \mathcal{S}_{\widetilde{G}} & \xrightarrow{u_{\widetilde{G}*}} & (\text{sheaves of } K\text{–spaces on } X_{\text{et}} \text{ with linear } \widetilde{G}\text{–action}) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{S}_{\mathcal{O}_{X/K}} & \xrightarrow{u_*} & (\text{sheaves of } K\text{–spaces on } X_{\text{et}}), \end{array} \tag{2.35.3}$$

where $u_{\widetilde{G}*}$ and u_* are the functors obtained from the projection from the convergent topos to the étale topos, and q and p are the functors which take $\widetilde{G}$ invariants.

The functors p and q have exact left adjoints obtained by sending a sheaf $\mathcal{F}$ to $\mathcal{F}$ with comodule structure given by the map

$$\mathcal{F} \simeq \mathcal{F} \otimes_K K \xrightarrow{1 \otimes j} \mathcal{F} \otimes_K \mathcal{O}_{\widetilde{G}}, \tag{2.35.4}$$

where j denotes the inclusion $K \hookrightarrow \mathcal{O}_{\widetilde{G}}$ obtained from the structure morphism $\widetilde{G} \to \text{Spec}(K)$. In particular, all the functors in the above diagram take injectives to injectives.

The usual yoga comparing crystalline and de Rham cohomology ([B-O], §7) gives an isomorphism in the derived category

$$\mathbb{R}u_{\widetilde{G}*}(\mathbb{L}(\mathcal{O}_{\widetilde{G}})) \simeq \mathbb{R}\lambda(C^*(\mathcal{V} \otimes \mathcal{O}_{\widetilde{G}})), \tag{2.35.5}$$

where $\lambda : U^\bullet_{\text{et}} \to X_{\text{et}}$ is the natural morphism of topoi. Furthermore, there is a natural isomorphism

$$\mathbb{R}\pi\mathbb{R}\Gamma(C^*(\mathcal{V} \otimes \mathcal{O}_{\widetilde{G}})) \simeq \mathbb{R}\Gamma\mathbb{R}p(\mathbb{R}u_{\widetilde{G}*}(\mathcal{V} \otimes \mathbb{L}(\mathcal{O}_{\widetilde{G}}))). \tag{2.35.6}$$

Hence it suffices to show that the natural map

$$\mathbb{R}u_*(\mathcal{V}) \longrightarrow \mathbb{R}p\mathbb{R}u_{\widetilde{G}*}(\mathcal{V}\otimes \mathbb{L}(\mathcal{O}_{\widetilde{G}})) \tag{2.35.7}$$

is a quasi–isomorphism. In fact, we prove the stronger statement that the natural map

$$\mathcal{V} \longrightarrow \mathbb{R}q_*(\mathcal{V}\otimes \mathbb{L}(\mathcal{O}_{\widetilde{G}})) \tag{2.35.8}$$

is a quasi–isomorphism.

For this observe that $\mathbb{R}q_*(\mathcal{V}\otimes\mathbb{L}(\mathcal{O}_{\widetilde{G}}))$ can be calculated as follows. For each enlargement T of some étale $U \to X$, the restriction of $\mathbb{R}q_*(\mathcal{V}\otimes\mathbb{L}(\mathcal{O}_{\widetilde{G}}))$ to T_{et} is simply the group cohomology of the $\mathcal{O}_{T_K} - \widetilde{G}$–module $\mathcal{V}_T\otimes\mathbb{L}(\mathcal{O}_{\widetilde{G}})$. By ([SGA3], 5.3), this cohomology can be described as follows. Let C^n be the isocrystal

$$(\mathcal{V}\otimes\mathbb{L}(\mathcal{O}_{\widetilde{G}}))\otimes\overbrace{\mathcal{O}_{\widetilde{G}}\otimes\cdots\otimes\mathcal{O}_{\widetilde{G}}}^{n}, \tag{2.35.9}$$

and let $d : C^n \to C^{n+1}$ be the differential given by the formula
(2.35.10)

$$\begin{aligned} d(f\otimes a_1\otimes\cdots\otimes a_n) &= \mu(f)\otimes a_1\otimes\cdots\otimes a_n + \textstyle\sum_{i=1}^n(-1)^i f\otimes a_1\otimes\cdots\otimes\Delta a_i\otimes\cdots\otimes a_n \\ &\quad +(-1)^{n+1}f\otimes a_1\otimes\cdots\otimes a_n\otimes 1, \end{aligned}$$

where

$$\mu : \mathcal{V}\otimes\mathbb{L}(\mathcal{O}_{\widetilde{G}}) \longrightarrow \mathcal{V}\otimes\mathbb{L}(\mathcal{O}_{\widetilde{G}})\otimes_K\mathcal{O}_{\widetilde{G}} \tag{2.35.11}$$

denotes the map giving the comodule structure and $\Delta : \mathcal{O}_{\widetilde{G}} \to \mathcal{O}_{\widetilde{G}}\otimes\mathcal{O}_{\widetilde{G}}$ is the map giving the group law on $\widetilde{G}$. The complex C^* represents $\mathbb{R}q(\mathcal{V}\otimes\mathbb{L}(\mathcal{O}_{\widetilde{G}}))$.

Each term C^n is by construction a direct limit of isocrystals of finite type, and hence the cohomology sheaves $\mathcal{H}^m(C^*)$ are also direct limits of isocrystals of finite type. We claim that $\mathcal{H}^m(C^*) = 0$ for $m \geq 1$ and that the map $\mathcal{V} \to \mathcal{H}^0(C^*)$ is a quasi–isomorphism. For this it suffices by ([Og1], 4.1) to show that $x^*\mathcal{H}^m(C^*) = 0$ for $m \geq 1$ and that the map $x^*\mathcal{V} \to x^*\mathcal{H}^0(C^*)$ is an isomorphism. Since the functor x^* is exact on the category of isocrystals of finite type ([Og1], 2.10), the natural map $x^*\mathcal{H}^m(C^*) \to H^m(x^*C^*)$ is an isomorphism. On the other hand, the complex x^*C^* on the category $\text{Isoc}(k/K) \simeq \text{Vec}_K$ is simply the complex computing the group cohomology of $V\otimes\mathcal{O}_{\widetilde{G}}$. Hence the result follows from (2.18). □

Proof of (2.27 (ii)). By (2.34 (ii)), the result holds for any representation of G. Since G is the reductive completion of $\pi_1(X_{\mathcal{C}})$, any representation of $\pi_1(X_{\mathcal{C}})$ admits a finite filtration whose associated graded pieces are representations of G. From this and the long exact sequence of cohomology, we see that to prove (2.27 (ii)) it suffices to construct, for every representation V of $\pi_1(X_{\mathcal{C}})$ as in (2.27 (ii)), a natural map

$$H^*(X_{\mathcal{C}}, V) \longrightarrow H^*_{\text{cris}}(X, \mathcal{V}). \tag{2.35.12}$$

Such a map is provided by the pullback map

$$H^*(X_{\mathcal{C}}, V) \longrightarrow H^*(\widetilde{X}, f^*V), \tag{2.35.13}$$

and (2.34 (i)). □

Proof of (2.27 (i)). We show that the map f_* in (2.30 (ii)) is an isomorphism. By Tannaka duality, it suffices to show that the functor

$$f^* : \mathrm{Rep}(\pi_1(X_{\mathcal{C}})) \longrightarrow \mathrm{Rep}(\pi_1(\mathcal{C}, \omega_x)) \tag{2.35.14}$$

is an equivalence of categories. By (2.30 (i)), every object in $\mathrm{Rep}(\pi_1(\mathcal{C}, \omega_x))$ admits a filtration whose associated graded pieces are in the essential image of f^*. Therefore, to prove that (2.35.14) is an equivalence, it suffices to show that for $V_1, V_2 \in \mathrm{Rep}(\pi_1(X_{\mathcal{C}}))$, the natural map

$$\mathrm{Ext}^i_{\mathrm{Rep}(\pi_1(X_{\mathcal{C}}))}(V_1, V_2) \longrightarrow \mathrm{Ext}^i_{\mathrm{Rep}(\pi_1(\mathcal{C},\omega_x))}(f^*V_1, f^*V_2) \tag{2.35.15}$$

is an isomorphism for $i = 0, 1$. Letting $M = V_1^{*c} \otimes V_2$, we see that what has to be shown is that the natural map

$$H^i(\mathrm{Rep}(\pi_1(X_{\mathcal{C}})), M) \longrightarrow H^i(\mathrm{Rep}(\pi_1(\mathcal{C}, \omega_x)), M) \tag{2.35.16}$$

is an isomorphism for $i = 0, 1$. Because $\mathcal{C}$ is closed under extensions, the right hand side of (2.35.16) is isomorphic to $H^i_{\mathrm{cris}}(X, \mathcal{M})$ where $\mathcal{M}$ denotes the isocrystal associated to M. By ([To1], 1.3.3), the left hand side is naturally isomorphic to $H^i(K(\pi_1(X_{\mathcal{C}}), 1), M)$, and $K(\pi_1(X_{\mathcal{C}}), 1)$ is isomorphic to the truncation $\tau_{\leq 1} X_{\mathcal{C}}$. Thus using (2.27 (ii)), what has to be shown is that the natural map

$$H^i(\tau_{\leq 1} X_{\mathcal{C}}, M) \longrightarrow H^i(X_{\mathcal{C}}, M) \tag{2.35.17}$$

is an isomorphism for $i = 0, 1$. This follows from the definition of cohomology. □

3. F–crystal structure on stacks

F–isocrystal structure on $X_{\mathcal{C}}$

3.1. If $F \in \mathrm{SPr}(K)$, define $F^\sigma \in \mathrm{SPr}(K)$ to be the simplicial presheaf which to any K–algebra R associates the simplicial set $X(R^{\sigma^{-1}})$, where $R^{\sigma^{-1}} := R \otimes_{K,\sigma^{-1}} K$. The functor $F \mapsto F^\sigma$ is an auto–equivalence of $\mathrm{SPr}(K)$, and is compatible with the model category structures. It therefore induces an auto–equivalence

$$\mathrm{Ho}(\mathrm{SPr}(K)) \longrightarrow \mathrm{Ho}(\mathrm{SPr}(K)), \quad F \mapsto F^\sigma. \tag{3.1.1}$$

Similarly, there is a natural Frobenius auto–equivalence on $\mathrm{Ho}(\mathrm{SPr}_*(K))$.

It follows from the definitions, that if $F \in \mathrm{Ho}(\mathrm{SPr}_*(K))$ is a pointed stack whose fundamental group is representable by a group scheme, then for every K–algebra A, there are natural isomorphisms

$$\pi_1(F^\sigma)(A) \simeq \pi_1(F)(A^{\sigma^{-1}}) \simeq \pi_1(F)^\sigma(A), \tag{3.1.2}$$

where $\pi_1(F)^\sigma$ denotes the group scheme $\pi_1(F) \otimes_{K,\sigma} K$. If V is a coherent local system on F, then the sheaf which to any K–algebra A associates $(V \otimes_{K,\sigma^{-1}} K) \otimes_K A$ admits a natural action of $\pi_1(F^\sigma)$. We thus obtain a functor

$$(\text{coherent local systems on } F) \to (\text{coherent local systems on } F^\sigma), \tag{3.1.3}$$

which we denote by $V \mapsto V^\sigma$

3.2. Let G/K be a group scheme with an isomorphism $\varphi_G : G \to G^\sigma$, where $G^\sigma := G \otimes_{K,\sigma} K$. Then there is a natural auto–equivalence

$$G - \mathrm{Alg}^\Delta \longrightarrow G - \mathrm{Alg}^\Delta, \quad A \mapsto A^\sigma := A \otimes_{K,\sigma} K, \tag{3.2.1}$$

where G acts on A^σ via the isomorphism φ_G. The auto–equivalence (3.2.1) is compatible with the model category structures, and hence induces an equivalence

$$\mathrm{Ho}(G - \mathrm{Alg}^\Delta) \longrightarrow \mathrm{Ho}(G - \mathrm{Alg}^\Delta), \quad A \mapsto A^\sigma. \tag{3.2.2}$$

This functor is compatible with the functor $[\mathbb{R}\mathrm{Spec}_G(-)/G]$ of (2.7) in the sense that for any $A \in \mathrm{Ho}(G - \mathrm{Alg}^\Delta)$, there is a natural isomorphism

$$[\mathbb{R}\mathrm{Spec}_G(A)/G]^\sigma \simeq [\mathbb{R}\mathrm{Spec}_G(A^\sigma)/G]. \tag{3.2.3}$$

3.3. Let $\mathcal{C} \subset \mathrm{Isoc}(X/K)$ be a Tannakian subcategory for which pullback by Frobenius induces an auto–equivalence on $\mathcal{C}$. There is then a 2–commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F^*} & \mathcal{C} \\ {\scriptstyle \omega_x}\downarrow & & \downarrow{\scriptstyle \omega_x} \\ \mathrm{Vec}_K & \xrightarrow{\otimes_\sigma K} & \mathrm{Vec}_K, \end{array} \tag{3.3.1}$$

and hence a natural isomorphism

$$G = \underline{\mathrm{Aut}}^\otimes(\omega_x) \simeq \underline{\mathrm{Aut}}^\otimes(\omega_x \circ F^*) \simeq \underline{\mathrm{Aut}}^\otimes((\otimes_\sigma K) \circ \omega_x) \simeq G \otimes_{K,\sigma} K. \tag{3.3.2}$$

Thus Frobenius induces an isomorphism $\varphi_G : G \to G \otimes_{K,\sigma} K$. For $\mathcal{V} \in \mathcal{C}$ with associated representation V of G, the pullback $F^*\mathcal{V}$ corresponds to the representation $V \otimes_\sigma K$ with action of G induced by the map φ_G. In particular there is a natural isomorphism of ind–isocrystals with right G–action $F^*\mathbb{L}(\mathcal{O}_G) \simeq \mathbb{L}(\mathcal{O}_G)$, where $\mathbb{L}(\mathcal{O}_G)$ is the ind–isocrystal defined in (2.15). If $U^\bullet \hookrightarrow Y^\bullet$ is an embedding system for X, then this Frobenius action induces an isomorphism

$$C^\bullet(\mathcal{O}_G) \otimes_{K,\sigma} K \simeq C^\bullet(\mathcal{O}_G) \tag{3.3.3}$$

in $G - \mathrm{dga}(Y^\bullet)$ (recall that by definition, an embedding system comes with a lifting of Frobenius). Applying the functors $\mathbb{R}\Gamma$ and $[\mathbb{R}\mathrm{Spec}_G(-)/G]$, we obtain an isomorphism

$$\varphi : X_{\mathcal{C}} \longrightarrow X_{\mathcal{C}}^\sigma. \tag{3.3.4}$$

In other words, the pair $(X_{\mathcal{C}}, \varphi_{X_{\mathcal{C}}})$ is an "F–isocrystal" in the category $\mathrm{Ho}(\mathrm{SPr}_*(K))$. The following result, whose proof follows from the construction, summarizes the basic properties of this structure:

Theorem 3.4. (i) *The isomorphism $\pi_1(X_{\mathcal{C}}) \to \pi_1(X_{\mathcal{C}})^\sigma$ is, via the identification of* (2.27 (i)), *the isomorphism $\pi_1(\mathcal{C}, \omega_x) \to \pi_1(\mathcal{C}, \omega_x)^\sigma$ obtained from Tannaka duality.*

(ii) *There is a natural equivalence of categories between the category of F–isocrystals (E, φ) on X/K with $E \in \mathcal{C}$, and the category of pairs (V, ϵ), where V is a local system on $X_{\mathcal{C}}$, and $\epsilon : \varphi_{X_{\mathcal{C}}}^*(V^\sigma) \to V$ is an isomorphism of local systems on $X_{\mathcal{C}}$.*

(iii) *If (V, ϵ) is as in* (ii) *with associated F–isocrystal (E, φ), then the isomorphism* (2.27.1)

$$H^*_{\mathrm{cris}}(X, E) \simeq H^*(X_{\mathcal{C}}, V) \tag{3.4.1}$$

is compatible with the F–isocrystal structures induced by φ and ϵ.

Remark 3.5. By ([Og1], 4.10), Frobenius pullback induces an autoequivalence of $\mathrm{Isoc}(X/K)$. Hence if $\mathcal{C} \subset \mathrm{Isoc}(X/K)$ is a Tannakian subcategory such that for every $\mathcal{V} \in \mathcal{C}$ the pullback $F^*\mathcal{V}$ is again in $\mathcal{C}$, then the functor $F^* : \mathcal{C} \to \mathcal{C}$ is fully faithful. If in addition, the category $\mathcal{C}$ is closed under extensions then the essential image of F^* is closed under extensions because for all $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{C}$ pullback by Frobenius induces an isomorphism

(3.5.1)

$$\mathrm{Ext}^1(\mathcal{V}_1, \mathcal{V}_2) \simeq H^1(X/K, \mathcal{V}_1^* \otimes \mathcal{V}_2) \xrightarrow{F^*} H^1(X/K, F^*(\mathcal{V}_1^* \otimes \mathcal{V}_2)) \simeq \mathrm{Ext}^1(F^*\mathcal{V}_1, F^*\mathcal{V}_2).$$

It follows that in this case F^* induces an autoequivalence of $\mathcal{C}$ if and only if F^* induces an autoequivalence of the full subcategory $\mathcal{C}^{\mathrm{ss}} \subset \mathcal{C}$ of semisimple objects. The essential image of F^* is also closed under direct summands because pullback by Frobenius induces for all $\mathcal{V} \in \mathcal{C}$ an isomorphism

$$\mathrm{End}(\mathcal{V}) \simeq H^0(X/K, \mathcal{V}^*\mathcal{V}) \xrightarrow{F^*} H^0(X/K, F^*(\mathcal{V}^* \otimes \mathcal{V})) \simeq \mathrm{End}(F^*\mathcal{V}). \tag{3.5.2}$$

Consequently if $\{(\mathcal{V}_i, \varphi_i)\}$ is a family of F–isocrystals and $\mathcal{C}$ is defined to be the smallest Tannakian subcategory of $\mathrm{Isoc}(X/K)$ closed under extensions and containing the $\mathcal{V}_i$, then pullback by Frobenius induces an autoequivalence of $\mathcal{C}$.

Remark 3.6. As explained to us by Toen, the action of Frobenius on $X_{\mathcal{C}}$ should by analogy with Hodge theory *not* be "continuous". Concretely, for us this means that the Frobenius action on $\mathbb{R}\Gamma(\mathbb{L}(\mathcal{O}_G))$ is not obtained from a cosimplicial algebra in the category of Ind–object in the category of F–isocrystals of finite type over K. For example, we do not expect that the infinite dimensional F–isocrystal $H^i(\mathbb{R}\Gamma(\mathbb{L}(\mathcal{O}_G)))$ can be written as a direct limit of F–isocrystals of finite type. This means that the F–isocrystal structure on $X_{\mathcal{C}}$ is not induced by an action of the Tannaka dual of the category $\mathrm{FIsoc}(k/K)$. In the next three subsections, however, we shall see that this is the case when $\mathcal{C}$ is generated by F–isocrystals as in (3.5).

Twisted equivariant theory

In order to deal with the fact that $\mathrm{FIsoc}(k/K)$ is a non–neutral Tannakian category, it will be necessary to develop some of the equivariant theory of ([KPT], §1) replacing BG by a gerbe.

3.7. Let C be a site and $\mathcal{G}$ a gerbe on C. We define the *$\mathcal{G}$–twisted site*, denoted $C_{\mathcal{G}}$ as follows. The objects of $C_{\mathcal{G}}$ are pairs (U, ω), where $U \in C$ and $\omega \in \mathcal{G}(U)$. A morphism $(U', \omega') \to (U, \omega)$ is a pair (f, ι), where $f : U' \to U$ is a morphism in C and $\iota : f^*\omega \to \omega'$ is an isomorphism in $\mathcal{G}(U')$. Covering families in $C_{\mathcal{G}}$ are defined to be families $\{(U', \omega') \to (U, \omega)\}$ for which the underlying family of maps $\{U' \to U\}$ is a covering family in C.

We can then define the category $\mathrm{SPr}(C_{\mathcal{G}})$, and we view it as a model category using the model structure defined in ([To1], 1.1.1).

Definition 3.8. A *$\mathcal{G}$–twisted stack on C* is an object $F \in \mathrm{Ho}(\mathrm{SPr}(C_{\mathcal{G}}))$.

Remark 3.9. Observe that if (U, ω) is an object in $C_{\mathcal{G}}$, then the category $\mathrm{SPr}(C_{\mathcal{G}})|_{(U,\omega)}$ is naturally equivalent to the category of $\underline{\mathrm{Aut}}(\omega)$–equivariant pre–sheaves on $C|_U$ defined in ([KPT], §1), and this is an equivalence of model categories.

Remark 3.10. If $\mathcal{G}$ is a trivial gerbe and $\omega_0 \in \mathcal{G}$ is a global object, then the site $C_{\mathcal{G}}$ is equivalent to the site $\underline{\mathrm{Aut}}(\omega_0)-C$ whose objects are the objects of C and for which a morphism

$U' \to U$ is a pair (f, g), where $f : U' \to U$ is a morphism in C and $g \in \underline{\mathrm{Aut}}(\omega_0)(U')$. Indeed, there is a functor

$$\underline{\mathrm{Aut}}(\omega_0) - C \longrightarrow C_{\mathcal{G}}, \quad U \mapsto (U, \omega_0) \tag{3.10.1}$$

which is an equivalence of sites since every object in $\mathcal{G}$ is locally isomorphic to ω_0. Hence in this case, the notion of $\mathcal{G}$–twisted stack is equivalent to the notion of $\underline{\mathrm{Aut}}(\omega_0)$–equivariant stack used in (loc. cit.).

3.11. The category $\mathrm{Ho}(\mathrm{SPr}(C_{\mathcal{G}}))$ can naturally be viewed as "stacks over $\mathcal{G}$" as follows.

Let $B\mathcal{G} \in \mathrm{SPr}(C)$ denote the simplicial presheaf which to any $U \in C$ associates the nerve of the category $\mathcal{G}(U)$, and let $\mathrm{SPr}(C)|_{B\mathcal{G}}$ denote the category of objects over $B\mathcal{G}$. As in ([Ho], 1.1.8) the category $\mathrm{SPr}(C)|_{B\mathcal{G}}$ has a natural model category structure.

Proposition 3.12. *Suppose that for every object $\omega \in \mathcal{G}(U)$, the sheaf*

$$(g : V \to U) \mapsto (\text{automorphisms of } g^*\omega) \tag{3.12.1}$$

is cofibrant viewed as a constant object in $\mathrm{SPr}(C|_U)$ (for example if it is representable). Then there is a natural equivalence of categories

$$\mathrm{Ho}(\mathrm{SPr}(C_{\mathcal{G}})) \simeq \mathrm{Ho}(\mathrm{SPr}(C)|_{B\mathcal{G}}). \tag{3.12.2}$$

Proof. Let G denote the presheaf of groups on $C_{\mathcal{G}}$ which to any (U, ω) associates the group of automorphisms of ω in $\mathcal{G}(U)$. For any two objects $(U', \omega'), (U, \omega)$ of $C_{\mathcal{G}}$, there is a natural action of $G(U', \omega')$ on

$$\mathrm{Hom}_{C_{\mathcal{G}}}((U', \omega'), (U, \omega)). \tag{3.12.3}$$

The action of $g \in G(U', \omega')$ sends (f, ι) to $(f, g \circ \iota)$. Let $\overline{C}_{\mathcal{G}}$ denote the site whose objects are the objects of $C_{\mathcal{G}}$ and whose morphisms are given by the quotient

$$G(U', \omega') \backslash \mathrm{Hom}_{C_{\mathcal{G}}}((U', \omega'), (U, \omega)). \tag{3.12.4}$$

Coverings are morphisms which can be represented by coverings in $C_{\mathcal{G}}$.

For any presheaf F on $C_{\mathcal{G}}$ there is a natural action of G on F. Namely, for each $(U, \omega) \in C_{\mathcal{G}}$ and $g \in G(U, \omega)$, the pair (id, g) defines an automorphism of (U, ω) and hence induces a map $g^* : F(U, \omega) \to F(U, \omega)$. The category of presheaves on $\overline{C}_{\mathcal{G}}$ is naturally identified with the full subcategory of the category of presheaves on $C_{\mathcal{G}}$ whose objects are the presheaves F for which the action of G on F is trivial. This gives an inclusion of categories

$$k_* : \mathrm{SPr}(\overline{C}_{\mathcal{G}}) \longrightarrow \mathrm{SPr}(C_{\mathcal{G}}). \tag{3.12.5}$$

The functor k_* has a left adjoint k^* sending a sheaf F to the simplicial presheaf $(U, \omega) \mapsto F(U, \omega)/G(U, \omega)$.

The functor $\lambda : \overline{C}_{\mathcal{G}} \to C$ sending (U, ω) to U defines a morphism of sites $C \to \overline{C}_{\mathcal{G}}$, and the induced morphism of topoi $j : C^{\sim} \to \overline{C}_{\mathcal{G}}^{\sim}$ is an equivalence. From this and the definition of the model category structures on $\mathrm{SPr}(C)$ and $\mathrm{SPr}(\overline{C}_{\mathcal{G}})$ ([To1], 1.1.1) it follows that (j^*, j_*) is a Quillen equivalence. In particular, there is an equivalence

$$\mathrm{Ho}(\mathrm{SPr}(C)|_{B\mathcal{G}}) \simeq \mathrm{Ho}(\mathrm{SPr}(\overline{C}_{\mathcal{G}})|_{j_*B\mathcal{G}}). \tag{3.12.6}$$

Let $E\mathcal{G} \in \mathrm{SPr}(C_{\mathcal{G}})$ be a cofibrant model for $*$ in $\mathrm{SPr}(C_{\mathcal{G}})$.

There is a canonical map $* \to k_* j_* B\mathcal{G}$ in $\mathrm{SPr}(C_{\mathcal{G}})$ defined on (U,ω) by the point of $B\mathcal{G}(U)$ defined by $\omega \in \mathcal{G}(U)$, and hence there is also an induced map $E\mathcal{G} \to k_* j_* B\mathcal{G}$. In particular, for any $F \in \mathrm{SPr}(C_{\mathcal{G}})$ there is a canonical map $k^*F \to j_*B\mathcal{G}$ defined as the composite

(3.12.7) $$k^*F \longrightarrow k^** \longrightarrow k^*k_*j_*B\mathcal{G} \xrightarrow{\text{can}} j_*B\mathcal{G}.$$

Therefore k^* defines a functor De (following the notation of ([KPT], 1.2))

(3.12.8) $$De : \mathrm{SPr}(\mathcal{C}_{\mathcal{G}}) \longrightarrow \mathrm{SPr}(\overline{\mathcal{C}}_{\mathcal{G}})|_{j_*B\mathcal{G}}$$

sending F to $k^*(F \times E\mathcal{G})$ with the natural map (3.12.7) to $j_*B\mathcal{G}$.

This functor k^* has a right adjoint Mo sending $H \to B\mathcal{G}$ to the simplicial presheaf $(U,\omega) \mapsto \underline{\mathrm{Hom}}_{k_*j_*B\mathcal{G}}(E\mathcal{G} \times (U,\omega), k_*H)$. Since $E\mathcal{G}$ is cofibrant as is the representable object (U,ω) the product $E\mathcal{G} \times (U,\omega)$ is also cofibrant. It follows that Mo preserves fibrations and trivial fibrations and hence (De, Mo) is a Quillen adjunction.

To prove the proposition, it suffices now to verify that (De, Mo) is a Quillen equivalence. For this it suffices by ([G-J], II.7.8) to show that if $X \in \mathrm{SPr}(C_{\mathcal{G}})$ and $Y \in \mathrm{SPr}(\overline{C}_{\mathcal{G}})|_{j_*B\mathcal{G}}$, then a map $X \to Mo(Y)$ is an equivalence if and only if the adjoint map $De(X) \to Y$ is an equivalence. For this, let $(U,\omega) \in C_{\mathcal{G}}$ be an object. Then it suffices to show that the morphism

(3.12.9) $$X|_{C_{\mathcal{G}}|_{(U,\omega)}} \longrightarrow Mo(Y)|_{C_{\mathcal{G}}|_{(U,\omega)}}$$

is an equivalence if and only if the morphism

(3.12.10) $$Mo(X)|_{\overline{C}_{\mathcal{G}}|_{(U,\omega)}} \longrightarrow Y|_{\overline{C}_{\mathcal{G}}|_{(U,\omega)}}$$

is an equivalence. But now we are reduced to the case of $\underline{\mathrm{Aut}}(\omega)$–equivariant sheaves on the site $C|_U$ which is the case discussed in the proof of ([To1], 1.3.3). □

Remark 3.13. Let $F \in \mathrm{Ho}(\mathrm{SPr}(C_{\mathcal{G}}))$ be an $\mathcal{G}$–twisted stack and $[F/\mathcal{G}]$ the corresponding object in $\mathrm{Ho}(\mathrm{SPr}(C)|_{B\mathcal{G}})$. For any object $(U,\omega) \in C_{\mathcal{G}}$ the restriction of F to $C_{\mathcal{G}}|_{(U,\omega)}$ defines an object $F_{(U,\omega)} \in \mathrm{Ho}(\mathrm{SPr}(C|_U))$ and chasing through the construction one sees that $F_{(U,\omega)}$ is the homotopy fiber product of the diagram

(3.13.1) $$\begin{array}{ccc} & & [F/\mathcal{G}] \\ & & \downarrow \\ U & \xrightarrow{\omega} & B\mathcal{G}. \end{array}$$

Remark 3.14. In what follows, we will for technical reasons prefer to work with $\mathrm{SPr}(C_{\mathcal{G}})$ rather than $\mathrm{SPr}(C)|_{B\mathcal{G}}$. The reader who so desires can translate our results using the above equivalence (3.12).

3.15. The case of interest in this paper is when C is the category of affine schemes over $\mathbb{Q}_p$ with the fpqc topology, and $\mathcal{G}$ is the gerbe which to any $\mathbb{Q}_p$–algebra R associates the groupoid of fiber functors

(3.15.1) $$\mathrm{FIsoc}(k/K) \longrightarrow \mathrm{Mod}_R$$

for some perfect field k of positive characteristic and K the field of fractions of the ring of Witt vectors of k. To ease notation we write simply $\mathrm{Aff}_{\mathbb{Q}_p,F}$ for the twisted site obtained from the category of $\mathbb{Q}_p$–algebras and the gerbe of fiber functors for $\mathrm{FIsoc}(k/K)$. Observe that there is a natural structure sheaf $\mathcal{O}$ on $\mathrm{Aff}_{\mathbb{Q}_p,F}$ which to any (R,ω) associates R.

Definition 3.16. A *vector space* on $\mathrm{Aff}_{\mathbb{Q}_p,F}$ is a sheaf of $\mathcal{O}$–modules $\mathcal{F}$ on $\mathrm{Aff}_{\mathbb{Q}_p,F}$ such that for each morphism $\Lambda : (R',\omega') \to (R,\omega)$ the pullback map $\mathcal{F}(R,\omega)\otimes_R R' \to \mathcal{F}(R',\omega')$ is an isomorphism.

Lemma 3.17. *The category of Ind–objects in* $\mathrm{FIsoc}(k/K)$ *is naturally equivalent to the category of vector spaces on* $\mathrm{Aff}_{\mathbb{Q}_p,F}$. *Moreover, this equivalence is compatible with the tensor structures.*

Proof. By ([Sa], III.3.2.2.3), the category of vector spaces on $\mathrm{Aff}_{\mathbb{Q}_p,F}$ is equivalent to the category of ind–objects in the category of vector spaces $\mathcal{F}$ on $\mathrm{Aff}_{\mathbb{Q}_p,F}$ with $\mathcal{F}(R,\omega)$ free of finite rank for every $(R,\omega)\in \mathrm{Aff}_{\mathbb{Q}_p,F}$. On the other hand, there is a natural equivalence between $\mathrm{FIsoc}(k/K)$ and this latter category which sends an F–isocrystal $\mathcal{V}$ to the sheaf which to any (R,ω) associates $\omega(\mathcal{V})$. □

3.18. Let us write $\varpi(\mathcal{V})$ for the vector space on $\mathrm{Aff}_{\mathbb{Q}_p,F}$ associated to an F–isocrystal $\mathcal{V}$. The fact that the equivalence (3.17) is compatible with the tensor structures, implies that any algebra (resp. coalgebra, Hopf algebra) in $\mathrm{FIsoc}(k/K)$ induces a sheaf of algebras (resp. coalgebras, Hopf algebras) on $\mathrm{Aff}_{\mathbb{Q}_p,F}$. If $\mathcal{A}$ is such an algebra (resp. coalgebra, Hopf algebra) we will write $\varpi(\mathcal{A})$ for the resulting sheaf on $\mathrm{Aff}_{\mathbb{Q}_p,F}$. If $\mathcal{A}$ is a Hopf algebra in $\mathrm{FIsoc}(k/K)$ and G is the associated group scheme, we sometimes also write $\varpi(G)$ for the group scheme on $\mathrm{Aff}_{\mathbb{Q}_p,F}$ associated to $\varpi(\mathcal{A})$.

3.19. Let $\sigma : K \to K$ the canonical map induced by Frobenius on k so that $\mathbb{Q}_p \subset K$ is the fixed field of σ. Suppose given a reductive group G/K together with an isomorphism $\varphi_G : G \to G^\sigma := G\otimes_{K,\sigma} K$. For any $V \in \mathrm{Rep}(G)$, we denote by $V^\sigma \in \mathrm{Rep}(G)$ the representation

$$G \xrightarrow{\varphi_G} G^\sigma \longrightarrow \mathrm{Aut}(V\otimes_{K,\sigma} K). \tag{3.19.1}$$

Definition 3.20. An *$F - G$–representation* is a pair (V,φ_V), where $V \in \mathrm{Rep}(G)$ and $\varphi_V : V^\sigma \to V$ is an isomorphism of representations such that the F–isocrystal obtained by forgetting the G–action is a direct limit of F–isocrystals of finite type. We denote the resulting category of $F-G$–representations by $\mathrm{Rep}(G)_F$.

Remark 3.21. If (V,φ) is a K–vector space with an isomorphism $V^\sigma \to V$ such that the pair (V,φ) is a direct limit of F–isocrystals of finite type, then any sub–object $(S,\varphi_S)\subset (V,\varphi)$ and quotient $(V,\varphi)\to (Q,\varphi_Q)$ is again a direct limit of F–isocrystals of finite type. Furthermore, if (W,ψ) is a second direct limit of F–isocrystals of finite type, then any extension of (V,φ) by (W,ψ) is again a direct limit of F–isocrystals of finite type. The verification of these assertions are left to the reader.

3.22. Note that there is a natural tensor structure on the category $\mathrm{Rep}(G)_F$. Denote by $G{-}\mathrm{Alg}_F^\Delta$ (resp. $G{-}\mathrm{dga}_F$) the category of cosimplicial algebras (resp. commutative differential graded algebras) in $\mathrm{Rep}(G)_F$.

Proposition 3.23. *There exists a simplicial, finitely generated closed model category structure on* $G-\mathrm{Alg}_F^\Delta$ *(resp.* $G-\mathrm{dga}_F$*) such that*

(i) *A map* $f : A \to B$ *is an equivalence if and only the map between the associated normalized complexes (resp. underlying complexes)* $N(f) : N(A) \to N(B)$ *is a quasi–isomorphism.*

(ii) *A map* $f : A \to B$ *is a fibration if and only if it is a level–wise epimorphism.*

Proof. This follows from the same argument used in ([KPT], 1.3.2). □

3.24. Suppose G is a group scheme in $\mathrm{FIsoc}(k/K)$ with associated sheaf of group $\varpi(G)$ on $\mathrm{Aff}_{\mathbb{Q}_p,F}$. Denote by $\varpi(G)-\mathrm{SPr}(\mathrm{Aff}_{\mathbb{Q}_p,F})$ the category of $\varpi(G)$–equivariant presheaves on $\mathrm{Aff}_{\mathbb{Q}_p,F}$. We view $\varpi(G)-\mathrm{SPr}(\mathrm{Aff}_{\mathbb{Q}_p,F})$ as a model category using ([To1], 1.1.1). There is a natural functor

$$\mathrm{Spec}_G : (G-\mathrm{Alg}_F^{\Delta})^{\mathrm{op}} \longrightarrow \varpi(G)-\mathrm{SPr}(\mathrm{Aff}_{\mathbb{Q}_p,F}) \tag{3.24.1}$$

sending A to the presheaf

$$(R,\omega) \mapsto ([n] \mapsto \mathrm{Hom}_{\mathbb{Q}_p}(\omega(A_n), R)) \tag{3.24.2}$$

with the natural action of $\varpi(G)(R,\omega) = \underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ on $\omega(A_n)$.

Lemma 3.25. *The functor* Spec_G *is a right Quillen functor.*

Proof. This follows from the same argument given in ([KPT], 1.3.6), using the Freyd adjoint functor theorem ([Ma]). □

3.26. We therefore have a derived version of Spec_G which we denote by

$$\mathbb{R}\mathrm{Spec}_G : \mathrm{Ho}(G-\mathrm{Alg}_F^{\Delta})^{\mathrm{op}} \longrightarrow \mathrm{Ho}(\mathrm{SPr}(\varpi(G)-\mathrm{Aff}_{\mathbb{Q}_p,F})). \tag{3.26.1}$$

As in ([KPT], 1.3), its left adjoint will be written

$$F \mapsto \mathbb{L}\mathcal{O}_G(F). \tag{3.26.2}$$

Note furthermore that by the same reasoning as in ([KPT], 1.2.1) there is a natural equivalence

$$\mathrm{Ho}(\varpi(G)-\mathrm{SPr}(\mathrm{Aff}_{\mathbb{Q}_p,F})) \longrightarrow \mathrm{Ho}(\mathrm{SPr}(\mathrm{Aff}_{\mathbb{Q}_p,F}|_{B\varpi(G)})), \tag{3.26.3}$$

where $B\varpi(G) \in \mathrm{SPr}(\mathrm{Aff}_{\mathbb{Q}_p,F})$ denotes the classifying space of $\varpi(G)$, defined for example in ([To1], 1.3). Composing with this equivalence we obtain a functor

$$[\mathrm{Spec}_G(-)/G] : \mathrm{Ho}(G-\mathrm{Alg}_F^{\Delta})^{\mathrm{op}} \longrightarrow \mathrm{Ho}(\mathrm{SPr}(\mathrm{Aff}_{\mathbb{Q}_p,F}|_{B\varpi(G)})). \tag{3.26.4}$$

Note that the forgetful fiber functor $\omega_0 : \mathrm{FIsoc}(k/K) \to \mathrm{Vec}_K$ induces a natural commutative diagram

$$\begin{array}{ccc} \mathrm{Ho}(G-\mathrm{Alg}_F^{\Delta})^{\mathrm{op}} & \xrightarrow{[\mathbb{R}\mathrm{Spec}_G(-)/G]} & \mathrm{Ho}(\mathrm{SPr}(\mathrm{Aff}_{\mathbb{Q}_p,F}|_{B\varpi(G)})) \\ \mathrm{res}\downarrow & & \mathrm{res}\downarrow \\ \mathrm{Ho}(\omega_0(G)-\mathrm{Alg}_K^{\Delta}) & \xrightarrow{[\mathbb{R}\mathrm{Spec}_{\omega_0(G)}(-)/\omega_0(G)]} & \mathrm{Ho}(\mathrm{SPr}(K)|_{B\omega_0(G)}), \end{array} \tag{3.26.5}$$

where res denotes the restriction functor to $\mathrm{Aff}_{\mathbb{Q}_p,F}|_{(K,\omega_0)} \simeq \mathrm{Aff}_K$. From this it follows that if $A \in G-\mathrm{Alg}_F^{\Delta}$ is an object with underlying equivariant K–algebra $\omega_0(A)$, then there is a natural isomorphism

$$\mathrm{res}([\mathbb{R}\mathrm{Spec}_G(A)/G]) \simeq [\mathbb{R}\mathrm{Spec}(\omega_0(A))/\omega_0(G)]. \tag{3.26.6}$$

For $A \in \mathrm{Aff}_{\mathbb{Q}_p,F}$ with $A \in \mathbb{U}$, we refer to $[\mathbb{R}\mathrm{Spec}_G(A)/G]$ as a *stack over the gerbe of fiber functors for* $\mathrm{FIsoc}(k/K)$.

Remark 3.27. Note also that the there are pointed versions of all the above results. In particular, if $A \in G-\mathrm{Alg}_F^{\Delta}$ has an augmentation $A \to K$, then $[\mathbb{R}\mathrm{Spec}_G(A)/G]$ is naturally a pointed stack over the gerbe of fiber functors for $\mathrm{FIsoc}(k/K)$.

Given $F \in \mathrm{Ho}(\mathrm{SPr}_*(\mathrm{Aff}_{\mathbb{Q}_p,F}))$, let $\pi_1(F)$ be the sheaf defined in ([To1], 1.1) on $\mathrm{Aff}_{\mathbb{Q}_p,F}$.

Definition 3.28. A *local system on F* is a vector space V on $\mathrm{Aff}_{\mathbb{Q}_p,F}$ together with an action of $\pi_1(F)$.

By ([To1], 1.3), if V is a local system on F, there are naturally defined cohomology groups $H^*(F,V)$.

Minimal models

We review some of the theory of minimal models (see for example ([DGMS]), ([Su])), and verify that the theory can be pushed through for equivariant algebras with Frobenius action.

Let k be a separably closed field of characteristic $p > 0$, W its ring of Witt vectors and K the field of fractions of W. Denote by $\sigma : K \to K$ the canonical automorphism induced by Frobenius on k, and let $\mathrm{Rep}(G)_F$, $G - \mathrm{dga}_F$, and $G - \mathrm{Alg}_F^\Delta$ be the categories defined in (3.22). The categories $G - \mathrm{Alg}_F^\Delta$ and $G - \mathrm{dga}_F$ are model categories with model structure given in (3.23).

Lemma 3.29. *The category $\mathrm{Rep}(G)_F$ is an abelian, semi–simple, $\mathbb{Q}_p$–linear tensor category.*

Proof. The only non–obvious statement is that $\mathrm{Rep}(G)_F$ is semi–simple. For this, we must show that any inclusion $V_1 \hookrightarrow V_2$ is split. Equivalently, that the surjection

$$V_2^* \otimes V_1 \longrightarrow V_1^* \otimes V_1 \tag{3.29.1}$$

remains a surjection after taking G and F invariants. In other words, given a surjection $V \to W$ in $\mathrm{Rep}(G)_F$, we must show that the induced map

$$(V^G)^{\varphi_V=1} \longrightarrow (W^G)^{\varphi_W=1} \tag{3.29.2}$$

is again surjective. Since G is reductive and $\mathrm{char}(K) = 0$, the map $V^G \to W^G$ is surjective, and hence since $\mathrm{FIsoc}(k/K)$ is semi–simple the map (3.29.2) is also surjective. □

3.30. For $V \in \mathrm{Rep}(G)_F$ and an integer $n > 0$, let $\Lambda(V)$ denote the tensor algebra on V if n is even, and the exterior algebra on V if n is odd. We view $\Lambda(V)$ as an object in $G - \mathrm{dga}_F$ with zero differential and V in degree n. We often write $\Lambda(V_n)$ to indicate the degree n of V. For $A \in G - \mathrm{dga}$, we define an *elementary extension* of A to be any algebra $(A \otimes \Lambda(V), d) \in G - \mathrm{dga}_F$, where $d|_A = d_A$ and $d(V) \subset A$. We say that the differential d_A of an algebra $A \in G - \mathrm{dga}_F$ is *decomposable* if for all $a \in A$ we have $d(a) \in A_+ \cdot A_+$, where $A_+ \subset A$ denotes the elements of degree ≥ 1. An algebra $M \in G - \mathrm{dga}_F$ is *minimal* if M can be written as a union

$$K \subset M_1 \subset M_2 \subset \cdots \subset \cup_{i>0} M_i = M, \tag{3.30.1}$$

where each $M_i \subset M_{i+1}$ is an elementary extension and d is decomposable.

A morphism $\rho : M \to A$ in $G - \mathrm{dga}_F$ is a *k–stage minimal model for A* if

(3.30.0.2) M is a minimal algebra generated in dimensions $\leq k$;

(3.30.0.3) ρ induces an isomorphism on cohomology in dimensions $\leq k$ and an injection in dimension $k + 1$.

Theorem 3.31. (i) *Any connected* $A \in G - \mathrm{dga}_F$ *admits a* k*–stage minimal model, for any integer* $k \in [0, \infty)$.

(ii) *For any* $A \in G-\mathrm{dga}_F$*, there exists an algebra* $M = \varinjlim M_k$ *and an equivalence* $\rho : M \to A$*, such that the restriction* $\rho|_{M_k} : M_k \to A$ *is a* k*–stage minimal model for* A*. Furthermore, there exists such an* M *so that the map* $M_k \to M_{k+1}$ *is obtained by a sequence (possibly infinite) of elementary extensions.*

Proof. (i) We proceed by induction on k. The case when $k = 0$ follows from the assumption that A is connected. So we assume given a k–stage minimal model $\rho_k : M_k \to A$ and construct a $k+1$–stage minimal model $\rho_{k+1} : M_{k+1} \to A$. Since $\mathrm{Rep}(G)_F$ is semi–simple, we can decompose $H^{k+1}(A)$ in the category $\mathrm{Rep}(G)_F$ as $H^{k+1}(M_k) \oplus V_{k+1}$. Furthermore, if Z_A^{k+1} denotes $\mathrm{Ker}(d : A^{k+1} \to A^{k+2})$, we can choose a map $V_{k+1} \to Z_A^{k+1}$ reducing to the inclusion into $H^{k+1}(A)$. Let $M_{k+1}^1 := M_k \otimes \Lambda(V_{k+1})$ with $d(V_{k+1}) = 0$, and let $\rho^1 : M_{k+1}^1 \to A$ be the map induced by $V_{k+1} \subset Z_A^{k+1}$. Then the map $H^i(M_{k+1}^1) \to H^i(A)$ is an isomorphism for $i \leq k+1$, but need not be an injection in degree $k+2$. Let $K_{k+1} \subset H^{k+2}(M_{k+1}^1)$ be the kernel of the map $H^{k+2}(M_{k+1}^1) \to H^{k+2}(A)$. Again since $\mathrm{Rep}(G)_F$ is semi–simple, we can choose a lifting $K_{k+1} \to Z^{k+2}_{M^1_{k+1}}$. Define $M^2_{k+1} := M^1_{k+1} \otimes \Lambda(K_{k+1})$, with $d|_{K_{n+1}} : K_{n+1} \to M^1_{k+1}$ being the inclusion into $Z^{k+2}_{M^1_{k+1}}$. Note that d is decomposable since M^1_{k+1} is generated in degrees $\leq k+1$.

Since the image of K_{k+1} in $H^{k+2}(A)$ is zero and $\mathrm{Rep}(G)_F$ is semi–simple, we can choose a map $K_{k+1} \to A_{k+1}$ such that the diagram

(3.31.1)
$$\begin{array}{ccc} K_{k+1} & \longrightarrow & Z^{k+2}_{M^1_{k+1}} \\ \downarrow & & \downarrow \\ A_{k+1} & \xrightarrow{d} & A_{k+2} \end{array}$$

commutes. Let $\rho^2 : M^2_{k+1} \to A$ be the resulting map. Then the algebra M^2_{k+1} is again generated in degrees $\leq k+1$. Repeating the construction inductively we obtain a sequence of minimal algebras in $G - \mathrm{dga}_F$

(3.31.2)
$$M_k \subset M_k^1 \subset M_k^2 \subset \cdots \longrightarrow A$$

so that each M_k^j is generated in degrees $\leq k+1$ and such that the image of $\mathrm{Ker}(H^{k+1}(M^j_{k+1}) \to H^{k+1}(A))$ in $H^{k+1}(M^{j+1}_{k+1})$ is zero. Defining $M_{k+1} := \varinjlim M^j_{k+1}$ we obtain the desired $(k+1)$–stage minimal model for A. Note that the construction of M_{k+1} only gives M_{k+1} as a direct limit over the ordinal 2ω. However, we leave it to the reader to verify that such a direct limit can be written as a union as in (3.30.1).

Statement (ii) follows from the proof of (i) by passage to the limit in k. □

Let us also note the following corollary of the construction:

Corollary 3.32. *If for every* $i \geq 0$*, the slopes of* $H^i(A)$ *are non–negative, then the algebra* M *in* (3.31 (ii)) *can be chosen so that the underlying* F*–isocrystal of* M *is a direct limit of finite–dimensional* F*–isocrystals with non–negative slopes.*

3.33. Let $(t, dt) \in G - \mathrm{dga}_F$ denote the differential graded algebra

$$d : K[t] \longrightarrow K[t] \cdot dt, \tag{3.33.1}$$

with trivial G–action and F–isocrystal structure defined by $t \mapsto t$. For any $B \in G - \mathrm{dga}_F$, we write $B(t, dt)$ for $B \otimes (t, dt)$. There are natural maps

$$s : B \longrightarrow B(t, dt), \quad p_i : B(t, dt) \longrightarrow B, \quad i = 0, 1, \tag{3.33.2}$$

where p_i is defined by sending dt to zero and t to i, and $s(b) = b \otimes 1$. Note that the composite $(p_0 \times p_1) \circ s : B \to B \times B$ is just the diagonal.

Lemma 3.34. *For any B, the algebra $B(t, dt)$ is a path object* ([G-J], p. 71) *for B in $G - \mathrm{dga}_F$.*

Proof. Since the natural morphism of complexes

$$K \longrightarrow (K[t] \to K[t]dt) \tag{3.34.1}$$

is a quasi–isomorphism, the map s is an equivalence. Furthermore, the map

$$p_0 \times p_1 : B(t, dt) \to B \times B \tag{3.34.2}$$

is easily seen to be an epimorphism and hence $p_0 \times p_1$ is a fibration. □

3.35. Define a *homotopy* between two maps $f, g : A \to B$ in $G - \mathrm{dga}_F$ to be a map

$$H : A \longrightarrow B(t, dt), \tag{3.35.1}$$

such that $H|_{t=0} = f$ and $H|_{t=1} = g$ (here we write $H|_{t=i}$ for $p_i \circ H$). The previous lemma combined with ([G-J], II.1.9) shows that if $A \in G - \mathrm{dga}_F$ is cofibrant and B is fibrant, then two maps $f, g : A \to B$ induce the same map in the homotopy category $\mathrm{Ho}(G - \mathrm{dga}_F)$ if and only if they are homotopic in the above sense.

Theorem 3.36. *Given a diagram in $G - \mathrm{dga}_F$*

$$\begin{array}{ccc} & & A \\ & & \big\downarrow{\scriptstyle \varphi} \\ M & \xrightarrow{f} & B, \end{array} \tag{3.36.1}$$

with M a minimal algebra and φ an equivalence, there exists a map $\tilde{f} : M \to A$ such that $\varphi \circ \tilde{f}$ is homotopic to f. Moreover, any two such liftings $\tilde{f}$ are homotopic and if φ is surjective then there exists a lifting $\tilde{f}$ such that $f = \varphi \circ \tilde{f}$.

Theorem (3.36) follows from the following (3.37) and (3.39).

Let $N = M \otimes \Lambda(V_n)$ be an elementary extension, $f, g : N \to A$ two maps, and $H : M \to A(t, dt)$ a homotopy between $f|_M$ and $g|_M$.

Proposition 3.37. *The map H extends to a homotopy $\overline{H} : N \to A(t, dt)$ between f and g if and only if the map*

$$V_n \longrightarrow H^n(A) \tag{3.37.1}$$

sending $x \in V_n$ to the class of

$$\sigma(x) := f(x) - g(x) + (-1)^{\deg(x)} \int_{t=0}^{t=1} H(dx) \tag{3.37.2}$$

is zero.

Proof. This follows from the same argument used in ([Mo], 5.7) and ([DGMS]) using the fact that $\mathrm{Rep}(G)_F$ is semisimple.

Let us just note that if the map (3.37.1) is zero the choice of a map $\rho : V_n \to A_{n-1}$ such that $d \circ \rho = \sigma$ defines an extension $\overline{H}$ by the formula

$$\overline{H}(x) := f(x) + (-1)^{\deg(x)} \left(\int_0^t H(dx) \right) - d(\rho(x) \otimes t). \tag{3.37.3}$$

□

Remark 3.38. Not every extension $\overline{H} : N \to A(t, dt)$ is obtained from a map $\rho : V_n \to A_{n-1}$ as in (3.37.3). However if $v \in V_n$, we can write

$$\overline{H}(v) = (\sum_i \alpha_i t^i, \sum_j \beta_j t^j \cdot dt) \in (A(t, dt))_n \simeq (A_n \otimes K[t]) \oplus (A_{n-1} \otimes K[t] \cdot dt), \tag{3.38.1}$$

so that

$$H(dv) = (\sum_i d(\alpha_i) t^i, \sum_{i \geq 1} (d(\beta_{i-1}) + (-1)^n i \alpha_i) t^{i-1} \cdot dt), \tag{3.38.2}$$

and

$$(-1)^n \int_0^t H(dv) = \sum_{i \geq 1} (\alpha_i + (-1)^n \frac{d\beta_{i-1}}{i}) t^i. \tag{3.38.3}$$

Let $\rho : V \to A_{n-1} \otimes K[t]$ be the map sending v to $(-1)^n \sum_j \frac{\beta_j}{j+1} t^j$. Then we see that

$$\alpha_0 + (-1)^n \int_0^t H(dv) - d(\rho(v) \otimes t) = (\sum \alpha_i t^i, \sum_j \beta_j t^j \cdot dt) = \overline{H}(v). \tag{3.38.4}$$

It follows that any extension $\overline{H}$ of H to N is given by the formula (3.37.3) for some map $\rho : V_n \to A_{n-1} \otimes K[t]$.

Proposition 3.39. *Let $M \to M \otimes_d \Lambda(V_n)$ be an elementary extension, and*

$$\begin{array}{ccc} M & \xrightarrow{\psi} & A \\ \downarrow & & \varphi \downarrow \\ M \otimes_d \Lambda(V_n) & \xrightarrow{\rho} & B \end{array} \tag{3.39.1}$$

a diagram in $G - \mathrm{dga}_F$, with φ an equivalence. Let $H : M \to B(t, dt)$ be a homotopy between $\varphi \circ \psi$ and $\rho|_M$. Then there exists a morphism $\lambda : M \otimes_d \Lambda(V_n) \to A$ extending ψ and a homotopy $\tilde{H}$ between $\varphi \circ \lambda$ and ρ extending H. If the diagram (3.39.1) *is commutative and if φ is also an epimorphism, then there exists a lifting λ such that $\varphi \circ \lambda = \rho$.*

Proof. This follows from the same argument used in ([DGMS], 1.2). □

Corollary 3.40. *A minimal algebra M is cofibrant in the model category $G - \mathrm{dga}_F$.*

Corollary 3.41. *Any two k–stage minimal models for an algebra $A \in G - \mathrm{dga}_F$ are isomorphic by an isomorphism unique up to homotopy.*

Proof. If $M \to A$ and $M' \to A$ are two k–stage minimal models for A, then (3.36) gives a canonical map $M \to M'$ (up to homotopy). That this map is an isomorphism can be verified after forgetting the G and F actions in which case it follows from ([DGMS], 1.6). □

Corollary 3.42. *Any object $A \in \mathrm{Ho}(G - \mathrm{dga}_F)$ admits a cofibrant model M which is a direct limit $\varinjlim M_k$ of cofibrant algebras M_k such that the morphism $M_k \to M_{k+1}$ is obtained by a sequence of elementary extensions.*

Proof. This follows from (3.31 (ii)) and (3.40). □

Remark 3.43. If G/K is any reductive group over a field K of characteristic 0, then all of the above results hold also for the category $G - \mathrm{dga}$ by exactly the same arguments. The assumptions on K are only used to assert that $\mathrm{Rep}(G)_F$ is semi–simple. In particular any $A \in G - \mathrm{dga}_F$ admits a cofibrant model whose underlying G–equivariant algebra is also cofibrant in $G - \mathrm{dga}$.

Remark 3.44. The preceding discussion could equally well have been carried out with G–equivariant algebras in $\mathbb{U}$ and the category $\mathrm{Ho}(G - \mathrm{dga}_F)_{\mathbb{U}}$ of objects isomorphic to algebras in $\mathbb{U}$.

Lifting Frobenius action in $\mathrm{Ho}(G - \mathrm{dga})$ to $G - \mathrm{dga}$.

3.45. Let $\mathcal{F}'$ denote the groupoid of objects $A \in \mathrm{Ho}(G - \mathrm{dga}_F)$ with $H^0(A) \simeq K$ (as an F–isocrystal), and let $\mathcal{F}$ denote the groupoid of pairs (F, φ), where $F \in \mathrm{Ho}(G - \mathrm{dga})$ and $\varphi : F^{(\sigma)} \to F$ is an isomorphism in $\mathrm{Ho}(G - \mathrm{dga})$ such that for every i the pair $(H^i(F), H^i(\varphi))$ is a direct limit of F–isocrystals of finite type and $H^0(F) \simeq K$ as an F–isocrystal. There is a natural functor

$$\Xi : \mathcal{F}' \longrightarrow \mathcal{F} \tag{3.45.1}$$

which sends A to the underlying G–equivariant algebra of A with the morphism $A^\sigma \to A$ induced by the F–isocrystal structure on A. In what follows, we refer to a morphism $\varphi : A^\sigma \to A$, where $A \in G - \mathrm{dga}$, as a *Frobenius structure* if it makes (A, φ) an object of $G - \mathrm{dga}_F$.

The main result of this subsection is the following:

Theorem 3.46. *The functor Ξ is an equivalence of categories.*

The proof of (3.46) will be in several steps (3.47)–(3.59).

3.47. First we prove that Ξ is faithful ((3.47)–(3.55)). For this, let $(M, \varphi) \in G - \mathrm{dga}_F$ be a connected algebra, and assume that $M = \varinjlim M_k$, where $M_k \to M$ is a k–stage minimal model for (M, φ) and $M_{k+1} = \varinjlim M_k^i$ with $M_k^0 = M_k$ and $M_k^{i+1} = M_k^i \otimes_d \Lambda(V_{k+1})$. By (3.39) (M, φ) is cofibrant in $G - \mathrm{dga}_F$ and by (3.42) every object of $\mathrm{Ho}(G - \mathrm{dga}_F)$ admits a representative of this form. Fix an automorphism $\xi : (M, \varphi) \to (M, \varphi)$ inducing the identity on $H^*(M)$.

Recall that a derivation (of degree 0) $D : M \to M$ is map compatible with the differential such that $D(ab) = aD(b) + D(a)b$ for all $a, b \in M$.

Lemma 3.48. *For every $m \in M$ there exists an integer N such that $(\xi - \mathrm{id})^N(m) = 0$. The resulting endomorphism $D := \log(\xi) : M \to M$ is a derivation compatible with Frobenius.*

Proof. We show by induction that the lemma holds for m in the image of M_k. The case $k = 0$ is trivial since $M_0 = K$ so we assume the result for M_k and prove it for M_{k+1}. For this it suffices in turn to show that the result holds for the image of M_k^i and this we show by induction on i. This reduces the proof to showing that if the result holds for M_k^i then it also holds for $M_k^{i+1} = M_k^i \otimes_d \Lambda(V_{k+1})$. For this it suffices to show that for every $v \in V_{k+1}$ there exists an integer N such that $(\xi - \mathrm{id})^N(v) = 0$. Since $dv \in M_k^i$ there exists by induction an integer N_1 such that $(\xi - \mathrm{id})^{N_1}(dv) = 0$. Let $N_2 = N_1 + 1$. Then $d(\xi - \mathrm{id})^{N_2}(v) = 0$ and the corresponding cohomology class in $H^{k+1}(M)$ is equal to $\xi - \mathrm{id}$ applied to the cohomology class of $(\xi - \mathrm{id})^{N_1}(v)$. Since ξ acts as the identity on cohomology there exists an element $b \in M$ of degree k with $db = (\xi - \mathrm{id})^{N_2}(v)$. Since $M_k \to M$ induces an isomorphism in degrees $\leq k$ it follows that there exists an integer M such that $(\xi - \mathrm{id})^M(b) = 0$ and hence setting $N = M + N_2$ we find that $(\xi - \mathrm{id})^N(v) = 0$. We leave to the reader the verification that $\log(\xi)$ is a derivation.

□

Lemma 3.49. *With notation as in the preceding lemma, if $\widetilde{D} : M(t, dt) \to M(t, dt)$ is a (t, dt)–linear derivation (not necessarily compatible with Frobenius) such that the composite*

$$M \xrightarrow{\Delta} M(t, dt) \xrightarrow{\widetilde{D}} M(t, dt) \xrightarrow{t \mapsto 1} M \tag{3.49.1}$$

equals D, then for every $m \in M(t, dt)$ there exists an integer N such that $\widetilde{D}^N(m) = 0$.

Proof. We show inductively that the result holds for the image of $M_k(t, dt)$. The case $k = 0$ is trivial since by assumption $\widetilde{D}$ kills $M_0(t, dt) = (t, dt)$. So assume the result holds for $M_k(t, dt)$. We then show inductively that the result holds for $M_k^i(t, dt)$. So we assume the result also holds for $M_k^i(t, dt)$ and show that it holds for $M_k^{i+1}(t, dt) = M_k^i(t, dt) \otimes_d \Lambda(V_{k+1})$. For this it suffices to show that for every $v \in V_{k+1}$ there exists an integer N such that $\widetilde{D}^N(v) = 0$. Since the result holds for dv by induction, there exists an integer N_1 such that $\widetilde{D}^{N_1}(v)$ is closed. Since D is nilpotent, there exists an integer M such that $D^M(\widetilde{D}^{N_1}(v)|_{t=1}) = 0$. Since the map $t \mapsto 1$ induces an isomorphism $H^{k+1}(M(t, dt)) \to H^{k+1}(M)$ it follows that the closed element $\widetilde{D}^{M+N_1}(v)$ is a boundary. Let $b \in M(t, dt)$ be an element of degree k such that $db = \widetilde{D}^{M+N_1}(v)$. Since b is in the image of $M_k(t, dt)$ there exists an integer N_2 with $\widetilde{D}^{N_2}(b) = 0$. Setting $N = M + N_1 + N_2$ we have $\widetilde{D}^N(v) = 0$. □

Lemma 3.50. *If $H : M \to M(t, dt)$ is a homotopy (not necessarily respecting Frobenius) between* id *and ξ then the induced (t, dt)–linear map $\overline{H} : M(t, dt) \to M(t, dt)$ is an isomorphism and $\overline{H} = \exp(\widetilde{D})$ for some derivation $\widetilde{D} : M(t, dt) \to M(t, dt)$ as in* (3.49).

Proof. Note first that for any $m \in M$ we have $H(m) = m + t\alpha + \beta dt$ for some $\alpha, \beta \in M(t, dt)$ since $H|_{t=0}$ is the identity. This implies that $\overline{H}$ is injective. To see this let $\omega = \sum_i \alpha_i t^i +$

$\sum_j \beta_j t^j dt$ be an element in the kernel of $\overline{H}$. If there exists $\alpha_i \neq 0$, let i_0 be the smallest integer for which $\alpha_i \neq 0$. Then

$$\overline{H}(\sum \alpha_i t^i + \sum_j \beta_j t^j dt) = \alpha_{i_0} t^{i_0} + \gamma t^{i_0+1} + \delta dt \tag{3.50.1}$$

for some $\gamma, \delta \in M(t, dt)$. This is a contradiction so all $\alpha_i = 0$. In this case if $\omega \neq 0$ define j_0 to be the smallest integer with $\beta_j \neq 0$. Then

$$\overline{H}(\sum_j \beta_j t^j dt) = (\beta_{j_0} t^{j_0} + \gamma t^{j_0+1})dt \tag{3.50.2}$$

for some $\gamma \in M(t, dt)$. This again is a contradiction so $\omega = 0$.

To prove that $\overline{H}$ is surjective, we show inductively that the image of $\overline{H}$ contains $M_k(t, dt)$. The case $k = 0$ is immediate since $\overline{H}$ is (t, dt)–linear. As before we are then reduced to showing that if the image of $\overline{H}$ contains $M_k^i(t, dt)$ then it also contains $M_k^{i+1}(t, dt) = M_k^i(t, dt) \otimes_d \Lambda(V_{k+1})$. For this it suffices to show that every element $v \in V_{k+1}$ is in the image. By induction dv is in the image so there exists an element $m \in M(t, dt)$ with $\overline{H}(m) = dv$. Since $\overline{H}$ is injective and $d^2 v = 0$ the element m must be closed, and since $\overline{H}$ induces an isomorphism on cohomology the element m must be a boundary. Thus there exists an element $n \in M(t, dt)$ such that $v - \overline{H}(n)$ is closed. Since $\overline{H}$ induces an isomorphism on cohomology there exists $n' \in M(t, dt)$ such that $v - \overline{H}(n + n')$ is a boundary. Let $b \in M(t, dt)$ be an element with $db = v - \overline{H}(n + n')$. Then since $M_k(t, dt) \to M(t, dt)$ induces an isomorphism in degrees $\leq k$ the element b is of the form $\overline{H}(b')$ so $v = \overline{H}(n+n') + \overline{H}(db')$. This proves that $\overline{H}$ is surjective and hence an isomorphism.

Since the isomorphism $\overline{H}$ is obtained from a homotopy between ξ and the identity, it induces the identity on $H^*(M(t, dt))$. By the same reasoning as in the proof of (3.48), it follows that for every element $m \in M(t, dt)$ there exists an integer N such that $(\overline{H} - \mathrm{id})^N(m) = 0$ (to start the induction note that $\overline{H}$ is the identity on $M_0(t, dt)$). Let $\widetilde{D} : M(t, dt) \to M(t, dt)$ denote $\log(\overline{H})$. As in the proof of (3.48) this is a derivation and by definition it satisfies the conditions of (3.49) and $\exp(\widetilde{D}) = \overline{H}$. □

3.51. Let $\mathcal{D}(M)$ (resp. $\mathcal{D}(M(t, dt))$) denote the K–vector space of K–linear (resp. (t, dt)–linear) derivations $M \to M$ (resp. $M(t, dt) \to M(t, dt)$) compatible with the action of G (but not necessarily respecting the Frobenius action). Let Φ_M (resp. $\Phi_{M(t,dt)}$) denote the composite

$$M \xrightarrow{\text{can}} M^\sigma \xrightarrow{\varphi} M \quad (\text{resp. } M(t, dt) \xrightarrow{\text{can}} M(t, dt)^\sigma \xrightarrow{\varphi_{M(t,dt)}} M(t, dt)). \tag{3.51.1}$$

There is a natural semi–linear automorphism $\Phi_{\mathcal{D}(M)}$ (resp. $\Phi_{\mathcal{D}(M(t,dt))}$) of $\mathcal{D}(M)$ (resp. $\mathcal{D}(M(t, dt))$) which sends a derivation ∂ to $\Phi_M \circ \partial \circ \Phi_M^{-1}$ (resp. $\Phi_{M(t,dt)} \circ \partial \circ \Phi_{M(t,dt)}^{-1}$).

Lemma 3.52. *The pairs $(\mathcal{D}(M), \varphi_{\mathcal{D}(M)})$ and $(\mathcal{D}(M(t, dt)), \varphi_{\mathcal{D}(M(t,dt))})$ have a natural structure of projective limits of ind–objects in the category* $\mathrm{FIsoc}(k/K)$.

Proof. Note that there are natural inclusions

$$\mathcal{D}(M) \subset \mathrm{Hom}_{\mathrm{Vec}_K}(M, M), \qquad \mathcal{D}(M(t, dt)) \subset \mathrm{Hom}_{\mathrm{Vec}_K}(M, M(t, dt)) \tag{3.52.1}$$

compatible with Frobenius structures. Since M is a direct limit of finite dimensional F–isocrystals the lemma follows. $\square$

3.53. Let $\mathrm{pr}_i : \mathcal{D}(M(t,dt)) \to \mathcal{D}(M)$ $(i = 0,1)$ be the map sending $\partial : M(t,dt) \to M(t,dt)$ to the composite

$$M \xrightarrow{\Delta} M(t,dt) \xrightarrow{\partial} M(t,dt) \xrightarrow{t \mapsto i} M, \tag{3.53.1}$$

and let

$$\rho : \mathcal{D}(M(t,dt)) \longrightarrow \mathcal{D}(M) \times \mathcal{D}(M) \tag{3.53.2}$$

be the map $\mathrm{pr}_0 \times \mathrm{pr}_1$ of pro–F–isocrystals. Denote by $I \subset \mathcal{D}(M) \times \mathcal{D}(M)$ the image of ρ. The space I is again a pro–F–isocrystal.

Since M is a direct limit of finite dimensional F–isocrystals, there exists an ordinal λ and a λ–sequence $\{V_n\}_{n\in\lambda}$ of sub–F–isocrystals $V_n \subset M$ (not necessarily finite dimensional) with $M = \varinjlim_\lambda V_n$ such that for every $n \in \lambda$ the quotient V_{n+1}/V_n is a finite dimensional F–isocrystal and such that for the smallest element $0 \in \lambda$ the space V_0 is finite dimensional. Furthermore, we may assume that if $\alpha \in \lambda$ is a limit ordinal then $V_\alpha = \varinjlim_{\alpha'<\alpha} V_{\alpha'}$. The projection $\rho : \mathcal{D}(M(t,dt)) \to I$ is then identified with the map obtained from a morphism of projective systems

(3.53.3)

$$\{\rho_{V_n}\} : \{\mathrm{Der}_{(t,dt)}(M_{V_n}(t,dt), M(t,dt))\} \to \{I_{V_n}\} \subset \{\mathrm{Der}_K(M_{V_n}, M) \times \mathrm{Der}_K(M_{V_n}, M)\},$$

where M_{V_n} denotes the differential graded subalgebra generated by V_n.

Lemma 3.54. *The map $\rho : \mathcal{D}(M(t,dt)) \to I$ admits a section compatible with the Frobenius structures.*

Proof. Let $\mathcal{D}_{V_n} \subset \mathrm{Der}_{(t,dt)}(M_{V_n}(t,dt), M(t,dt))$ be the image of $\mathrm{Der}_{(t,dt)}(M(t,dt), M(t,dt))$ and let $I'_{V_n} \subset I_{V_n}$ be the image of I. For every $n \in \lambda$, the kernel K_{n+1} (resp. K^*_{n+1}) of $\mathcal{D}_{V_{n+1}} \to \mathcal{D}_{V_n}$ (resp. $I'_{V_{n+1}} \to I_{V_n}$) injects into $\mathrm{Hom}_{\mathrm{Vec}_K}(V_{n+1}/V_n, M(t,dt))$ (resp. $\mathrm{Hom}_{\mathrm{Vec}_K}(V_{n+1}/V_n, M)$) and in particular is an ind–object in $\mathrm{FIsoc}(k/K)$.

Let $W_{n+1} \subset V_{n+1}$ be a finite dimensional F–isocrystal such that the projection $W_{n+1} \to V_{n+1}/V_n$ is surjective, and set $W_n := W_{n+1} \cap V_n$. Then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & K_{n+1} & \to & \mathcal{D}_{V_{n+1}} & \to & \mathcal{D}_{V_n} & \to & 0 \\ & & \downarrow{\scriptstyle j} & & \downarrow & & \downarrow & & \\ 0 & \to & \widetilde{K}_{n+1} & \to & \mathrm{Der}(M_{W_{n+1}}, M(t,dt)) & \xrightarrow{\delta} & \mathrm{Der}(M_{W_n}, M(t,dt)) & \to & 0, \end{array} \tag{3.54.1}$$

where $\widetilde{K}_{n+1}$ is defined to be the kernel of δ. Moreover, by construction the map j is injective. Since the category of ind–objects in $\mathrm{FIsoc}(k/K)$ is semisimple we can choose a complement $\widetilde{K}_{n+1} = K_{n+1} \oplus K'_{n+1}$. Let

$$0 \longrightarrow K_{n+1} \longrightarrow P \xrightarrow{\delta'} \mathrm{Der}(M_{W_n}, M(t,dt)) \longrightarrow 0 \tag{3.54.2}$$

be the exact sequence obtained from the bottom row of (3.54.1) by pushout along the projection $\widetilde{K}_{n+1} \to K_{n+1}$ determined by the splitting. Then the diagram

$$\begin{array}{ccc} \mathcal{D}_{V_{n+1}} & \longrightarrow & \mathcal{D}_{V_n} \\ \downarrow & & \downarrow \\ P & \xrightarrow{\delta'} & \mathrm{Der}(M_{W_n}, M(t,dt)) \end{array} \tag{3.54.3}$$

is cartesian. Furthermore, δ' is a map of ind–objects in $\mathrm{FIsoc}(k/K)$ and in particular the projection $P \to \delta'(P)$ admits a section. This implies that the map $\mathcal{D}_{V_{n+1}} \to \mathcal{D}_{V_n}$ also admits a section.

By the same argument the projection $I'_{V_{n+1}} \to I'_{V_n}$ also admits a section.

To construct the section s of ρ it suffices to show that for every $n \in \lambda$ and section s_n of $\mathcal{D}_{V_n} \to I'_{V_n}$ there exists a section s_{n+1} of $\mathcal{D}_{V_{n+1}} \to I'_{V_{n+1}}$ compatible with s_n. To see that this suffices assume to the contrary that this holds but no section of ρ exists. Let $\mathcal{S}$ be the set of pairs (n, s_n), where $n \in \lambda$ and s_n is a section of $\mathcal{D}_{V_n} \to I'_{V_n}$. Define a partial order on $\mathcal{S}$ by declaring $(n, s_n) \geq (n', s_{n'})$ if $n \geq n'$ and s_n is compatible with $s_{n'}$. If $T \subset \mathcal{S}$ is a totally ordered subset then there exists $\alpha \in \lambda$ such that $\alpha > t$ for all $t \in T$. For if no such α exists then there is an isomorphism of functors $\varprojlim_\lambda = \varprojlim_T$ and a section of ρ exists. Thus there exists a minimal element α_0 with the property that $\alpha_0 > t$ for all $t \in T$. On the other hand, since α_0 is minimal there exists an element $t \in T$ such that either $\alpha_0 = t + n$ for some $n \in \mathbb{N}$ or α_0 is a limit ordinal and $V_{\alpha_0} = \varinjlim_{n \in \mathbb{N}} V_{t+n}$. In either case it follows that the map $\mathcal{D}_{V_{\alpha_0}} \to I'_{V_{\alpha_0}}$ admits a section compatible with the sections defined by T. Thus every totally ordered subset $T \subset \mathcal{S}$ admits an upper bound in $\mathcal{S}$ and so by Zorn's lemma there exists a maximal element $(n_{\max}, s_{n_{\max}}) \in \mathcal{S}$. Since ρ by assumption does not admit a section, there exists an element $\alpha \in \lambda$ with $\alpha > n_{\max}$. But then repeating the argument of why every totally ordered subset of $\mathcal{S}$ admits an upper bound we see that if we choose α minimal then there exists a section of $\mathcal{D}_{V_\alpha} \to I'_{V_\alpha}$. This contradicts the maximality of $(n_{\max}, s_{n_{\max}})$ so a section of ρ must exist.

So we assume s_n has been constructed giving a decomposition $\mathcal{D}_{V_n} = T_n \oplus I'_{V_n}$ and construct s_{n+1}. Let $W \subset \mathcal{D}_{V_{n+1}}$ be the inverse image of $T_n \subset \mathcal{D}_{V_n}$.

The choice of a section of $\mathcal{D}_{V_{n+1}} \to \mathcal{D}_{V_n}$ induces a splitting $\mathcal{D}_{V_{n+1}} = W \oplus I'_{V_n}$ which via the projection to $I'_{V_{n+1}}$ induces a splitting $I'_{V_{n+1}} \simeq I'_{V_n} \oplus K^*_{n+1}$. Furthermore, the map $W \to K^*_{n+1}$ is surjective. Since K^*_{n+1} is an ind–object in $\mathrm{FIsoc}(k/K)$ this map admits a section, and the choice of such a section gives the desired extension s_{n+1}. □

Corollary 3.55. *The functor Ξ is faithful.*

Proof. Suppose $f, g : (M, \varphi) \to (M', \varphi')$ are two isomorphisms in $\mathrm{Ho}(G - \mathrm{dga}_F)$ inducing the same map in $\mathrm{Ho}(G - \mathrm{dga})$, where (M, φ) is as in (3.47). Then $\xi := g^{-1} \circ f$ is an automorphism of (M, φ) in $\mathrm{Ho}(G - \mathrm{dga}_F)$ homotopic to the identity in $\mathrm{Ho}(G - \mathrm{dga})$. Since (M, φ) is both fibrant and cofibrant in $G - \mathrm{dga}_F$ (3.40), the map ξ can be represented by an automorphism (denoted by the same letter) $\xi : (M, \varphi) \to (M, \varphi)$ in $G - \mathrm{dga}_F$ such that there exists a homotopy $H : M \to M(t, dt)$ between id and ξ in $G - \mathrm{dga}$. By (3.48) there exists a derivation $D : M \to M$ compatible with φ such that $\xi = \exp(D)$. Since ξ is homotopic to the identity in $G - \mathrm{dga}$, the pair $(0, D)$ is by (3.50) in the image of the map

$\rho : \mathcal{D}(M(t,dt)) \to I \subset \mathcal{D}(M) \times \mathcal{D}(M)$. By (3.54), there exists a (t,dt)–linear derivation $\widetilde{D} : M(t,dt) \to M(t,dt)$ which is compatible with the Frobenius structures and maps to $(0,D)$ under ρ. By (3.49) we can therefore define a homotopy $H : M \to M(t,dt)$ between ξ and id in $G - \mathrm{dga}_F$ by $m \mapsto \exp(\widetilde{D})(m)$ (we leave to the reader the verification that $\exp(\widetilde{D})$ is compatible with the algebra structures). □

To prove that Ξ is fully faithful, we actually prove something stronger:

Proposition 3.56. *Let $(M,\varphi_M),(D,\varphi_D) \in G - \mathrm{dga}_F$ be two objects with (M,φ_M) as in (3.47), and (f,H) a pair consisting of a morphism $f : M \to D$ in $G - \mathrm{dga}$ and a homotopy $H : M^\sigma \to D(t,dt)$ between $\varphi_D \circ f^\sigma$ and $\varphi_M \circ f$. Then the morphism in $\mathrm{Ho}(G - \mathrm{dga})$ induced by f is obtained from a morphism $(M,\varphi_M) \to (D,\varphi_D)$ in $\mathrm{Ho}(G - \mathrm{dga}_F)$.*

Proof. Write $M = \varinjlim M_k$ with $M_{k+1} = \varinjlim M_k^i$ as in (3.47).

It suffices to find a factorization of f

(3.56.1) $$M \xrightarrow{\epsilon} C \xrightarrow{\rho} D,$$

Frobenius structures φ_C and φ'_C on C such that the diagrams

(3.56.2) $$\begin{array}{ccccccc} M^\sigma & \xrightarrow{\epsilon^\sigma} & C^\sigma & & C^\sigma & \xrightarrow{\rho^\sigma} & D^\sigma \\ {\scriptstyle\varphi_M}\downarrow & & \downarrow{\scriptstyle\varphi_C} & & \downarrow{\scriptstyle\varphi'_C} & & \downarrow{\scriptstyle\varphi_D} \\ M & \xrightarrow{\epsilon} & C & & C & \xrightarrow{\rho} & D \end{array}$$

commute and a homotopy

(3.56.3) $$H_C : C^\sigma \longrightarrow C(t,dt)$$

between φ_C and φ'_C such that the induced (t,dt)–linear map

(3.56.4) $$C^\sigma(t,dt) \longrightarrow C(t,dt)$$

is an isomorphism, and the pair $(E,\varphi_E) := (C(t,dt),(3.56.4))$ is a direct limit of F–isocrystals of finite type (i.e. an object of $G - \mathrm{dga}_F$). For then the morphism in $\mathrm{Ho}(G - \mathrm{dga}_F)$ defined by the maps

(3.56.5) $$(M,\varphi_M) \longrightarrow (C,\varphi_C) \xleftarrow{t=0} (E,\varphi_E) \xrightarrow{t=1} (C,\varphi'_C) \longrightarrow (D,\varphi_D)$$

is a morphism in $\mathrm{Ho}(G - \mathrm{dga}_F)$ inducing f.

To construct the above factorization of $M \to D$, it suffices to inductively construct compatible factorizations of the maps $M_k \to D$. We can therefore rephrase the problem as follows. Suppose given a morphism $f : A \to D$ in $G - \mathrm{dga}$, Frobenius structures φ_A and φ_D on A and D respectively, and a homotopy

(3.56.6) $$H_A : A^\sigma \longrightarrow D(t,dt)$$

between the two maps

(3.56.7) $$f \circ \varphi_A, \varphi_D \circ f^\sigma : A^\sigma \longrightarrow D.$$

Let $\tilde{A} = A \otimes_d \Lambda(V_n)$ be an elementary extension with a Frobenius structure $\varphi_{\tilde{A}}$ restricting to φ_A on A, and suppose given an extension of the map f to a map $\tilde{f} : \tilde{A} \to D$ such that the maps

$$\tilde{f} \circ \varphi_{\tilde{A}}, \varphi_D \circ \tilde{f}^\sigma : \tilde{A}^\sigma \longrightarrow D \tag{3.56.8}$$

are homotopic by a homotopy

$$H_{\tilde{A}} : \tilde{A}^\sigma \longrightarrow D(t, dt) \tag{3.56.9}$$

extending H_A.

Suppose given a factorization of f

$$A \xrightarrow{\epsilon} C \xrightarrow{\rho} D, \tag{3.56.10}$$

and Frobenius structures φ_C and φ'_C on C such that the diagrams

$$\begin{array}{ccccccc} A^\sigma & \xrightarrow{\epsilon^\sigma} & C^\sigma & & C^\sigma & \xrightarrow{\rho^\sigma} & D^\sigma \\ \varphi_A \downarrow & & \downarrow \varphi_C & & \downarrow \varphi'_C & & \downarrow \varphi_D \\ A & \xrightarrow{\epsilon} & C & & C & \xrightarrow{\rho} & D \end{array} \tag{3.56.11}$$

commute. Suppose further given a homotopy

$$H_C : C^\sigma \longrightarrow C(t, dt) \tag{3.56.12}$$

between φ_C and $\varphi_{C'}$ such that the diagram

$$\begin{array}{ccc} C^\sigma & \xrightarrow{H_C} & C(t, dt) \\ \uparrow & & \downarrow \\ A^\sigma & \xrightarrow{H_A} & D(t, dt) \end{array} \tag{3.56.13}$$

commutes.

The following lemma and induction now proves (3.56). □

Lemma 3.57. *There exists a commutative diagram*

$$\begin{array}{ccccc} \tilde{A} & \xrightarrow{\tilde{\epsilon}} & \widetilde{C} & \xrightarrow{\tilde{\rho}} & D \\ \uparrow & & \lambda \uparrow & & \text{id} \uparrow \\ A & \xrightarrow{\epsilon} & C & \xrightarrow{\rho} & D \end{array} \tag{3.57.1}$$

such that $\tilde{\rho} \circ \tilde{\epsilon} = \tilde{f}$, Frobenius structures $\varphi_{\widetilde{C}}$ and $\varphi'_{\widetilde{C}}$ on $\widetilde{C}$ compatible with φ_C and φ'_C, such that the morphisms

$$(\tilde{A}, \varphi) \longrightarrow (\widetilde{C}, \varphi_{\widetilde{C}}), \quad (\widetilde{C}, \varphi'_{\widetilde{C}}) \longrightarrow (D, \varphi_D) \tag{3.57.2}$$

are morphisms in $G - \mathrm{dga}_F$. In addition, we can find data as above such that there exists a homotopy between $\varphi_{\widetilde{C}}$ and $\varphi'_{\widetilde{C}}$

$$H_{\widetilde{C}} : \widetilde{C}^\sigma \longrightarrow \widetilde{C}(t, dt) \tag{3.57.3}$$

for which the diagrams

(3.57.4)
$$\begin{array}{ccccccc} \widetilde{C}^\sigma & \xrightarrow{H_{\widetilde{C}}} & \widetilde{C}(t,dt) & & \widetilde{C}^\sigma & \xrightarrow{H_{\widetilde{C}}} & \widetilde{C}(t,dt) \\ \uparrow & & \uparrow & & \uparrow & & \downarrow \\ C^\sigma & \xrightarrow{H_C} & C(t,dt) & & \tilde{A}^\sigma & \xrightarrow{H_{\tilde{A}}} & D(t,dt) \end{array}$$

commute. If H_C induces an isomorphism

(3.57.5)
$$C^\sigma(t,dt) \longrightarrow C(t,dt),$$

then we can choose $\widetilde{C}$ and $H_{\widetilde{C}}$ such that the induced map

(3.57.6)
$$\widetilde{C}^\sigma(t,dt) \longrightarrow \widetilde{C}(t,dt)$$

is an isomorphism as well. If in addition the pair $(C(t,dt),(3.57.5))$ is in $G-\mathrm{dga}_F$ then we can choose $\widetilde{C}$ and $H_{\widetilde{C}}$ so that $(\widetilde{C}(t,dt),(3.57.6))$ is also in $G-\mathrm{dga}_F$.

Proof. Let B^n_D denote the boundaries of D in degree n. Define $C_1 := C\otimes_0 \Lambda(B^n_D)$ with $\varphi_{C_1} = \varphi_C\otimes\varphi_D$, $\varphi'_{C_1} = \varphi'_C\otimes\varphi_D$. Then there is a natural map $C_1 \to D$ in $G-\mathrm{dga}_F$ and a homotopy

(3.57.7)
$$H_{C_1} : C_1^\sigma \longrightarrow C_1(t,dt)$$

between φ_{C_1} and φ'_{C_1} extending H_C. In fact, we take for H_{C_1} the map extending H_C which sends $b\in B^n_D$ to $\varphi_D(b)$. Since φ_D induces an automorphism of B^n_D, if the map (3.57.5) is an isomorphism then so is the induced map

(3.57.8)
$$C_1^\sigma(t,dt) \longrightarrow C_1(t,dt),$$

and if $(C(t,dt),(3.57.5))$ is in $G-\mathrm{dga}_F$ then so is $(C_1(t,dt),(3.57.8))$.

Now define $C_2 := C_1\otimes_d \Lambda(D_{n-1})$, where $d : D_{n-1}\to B^n_D$ is the differential of D, and define

(3.57.9)
$$\varphi_{C_2} := \varphi_{C_1}\otimes\varphi_D, \qquad \varphi'_{C_2} := \varphi'_{C_1}\otimes\varphi_D.$$

Then there is a natural map $(C_2,\varphi'_{C_2}) \to (D,\varphi_D)$ of Frobenius algebras. Also, define

(3.57.10)
$$H_{C_2} : C_2^\sigma \longrightarrow C_2(t,dt)$$

to be the unique homotopy extending H_{C_1} which restrict to the map $h\mapsto \varphi_D(h)$ on D_{n-1}. Again since φ_D is an automorphism of D, if (3.57.5) is an isomorphism, then the map

(3.57.11)
$$C_2^\sigma(t,dt) \longrightarrow C_2(t,dt)$$

is also an isomorphism, and $(C_2(t,dt),(3.57.11)) \in G-\mathrm{dga}_F$ if $(C_1(t,dt),(3.57.8)) \in G-\mathrm{dga}_F$.

Set $\widetilde{C} := C_2\otimes_d \Lambda(V_n) = C_2\otimes_A \widetilde{A}$, and let $\varphi_{\widetilde{C}}$ be the extension of φ_{C_2} obtained from $\varphi_{\tilde{A}}$. Since $H_{\tilde{A}}$ exists, there exists by (3.37) a unique map $b : V_n \to B^n_D$ such that for each $v\in V_n$ we have equality in D_n

(3.57.12)
$$\varphi_D(f(v)) = f(\varphi_{\tilde{A}}(v)) + (-1)^n \int_{t=0}^{t=1} H_{\tilde{A}}(dv) + b(v).$$

Define $\varphi'_{\widetilde{C}}$ to be the unique extension of φ'_{C_2} which sends v to

(3.57.13)
$$\varphi_{\widetilde{C}}(v) + (-1)^n \int_{t=0}^{t=1} H_{C_2}(dv) + b(v).$$

Note that if
(3.57.14)
$$H_{C_2}(dv) = (\sum_i \alpha_i t^i, \sum_j \beta_j t^j \cdot dt) \in (C_2(t,dt))_{n+1} \simeq (C_{2,n+1} \otimes K[t]) \oplus (C_{2,n} \otimes K[t]) \cdot dt,$$

then the fact that $dH_{C_2}(dv) = 0$ implies that $\alpha_{j+1} = (-1)^n \frac{d\beta_j}{j+1}$ for every $j \geq 0$, and a calculation shows that

$$(-1)^n \int_0^1 H_{C_2}(dv) = (-1)^n \sum_j \frac{\beta_j}{j+1}. \quad (3.57.15)$$

Therefore,

$$d(\varphi'_{\widetilde{C}}(v)) = \alpha_0 + \sum_{i \geq 1} \alpha_i = H_{C_2}(dv)|_{t=1} = \varphi'_{C_2}(dv), \quad (3.57.16)$$

so $\varphi'_{\widetilde{C}}$ is a semi–linear endomorphism of the differential graded algebra $\widetilde{C}$.

In fact, $\varphi'_{\widetilde{C}}$ is an automorphism. To see this, note that $\varphi'_{\widetilde{C}}$ restricted to V_n is of the form $\gamma + \delta$, where γ is an automorphism of V_n and $\delta : V_n \rightarrow C_2$ is a map. Now the degree r piece of $\widetilde{C}$ admits a filtration, which is stable under $\varphi'_{\widetilde{C}}$, whose graded pieces are of the form $C_{2,r-ns} \otimes \Lambda^s(V_n)$, where we write $\Lambda^s(V_n)$ for the symmetric power if n is even and the exterior power if n is odd. The endomorphism induced by $\varphi'_{\widetilde{C}}$ on these graded pieces is just that induced by φ'_{C_2} and γ, and hence $\varphi'_{\widetilde{C}}$ is an automorphism.

Furthermore, the pair $(\widetilde{C}, \varphi'_{\widetilde{C}})$ is a direct limit of F–isocrystals of finite type. For this note that this is true for (C_2, φ_{C_2}) so it suffices to show that for any $v \in V_n$ there exists a finite–dimensional subspace $\widetilde{W} \subset \widetilde{C}$ stable by $\varphi'_{\widetilde{C}}$ containing v. Choose first a finite–dimensional subspace $W \subset \widetilde{C}$ stable under $\varphi_{\widetilde{C}}$ and containing v. Let $w_1, \ldots, w_r \in W$ be a basis for W and set $\kappa_i = \varphi'_{\widetilde{C}}(w_i) - \varphi_{\widetilde{C}}(w_i)$. By definition of $\varphi'_{\widetilde{C}}$ the elements κ_i lie in C_2. Thus there exists a finite–dimensional subspace $W' \subset C_2$ stable under $\varphi'_{\widetilde{C}}$ containing all the κ_i. Let $\widetilde{W} \subset \widetilde{C}$ be the subspace generated by W and W'. Any element $w \in \widetilde{W}$ can be written as a sum $\sum_{i=1}^r a_i w_i + \gamma$, where $a_i \in K$ and $\gamma \in W'$. We then have

$$\varphi'_{\widetilde{C}}(w) = \sum_{i=1}^r a_i \varphi_{\widetilde{C}}(w_i) + \sum_{i=1}^r a_i \kappa_i + \varphi'_{\widetilde{C}}(\gamma), \quad (3.57.17)$$

and by construction $\varphi_{\widetilde{C}}(w_i) \in W$, $\kappa_i \in W'$, and $\varphi'_{\widetilde{C}}(\gamma) \in W'$ whence $\varphi'_{\widetilde{C}}(w) \in \widetilde{W}$.

By (3.38) there exists a map

$$\rho : V_n \longrightarrow D_{n-1} \otimes K[t] \quad (3.57.18)$$

with $d(\rho(v)) = -b(v)$ for every $v \in V_n$ such that $H_{\widetilde{A}}$ is given by the formula (3.37.3). Using ρ, we obtain a homotopy $H_{\widetilde{C}}$ as in the lemma by the formula

$$H_{\widetilde{C}}(v) := \varphi_{\widetilde{C}}(v) + (-1)^{\deg(v)} \left(\int_0^t H_{C_2}(dv) \right) - d(\rho(v) \otimes t). \quad (3.57.19)$$

We claim that if the map (3.57.11) is an isomorphism, then so is the induced map

$$\theta : \widetilde{C}^\sigma(t,dt) \longrightarrow \widetilde{C}(t,dt). \quad (3.57.20)$$

To see this, note that there exists an automorphism ψ of V such that the map $H_{\widetilde{C}}$ sends $v \in V_n$ to $\psi(v)$ plus an element of $C_2(t, dt)$. Also observe that there is a natural isomorphism

$$\widetilde{C}(t, dt) \simeq (C_2(t, dt)) \otimes \Lambda(V_n). \tag{3.57.21}$$

As before, it follows that for each integer r, there exists a filtration F on $(\widetilde{C}(t, dt))_r$ for which $\theta(F^i) \subset F^i$ and such that the graded pieces F^i/F^{i-1} are isomorphic to $(C_2(t, dt))_{r-sn} \otimes \Lambda^s V_n$. Furthermore, the endomorphism on $(C_2(t, dt))_{r-sn} \otimes \Lambda^s V_n$ induced by θ is just that induced by $\theta|_{(C_2(t,dt))_{r-sn}}$ and ψ. Hence θ is an isomorphism.

Finally note that if $v \in V_n$ then $H_{\widetilde{C}}(v)$ is equal to $\varphi_{\widetilde{C}}(v)$ plus an element of $C_2(t, dt)$. Repeating the argument proving that $(\widetilde{C}, \varphi'_{\widetilde{C}})$ is a direct limit of F–isocrystals of finite type it follows that if $(C_2(t, dt), (3.57.11))$ is a direct limit of F–isocrystals of finite type then so is $(\widetilde{C}(t, dt), \theta)$. □

3.58. To complete the proof of (3.46), it remains only to show that Ξ is essentially surjective. For this let $A \in G-\mathrm{dga}$ be a connected object and $\varphi_A : A^\sigma \to A$ an equivalence. Without loss of generality, we can assume that A is as in (3.47) in which case φ_A is an isomorphism by ([DGMS], 1.6). Hence the following Lemma completes the proof of (3.46):

Lemma 3.59. *Let $M \in G-\mathrm{dga}$ be a minimal algebra as in (3.47), and $\varphi : M^\sigma \to M$ an isomorphism in $G-\mathrm{dga}$ such that for all $i \geq 0$ the pair $(H^i(M), H^i(\varphi))$ is a direct limit of F–isocrystals of finite type. Then (M, φ) is a direct limit of F–isocrystals of finite type.*

Proof. Write $M = \varinjlim M_k$, $M_{k+1} = \varinjlim M_k^i$ with $M_k^{i+1} = M_k^i \otimes_d \Lambda(V_{k+1})$ as in (3.47). We show inductively that every finite collection of elements of M contained in M_k is contained in a finite–dimensional φ–stable subspace. The case $k = 0$ is trivial, so we assume the result holds for M_k and prove it for M_{k+1}. Another induction therefore reduces us to proving the result for M_k^{i+1} assuming the result holds for M_k^i. For this it suffices to show that if $v_1, \ldots, v_r \in M_k^{i+1}$ are elements of degree $k+1$, then there exist a finite dimensional subspace $W \subset M_k^{i+1}$ containing the v_i which is φ–stable.

Consider first the case when the v_i are all closed. By assumption, the corresponding cohomology classes are contained in a φ–stable finite–dimensional subspace of $H^*(M)$. Choosing representatives for a basis of such a finite–dimensional subspace we can therefore find a finite–dimensional subspace of closed elements $W \subset M$ containing all the v_i such that for any element $w \in W$ there exists $w' \in W$ such that $\varphi(w) = w' + d\lambda$ for some $\lambda \in M$ Since $M_k \to M$ induces an isomorphism in degrees $\leq k$ the element λ is in M_k. Let $\{w_i\}$ be a basis for W and write $\varphi(w_i) = w_i' + d\lambda_i$ as above. Then by induction there exists a finite–dimensional φ–stable subspace $W' \subset M$ containing the $d\lambda_i$. Let $\widetilde{W} = W + W'$. Then $\widetilde{W}$ is a finite–dimensional φ–stable subspace containing the v_i. This completes the case when the v_i are all closed. Note that this argument only requires the induction hypothesis for M_k so this proves the result for arbitrary closed elements of degree $k+1$ in M.

For the general case, note first that since the result holds by assumption for M_k^i there exists a finite dimensional φ–stable subspace W of M containing the elements dv_i. Furthermore, since the image of d is φ–stable we can choose W to be in the image of d. We can therefore find a finite–dimensional subspace $W' \subset M$ containing the v_i such that for any $w \in W'$ there exists $w' \in W'$ and closed element z with $\varphi(w) = w' + z$. Choose a basis $w_1, \ldots, w_l$ for W'

and write $\varphi(w_i) = w'_i + z_i$. By the case already considered, there exists a finite dimensional φ–stable subspace $W'' \subset M$ containing all the z_i. The space $\widetilde{W} := W' + W''$ is then a finite dimensional φ-stable subspace containing all the v_i. □

Remark 3.60. It follows from the preceding constructions that the equivalence (3.46) induces an equivalence between the full subcategories of objects isomorphic to algebras in $\mathbb{U}$.

Let us note the following corollary of (3.46) and (3.32) which will be used later:

Corollary 3.61. *Let (F, φ) be an object of $\mathcal{F}$ such that the F–isocrystal $(H^i(F), H^i(\varphi))$ has non–negative slopes for all $i \geq 0$. Then there exists a cofibrant representative $A \in G - \mathrm{dga}_F$ with decomposable differential such that the slopes appearing in the underlying Ind–F–isocrystal of A are all non–negative.*

The stacks $F^0_{\mathcal{C}}$

3.62. Let k be a separably closed field, W its ring of Witt vectors, $K = \mathrm{Frac}(W)$, and X/k a proper smooth scheme. Let $\mathcal{C} \subset \mathrm{Isoc}(k/K)$ be a Tannakian subcategory closed under extensions, and assume that $\mathcal{C}$ is generated by a family of F–isocrystals $\{(\mathcal{V}_i, \varphi_i)\}$ in the sense that $\mathcal{C}$ is the smallest Tannakian subcategory of $\mathrm{Isoc}(k/K)$ which is closed under extensions and contains all the $\mathcal{V}_i$. Let G denote the pro–reductive completion of $\pi_1(\mathcal{C})$. Then G is naturally viewed as a group scheme G_0 in $\mathrm{FIsoc}(k/K)$ using (3.3), and by (3.46) the cohomology algebra $\mathbb{R}\Gamma(\mathbb{L}(\mathcal{O}_G))$ has a natural structure of an object in $G_0 - \mathrm{dga}_F$. Just as in ([KPT], p. 49), the functor of Thom–Sullivan cochains gives an equivalence $\mathrm{Ho}(G_0 - \mathrm{dga}_F) \simeq \mathrm{Ho}(G_0 - \mathrm{Alg}^{\Delta}_{\mathbb{Q}_p,F})$. We denote by $A(\mathcal{C}) \in \mathrm{Ho}(G_0 - \mathrm{Alg}^{\Delta}_{\mathbb{Q}_p,F})$ the object obtained from $\mathbb{R}\Gamma(\mathbb{L}(\mathcal{O}_G))$ and define

(3.62.1) $$F^0_{\mathcal{C}} := [\mathbb{R}\mathrm{Spec}_{G_0}(A(\mathcal{C}))/G_0] \in \mathrm{Ho}(\mathrm{SPr}_*(\mathrm{Aff}_{\mathbb{Q}_p,F})).$$

As remarked in (3.26), there is a natural isomorphism between $\mathrm{res}(F^0_{\mathcal{C}}) \in \mathrm{Ho}(\mathrm{SPr}(K)|_{BG})$ and $X_{\mathcal{C}}$ defined in (2.20).

Theorem 3.63. (i) *There is a natural equivalence between the category of F–isocrystals $(\mathcal{V}, \varphi)$ with $\mathcal{V} \in \mathcal{C}$ and the category of local systems on $F^0_{\mathcal{C}}$.*

(ii) *If V is the local system on $F^0_{\mathcal{C}}$ associated to some F–isocrystal $(\mathcal{V}, \varphi)$, then there is a natural isomorphism*

(3.63.1) $$H^*(F^0_{\mathcal{C}}, V) \simeq H^*_{\mathrm{cris}}(X/K, \mathcal{V})^{\varphi=1}.$$

Proof. To see (i), note that by (3.17), to give a local system on $F^0_{\mathcal{C}}$ is equivalent to giving an F–isocrystal (V, φ) in $\mathrm{FIsoc}(k/K)$ with an action of $\pi_1(\mathcal{C})$ which is compatible with the Frobenius structures. By Tannaka duality this is in turn is equivalent to the data of an F–isocrystal on X/K.

As for (ii), note first that there is a natural map

(3.63.2) $$H^*(F^0_{\mathcal{C}}, \mathcal{V}) \longrightarrow H^*(X_{\mathcal{C}}, \mathcal{V}) \simeq H^*_{\mathrm{cris}}(X/K, \mathcal{V}).$$

To prove that this gives an isomorphism as in (3.63.1), it suffices by filtering $\mathcal{V}$ and considering the associated long exact sequences to consider the case when $\mathcal{V}$ is induced by a representation V of G with a Frobenius structure $\varphi_V : V^{\sigma} \to V$ compatible with the Frobenius structure on G. Now in this case, one shows just as in the proof of (2.34) that there is a natural isomorphism

$$H^*(F^0_{\mathcal{C}}, \mathcal{V}) \simeq ((H^*(\mathbb{R}\Gamma(\mathbb{L}(\mathcal{O}_G))) \otimes V^*)^G)^{\varphi=1}. \tag{3.63.3}$$

From this the result follows. □

Let us also note the following consequence of our discussion so far:

Corollary 3.64. *The coordinate ring $\mathcal{O}_{\pi_1(X_{\mathcal{C}})}$ with the semi–linear automorphism $\varphi^*_{\pi_1(X_{\mathcal{C}})}$: $\mathcal{O}_{\pi_1(X_{\mathcal{C}})} \to \mathcal{O}_{\pi_1(X_{\mathcal{C}})}$ obtained from Frobenius pullback is a direct limit of finite–dimensional F–isocrystals.*

Proof. In fact, the ring $\mathcal{O}_{\pi_1(X_{\mathcal{C}})}$ is naturally the evaluation of a vector space $\mathcal{A}$ on $\mathrm{Aff}_{\mathbb{Q}_p,F}$. Indeed, for every $(R,\omega) \in \mathrm{Aff}_{\mathbb{Q}_p,F}$, let $\mathcal{A}(R,\omega)$ be the coordinate ring of the fundamental group of $F^0_{\mathcal{C}}$ restricted to $\mathrm{Ho}(\mathrm{SPr}_*(R)|_{B\underline{\mathrm{Aut}}^{\otimes}(\omega)})$. Then $\mathcal{A}$ defines a vector space on $\mathrm{Aff}_{\mathbb{Q}_p,F}$ which under the equivalence (3.17) corresponds to $\mathcal{O}_{\pi_1(X_{\mathcal{X}})}$ with the semi–linear automorphism $\varphi^*_{\pi_1(X_{\mathcal{C}})}$. □

4. Applications

In this section all algebras, vector spaces etc. are in $\mathbb{U}$.

Quadratic Presentations

4.1. Let k be a finite field with $q = p^a$ elements, W the ring of Witt vectors of k and K the field of fractions of W. Fix an embedding $\iota : K \hookrightarrow \mathbb{C}$ and let $\overline{K}$ be the algebraic closure of K in $\mathbb{C}$.

We call an automorphism $T : V \to V$ of a finite dimensional K–vector space V is *ι–pure of weight w* if every eigenvalue λ of $T_{\mathbb{C}} : V \otimes_\iota \mathbb{C} \to V \otimes_\iota \mathbb{C}$ has absolute value $p^{aw/2}$. More generally, if (V,T) is a direct limit $\varinjlim(V_i,T_i)$ of finite dimensional K–vector spaces with automorphisms, then we say that (V,T) is ι–pure of weight w if the image of each V_i in V with the restriction of T is ι–pure of weight w. This is easily seen to be independent of the presentation of (V,T) as a direct limit.

Let X/K be a smooth proper scheme, $x \in X(k)$ a point, and assume $(\mathcal{V},\varphi)$ is an ι–pure F–isocrystal on X/K in the sense of ([Ke], 5.1). Let G be the pro–reductive completion of $\pi_1(\langle\mathcal{V}\rangle_\otimes,\omega_x)$, and let $\mathcal{C}$ denote the smallest full Tannakian subcategory of $\mathrm{Isoc}(X/K)$ closed under extensions and containing $\mathcal{V}$. Set $\widetilde{G} := \pi_1(\mathcal{C},\omega_x)$, and define

$$\mathcal{U} := \mathrm{Ker}(\widetilde{G} \longrightarrow G). \tag{4.1.1}$$

The group $\mathcal{U}$ is a pro–unipotent group scheme. We denote by $\mathfrak{u}$ its Lie algebra. The following result is a p–adic version of a result of Hain ([Ha1], 13.14).

Theorem 4.2. *The Lie algebra $\mathfrak{u}$ admits a quadratic presentation. In other words, there exists a (possibly infinite dimensional) K–vector space V and a surjection $\pi : \mathbb{L}(V) \to \mathfrak{u}$, where $\mathbb{L}(V)$ denote the free pro–Lie algebra on V, such that* $\mathrm{Ker}(\pi)$ *is generated by degree* 2 *elements in* $\mathbb{L}(V)$*.*

The proof is in several steps (4.3)–(4.12).

4.3. Let us first observe that it suffices to consider the case when G is actually equal to $\pi_1(\langle \mathcal{V}\rangle_\otimes, \omega_x)$. For this, note that the group G is the reductive completion of $\pi_1(\langle \mathcal{V}\rangle_\otimes, \omega_x)$, and hence any representation W of $\pi_1(\langle \mathcal{V}\rangle_\otimes, \omega_x)$ admits a canonical filtration whose successive quotients are representations of G. In particular, the representation W_0 corresponding to $\mathcal{V}$ has a canonical filtration F such that the quotients F^i/F^{i-1} are representations of G. Furthermore, because the isomorphism

$$\pi_1(\langle \mathcal{V}\rangle_\otimes, \omega_x) \longrightarrow \pi_1(\langle \mathcal{V}\rangle_\otimes, \omega_x)^\sigma \tag{4.3.1}$$

necessarily maps the unipotent radical to itself, the filtration F induces on $\mathcal{V}$ a filtration $\mathcal{F}$ by sub–F–isocrystals. Replacing $\mathcal{V}$ by $\oplus_i \mathrm{gr}^i\mathcal{V}$ we are therefore reduced to the case when $\pi_1(\langle \mathcal{V}\rangle_\otimes, \omega_x) = G$.

We make this assumption in the rest of the proof.

Lemma 4.4. *Let $\mathbb{L}(\mathcal{O}_G)$ be the ind–F–isocrystal associated to G as in* (2.15)*. Then $\mathbb{L}(\mathcal{O}_G)$ is a direct limit of ι–pure F–isocrystals of weight* 0.

Proof. Let $V = \omega_x(\mathcal{V})$. Since $\mathcal{C}$ is generated by the F–isocrystal $(\mathcal{V}, \varphi)$, there is a closed immersion $G \hookrightarrow GL(V)$ which is compatible with the F–isocrystal structures, where $GL(V)$ is given an F–isocrystal structure $\varphi_{GL(V)}$ via the isomorphism $\omega_x(\varphi) : V^\sigma \to V$. Let L denote the dual of the left G–representation $\bigwedge^r V$, where r is the rank of V, and let $\mathcal{L}$ denote the corresponding F–isocrystal. There is then a natural surjection

$$\mathrm{Sym}^\bullet(V \boxtimes V^*) \otimes_K \mathrm{Sym}^\bullet(L \boxtimes L^*) \longrightarrow \mathcal{O}_G \tag{4.4.1}$$

of ind–F–isocrystals in (G, G)–bimodules induced by the inclusion $G \subset \mathrm{End}(V)$ and the inverse of the determinant map $G \to \mathrm{End}(\bigwedge^r V^*)$. This map induces a surjection of ind–F–isocrystals on X/K

$$\mathrm{Sym}^\bullet(\mathcal{V} \boxtimes V^*) \otimes_K \mathrm{Sym}^\bullet(\mathcal{L} \boxtimes L^*) \longrightarrow \mathbb{L}(\mathcal{O}_G). \tag{4.4.2}$$

From this it follows that it suffices to show that the F–isocrystals $\mathcal{V} \otimes V^*$ and $\mathcal{L} \otimes L^*$ are ι–pure of weight 0. This is immediate for if w is the ι–weight of $\mathcal{V}$, then $\mathcal{O}_{X/K} \otimes_K V^*$ is ι–pure of weight $-w$, $\mathcal{L}$ is ι–pure of weight $-rw$, and $\mathcal{O}_{X/K} \otimes_K L^*$ is ι–pure of weight rw. □

4.5. Let $X_\mathcal{C}$ denote the pointed stack associated to $\mathcal{C}$ as in (2.20), and let $\pi : X_\mathcal{C} \to BG$ be the natural projection. The homotopy fiber F of π is by (2.11) an affine stack isomorphic to $\mathbb{R}\mathrm{Spec}(\mathbb{R}\Gamma(\mathbb{L}(\mathcal{O}_G)))$, and by the long exact sequence of homotopy groups associated to a fibration there is a natural isomorphism $\pi_1(F) \simeq \mathcal{U}$. From this and ([To1], 1.5.1), we see that for each $i \geq 0$ there is a natural isomorphism

$$H^i(\mathfrak{u}) \simeq H^i(\tau_{\leq 1}F, \mathbb{G}_a), \tag{4.5.1}$$

where $H^i(\mathfrak{u})$ denotes Lie algebra cohomology ([C-E], XIII). By ([To1],2.4.9), the natural map

$$H^i(\tau_{\leq 1}F, \mathbb{G}_a) \longrightarrow H^i(F, \mathbb{G}_a) \tag{4.5.2}$$

is an isomorphism for $i = 1$ and an injection for $i = 2$. Using (2.27 (ii)) and taking duals, we obtain a map

$$H^i_{\mathrm{cris}}(X, \mathbb{L}(\mathcal{O}_G))^* \longrightarrow H_i(\mathfrak{u}) \tag{4.5.3}$$

which is an isomorphism for $i = 1$ and a surjection for $i = 2$. Here we have identified the dual of $H^i(\mathfrak{u})$ with $H_i(\mathfrak{u})$ using ([C-E], XI.3.1). Note also that this map is natural, and so in particular is compatible with the F–isocrystal structures. By ([Ke], 5.6.2) the F–isocrystal $H^i(X/K, \mathbb{L}(\mathcal{O}_G))$ is ι–pure of weight i, and therefore $H_i(\mathfrak{u})$ is also ι–pure of weight $-i$ for $i = 1, 2$.

4.6. Set

$$\mathbb{S} := \{z \in \mathbb{C}^* | |\sigma(z)| = |z| \text{ for every } \sigma \in \mathrm{Aut}(\mathbb{C}/K)\}. \tag{4.6.1}$$

We view $\mathbb{S}$ as a group under multiplication, and write $D(\mathbb{S})_{\mathbb{C}}$ for the associated diagonalizable group scheme over $\mathbb{C}$. To give an action of $D(\mathbb{S})_{\mathbb{C}}$ on a vector space V is equivalent to giving a decomposition $V = \oplus_{s \in \mathbb{S}} V_s$. There is a natural group homomorphism

$$\delta : \mathbb{S} \longrightarrow \mathbb{R}, \quad z \mapsto 2 \log_q |z|, \tag{4.6.2}$$

where q is the order of k. This map induces a morphism of diagonalizable group schemes $D(\mathbb{R})_{\mathbb{C}} \to D(\mathbb{S})_{\mathbb{C}}$.

4.7. Let V be a K–vector space and $T : V \to V$ an automorphism, and assume that the pair (V, T) is a direct limit of finite dimensional vector spaces with automorphisms. Tensoring with $\mathbb{C}$, there is a canonical decomposition $V_{\mathbb{C}} = \oplus_{z \in \mathbb{C}^*} V_{\mathbb{C},z}$ into generalized eigenspaces. If $V_{\mathbb{C},z} = 0$ for $z \notin \mathbb{S}$, we can write this decomposition as

$$V_{\mathbb{C}} = \oplus_{s \in \mathbb{S}} V_{\mathbb{C},s} = \oplus_{r \in \mathbb{R}} (\oplus_{\delta(s) = r} V_{\mathbb{C},s}). \tag{4.7.1}$$

The right hand decomposition is invariant under $\mathrm{Aut}(\mathbb{C}/K)$, and hence by descent theory induces an action of $D(\mathbb{R})_K$ (diagonalizable group scheme over K) on V.

Lemma 4.8. *The pro–Lie algebra $(\mathfrak{u}, \varphi_{\mathfrak{u}})$ with semi–linear automorphism can be written as a projective limit $\varprojlim(\mathfrak{u}_i, \varphi_{\mathfrak{u}_i})$ of finite dimensional Lie algebras with Frobenius automorphism.*

Proof. Let S be the dual space of $\mathfrak{u}$. The space S is equal to I/I^2, where $I \subset \mathcal{O}_{\mathcal{U}}$ is the ideal of the identity. The Lie algebra structure on $\mathfrak{u}$ is induced by a morphism

$$\rho : S \longrightarrow S \otimes S. \tag{4.8.1}$$

Moreover, since $\mathcal{U}$ is pro–unipotent, we can write $S = \varinjlim S_i$ where S_i is finite–dimensional and $\rho(S_i) \subset S_i \otimes S_i$. The Frobenius automorphism on $\mathfrak{u}$ is induced by a semi–linear isomorphism $\varphi_S : S \to S$. Moreover, by (3.64) we can write $S = \varinjlim M_j$, where each M_j is a finite–dimensional subspace of S stable under Frobenius. For each S_i let $\overline{S}_i$ denote the intersection of all Frobenius stable subspaces of S containing S_i. Since S_i is contained in some M_j the space $\overline{S}_i$ is finite–dimensional. Moreover, $\rho^{-1}(\overline{S}_i \otimes \overline{S}_i)$ is Frobenius stable and contains S and hence ρ sends $\overline{S}_i$ to $\overline{S}_i \otimes \overline{S}_i$. Let $(\mathfrak{u}_i, \varphi_{\mathfrak{u}_i})$ be the Lie algebra with Frobenius automorphism corresponding to $\overline{S}_i$. □

4.9. We apply the discussion of (4.7) to the Lie algebras $\mathfrak{u}_i$ with the automorphisms $T : \mathfrak{u}_i \to \mathfrak{u}_i$ induced by the a–th power of Frobenius. We claim that the generalized eigenvalues of $T_{\mathbb{C}}$ all are in $\mathbb{S}$. To see this, consider the lower central series defined inductively by

$$\mathrm{Fil}_0 = \mathfrak{u}_i \quad \mathrm{Fil}_n = [\mathrm{Fil}_{n-1}, \mathfrak{u}_i]. \tag{4.9.1}$$

Since $\mathfrak{u}_i$ is unipotent, it suffices to verify that the induced action of $T_{\mathbb{C}}$ on the graded pieces $\mathrm{gr}^n_{\mathrm{Fil}}(\mathfrak{u}_i)$ has eigenvalues in $\mathbb{S}$. By definition of Fil, the surjection

$$[\cdot,\cdot] : \mathrm{Fil}_{n-1} \otimes \mathfrak{u}_i \longrightarrow \mathrm{Fil}_n/\mathrm{Fil}_{n+1} \tag{4.9.2}$$

factors through $\mathrm{gr}^{n-1}_{\mathrm{Fil}}(\mathfrak{u}_i) \otimes \mathrm{gr}^0_{\mathrm{Fil}}(\mathfrak{u}_i)$. Hence by induction it suffices to verify the claim for $\mathrm{gr}^0_{\mathrm{Fil}}(\mathfrak{u}_i)$ which by the discussion in (4.5) is isomorphic to a quotient of the dual of $H^1_{\mathrm{cris}}(X/K, \mathbb{L}(\mathcal{O}_G))$. The claim therefore holds by ([Ke], 5.6.2).

4.10. It follows that there is a natural action of $D(\mathbb{R})_K$ on the pro–Lie algebra $\mathfrak{u}$ such that the morphism (4.5.3) is compatible with the $D(\mathbb{R})_K$–actions, where the action of $D(\mathbb{R})_K$ on $H^i_{\mathrm{cris}}(X/K, \mathbb{L}(\mathcal{O}_G))$ is obtained from the a–th power of Frobenius.

Since the category $\mathrm{Rep}(D(\mathbb{R})_K)$ is semi–simple, the surjection

$$q : \mathfrak{u} \longrightarrow \mathfrak{u}/[\mathfrak{u},\mathfrak{u}] \simeq H_1(\mathfrak{u}) \simeq H^1_{\mathrm{cris}}(X/K, \mathbb{L}(\mathcal{O}_G))^* \tag{4.10.1}$$

admits a section $\sigma : H^1_{\mathrm{cris}}(X/K, \mathbb{L}(\mathcal{O}_G))^* \to \mathfrak{u}$ in the category of pro–objects in $\mathrm{Rep}(D(\mathbb{R})_K)$. The section σ induces a surjection of pro–Lie algebras in $\mathrm{Rep}(D(\mathbb{R})_K)$

$$\pi : \mathfrak{f} \longrightarrow \mathfrak{u}, \tag{4.10.2}$$

where $\mathfrak{f}$ denotes the free pro–Lie algebra on $H^1(X/K, \mathbb{L}(\mathcal{O}_G))^*$. We claim that the ideal $\mathfrak{k} := \mathrm{Ker}(\pi)$ is generated by elements of degree 2 in $\mathfrak{f}$.

Lemma 4.11 ([Ha2], 5.6)**.** *There is a natural isomorphism $H_0(\mathfrak{f},\mathfrak{k}) \simeq H_2(\mathfrak{u})$ compatible with the actions of $D(\mathbb{R})_K$.*

Proof. By functoriality, the group $D(\mathbb{R})_K$ acts on the associated change of rings spectral sequence ([C-E], XVI.7 (6a))

$$\mathrm{E}_2^{p,q} = H_p(\mathfrak{u}, H_q(\mathfrak{k})) \implies H_{p+q}(\mathfrak{f}). \tag{4.11.1}$$

Consider the map

$$d_2 : H_0(\mathfrak{u}, H_1(\mathfrak{k})) \longrightarrow H_2(\mathfrak{u}) \tag{4.11.2}$$

in this spectral sequence. Since the map $H_1(\mathfrak{f}) \to H_1(\mathfrak{u})$ is an isomorphism, the map d_2 is injective. On the other hand since $\mathfrak{f}$ is a free Lie algebra $H_2(\mathfrak{f}) = 0$ (this follows for example from ([C-E], XIV.2.1) and duality), and hence d_2 must also be surjective. The lemma now follows from the natural isomorphism

$$H_0(\mathfrak{u}, H_1(\mathfrak{k})) \simeq H_0(\mathfrak{f},\mathfrak{k}). \tag{4.11.3}$$

□

4.12. It follows that the only non–trivial factor occurring in the decomposition $\mathfrak{k}/[\mathfrak{f},\mathfrak{k}] = H_0(\mathfrak{f},\mathfrak{k}) = \oplus_{\lambda\in\mathbb{R}} H_0(\mathfrak{f},\mathfrak{k})_\lambda$ corresponding to the action of $D(\mathbb{R})_K$ is the $\lambda = 2$ factor. Again since the category $\mathrm{Rep}(D(\mathbb{R})_K)$ is semi–simple, we can choose a splitting in the category $\mathrm{Rep}(D(\mathbb{R})_K)$ of the projection

$$\mathfrak{k} \longrightarrow \mathfrak{k}/[\mathfrak{f},\mathfrak{k}]. \tag{4.12.1}$$

From this it follows that $\mathfrak{k}$ is generated by elements of $\mathfrak{f} = \oplus_{\lambda\in\mathbb{R}}\mathfrak{f}_\lambda$ lying in $\mathfrak{f}_2$. Since $\mathfrak{f}$ is generated by $\mathfrak{f}_1$, it follows that $\mathfrak{k}$ is generated by elements in degree 2.

Comparison with étale sheaves

4.13. Let k be a separably closed field of characteristic $p > 0$, and $(X, x)/k$ a pointed, proper, smooth, connected k–scheme. Our aim in this subsection is to generalize to the level of homotopy types the classical equivalence ([Cr1], 2.1) between unit root F–isocrystals on X/K and étale local systems of $\mathbb{Q}_p$–spaces on X_{et}.

Let $\{(\mathcal{V}_i, \varphi_i)\}_{i\in I}$ be a family of unit–root F–isocrystals on X/K (for example all unit–root F–isocrystals), and let $\mathcal{C} \subset \text{Isoc}(X/K)$ denote the smallest Tannakian subcategory closed under extensions and containing the $\mathcal{V}_i$. Denote by $\mathcal{C}^{\text{ss}} \subset \mathcal{C}$ the full subcategory of semi–simple objects, and by G the group $\pi_1(\mathcal{C}^{\text{ss}}, x)$. Equivalently, G is the pro–reductive completion of $\pi_1(\mathcal{C}, x)$.

The following was proven in the rank 1 case by R. Crew ([Cr2], discussion preceding (5.6)).

Proposition 4.14. *Let $(\mathcal{V}, \varphi)$ be an F–isocrystal on X/K with $\mathcal{V} \in \mathcal{C}^{\text{ss}}$. If $(\mathcal{V}(x), \varphi_x) \in$ $\text{FIsoc}(k/K)$ is unit root for some $x \in X(k)$, then $(\mathcal{V}, \varphi)$ is a unit root F–isocrystal.*

Proof. Write $\mathcal{V} \simeq \oplus_i \mathcal{V}_i^{r_i}$ where each $\mathcal{V}_i$ is irreducible and $\mathcal{V}_i \not\simeq \mathcal{V}_j$ for $i \neq j$. Since $\mathcal{V}^\sigma \simeq \mathcal{V}$, there exists for each i a unique i' such that $\mathcal{V}_i^\sigma \simeq \mathcal{V}_{i'}$, and $r_i = r_{i'}$. It follows that there exists a canonical decomposition of $(\mathcal{V}, \varphi)$ as an F–isocrystal

$$(\mathcal{V}, \varphi) \simeq \oplus_j (\mathcal{W}_j^{r_j}, \varphi_j), \tag{4.14.1}$$

where $\mathcal{W}_j = \mathcal{V}_i \oplus \mathcal{V}_i^\sigma \oplus \cdots \oplus \mathcal{V}_i^{\sigma^l}$. Here $\mathcal{V}_i^{\sigma^{l+1}} \simeq \mathcal{V}_i$, and $l+1$ is the smallest integer with this property. We may therefore assume that $(\mathcal{V}, \varphi) = (\mathcal{W}_j^{r_j}, \varphi_j)$ for some j.

Since $\mathcal{V} \in \mathcal{C}^{\text{ss}}$, and $\mathcal{C}$ is generated by unit root F–isocrystals, there exists a unit root F–isocrystal $(\mathcal{Z}, \varphi_{\mathcal{Z}})$ such that $\mathcal{Z} \simeq \mathcal{V}_i^r \oplus \mathcal{Z}'$, where $\text{Hom}(\mathcal{V}_i, \mathcal{Z}') = 0$. Since $\mathcal{Z}^\sigma \simeq \mathcal{Z}$, there in fact exists a decomposition $\mathcal{Z} \simeq \mathcal{W}_j^r \oplus \mathcal{Z}''$ with $\text{Hom}(\mathcal{W}_j, \mathcal{Z}'') = 0$. It follows that $\varphi_{\mathcal{Z}}$ preserves $\mathcal{W}_j^r$, and hence there exists a unit root F–isocrystal structure on $\mathcal{W}_j^r$.

Now to prove the proposition for $(\mathcal{V}, \varphi)$, it suffices to prove the proposition for $(\mathcal{V}, \varphi)^r$, and hence we can assume that $\mathcal{V}$ admits a unit root F–isocrystal structure ψ.

We claim that under these assumptions, for any two points $x, x' \in X(k)$ the F–isocrystals $(\mathcal{V}(x), \varphi_x)$ and $(\mathcal{V}(x'), \varphi_{x'})$ are isomorphic. This certainly suffices for the proof of the proposition.

To see this, let $\mathcal{R} = \text{End}(\mathcal{V})$ and let $A \in \mathcal{R}$ be the element $\varphi \circ \psi^{-1}$. If V_0 denotes the étale local system of $\mathbb{Q}_p$–spaces associated to the unit root F–isocrystal $(\mathcal{V}, \psi)$, then if $\mathcal{R}_0 := H^0(X_{\text{et}}, V_0 \otimes V_0^*)$ we have $\mathcal{R}_0 \otimes_{\mathbb{Q}_p} K \simeq \mathcal{R}$. This implies in particular that the algebra $\mathcal{R}_0$ is semi–simple. Indeed by ([La], XVII.6.1) it suffices to show that the radical of $\mathcal{R}_0$ is zero which can be verified after base change to K, and

$$\mathcal{R} \simeq M_r(\text{End}(\mathcal{V}_i)) \times \cdots \times M_r(\text{End}(\mathcal{V}_i^{\sigma^l})), \tag{4.14.2}$$

where each $\text{End}(\mathcal{V}_i^{\sigma^j})$ is a division ring since $\mathcal{V}_i$ is simple.

Now consider the two representations $V_{0,x}$ and $V_{0,x'}$ of $\mathcal{R}_0$. Since $\mathcal{R}_0$ is semi–simple, these representations are determined up to isomorphism by the associated trace maps ([La], XVII.3.8)

$$t_x, t_{x'} : \mathcal{R}_0 \longrightarrow \mathbb{Q}_p. \tag{4.14.3}$$

But both of these maps are equal to the global trace map

$$t : \mathcal{R}_0 \longrightarrow H^0(X_{\text{et}}, \mathbb{Q}_p) \simeq \mathbb{Q}_p, \tag{4.14.4}$$

and hence $t_x = t_{x'}$. It follows that there exists an isomorphism of $\mathcal{R}_0$–modules $\iota : V_{0,x} \simeq V_{0,x'}$ such that the diagram

$$\begin{array}{ccccc} V_{0,x} \otimes_{\mathbb{Q}_p} K & \xrightarrow{1\otimes\sigma} & V_{0,x} \otimes_{\mathbb{Q}_p} K & \xrightarrow{A_x} & V_{0,x} \otimes K \\ \downarrow{\scriptstyle \iota\otimes 1} & & \downarrow{\scriptstyle \iota\otimes 1} & & \downarrow{\scriptstyle \iota\otimes 1} \\ V_{0,x'} \otimes_{\mathbb{Q}_p} K & \xrightarrow{1\otimes\sigma} & V_{0,x'} \otimes_{\mathbb{Q}_p} K & \xrightarrow{A_{x'}} & V_{0,x'} \otimes K \end{array} \tag{4.14.5}$$

commutes. In other words, ι gives an isomorphism of F–isocrystals $(\mathcal{V}(x), \varphi_x) \simeq (\mathcal{V}(x'), \varphi_{x'})$. □

Let $\mathcal{C}_{\text{et}}$ denote the smallest Tannakian subcategory of the category of smooth $\mathbb{Q}_p$–sheaves on X_{et} closed under extensions and containing the smooth sheaves corresponding (via ([Cr1], 2.1)) to the unit–rot F–isocrystals $\{(\mathcal{V}_i, \varphi_i)\}$. The point x induces a geometric point $\bar{x} \to X$ which defines a fiber functor $\mathcal{C}_{\text{et}} \to \text{Vec}_{\mathbb{Q}_p}$. Denote by $\pi_1(\mathcal{C}_{\text{et}}, \bar{x})$ the Tannaka dual of $\mathcal{C}_{\text{et}}$ and let G^{et} the pro–reductive completion of $\pi_1(\mathcal{C}_{\text{et}}, \bar{x})$.

Proposition 4.15. *There is a natural isomorphism $G^{\text{et}} \otimes_{\mathbb{Q}_p} K \simeq G$ of group schemes with F–isocrystal structure.*

Proof. Let $\text{Rep}(G)_F^{\text{ur}}$ denote the category of $F - G$ representations (in the sense of (3.20)) for which the underlying F–isocrystal is a unit–root F–isocrystal. The category $\text{Rep}(G)_F^{\text{ur}}$ is a Tannakian subcategory of $\text{Rep}(G)_F$, and taking invariants under Frobenius defines a fiber functor

$$q : \text{Rep}(G)_F^{\text{ur}} \longrightarrow \text{Vec}_{\mathbb{Q}_p}. \tag{4.15.1}$$

By ([Cr2], 3.6), there is a natural isomorphism

$$\underline{\text{Aut}}^{\otimes}(q) \otimes_{\mathbb{Q}_p} K \simeq G. \tag{4.15.2}$$

On the other hand, (4.14) shows that the category $\text{Rep}(G)_F^{\text{ur}}$ is equivalent to the category of unit–root F–isocrystals $(\mathcal{V}, \varphi)$ with $\mathcal{V} \in \mathcal{C}^{\text{ss}}$. Thus to prove the proposition it suffices to show that if V is a $\mathbb{Q}_p$–local system on X_{et} corresponding to a unit root F–isocrystal $(\mathcal{V}, \varphi)$, then $\mathcal{V} \in \mathcal{C}^{\text{ss}}$ if and only if V is semi–simple.

For the "only if" direction, suppose given an extension

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0 \tag{4.15.3}$$

of étale sheaves giving rise to an extension of unit-root F–isocrystals

$$0 \longrightarrow (\mathcal{V}_1, \varphi_1) \longrightarrow (\mathcal{V}, \varphi) \xrightarrow{\pi} (\mathcal{V}_2, \varphi_2) \longrightarrow 0 \tag{4.15.4}$$

of isocrystals. The existence of a section of $\pi : \mathcal{V} \to \mathcal{V}_2$ is equivalent to the statement that the image of the map

$$\rho : H^0(X/K, \mathcal{V}_2^* \otimes \mathcal{V}) \longrightarrow H^0(X/K, \mathcal{V}_2^* \otimes \mathcal{V}_2) \tag{4.15.5}$$

contains the element $s \in H^0(X/K, \mathcal{V}_2^* \otimes \mathcal{V}_2)$ corresponding to the identity element $\mathcal{V}_2 \to \mathcal{V}_2$. By ([Et], 2.1) there is a commutative diagram

$$\begin{array}{ccc} H^0(X/K, \mathcal{V}_2^* \otimes \mathcal{V}) & \xrightarrow{\rho} & H^0(X/K, \mathcal{V}_2^* \otimes \mathcal{V}_2) \\ \simeq\uparrow & & \uparrow\simeq \\ H^0(X_{\text{et}}, V_2^* \otimes V) \otimes_{\mathbb{Q}_p} K & \xrightarrow{\rho_{\text{et}}\otimes K} & H^0(X_{\text{et}}, V_2^* \otimes V_2) \otimes_{\mathbb{Q}_p} K \end{array} \tag{4.15.6}$$

with the vertical arrows isomorphisms. Therefore if the image of ρ contains s then the image of ρ_{et} contains the element of $H^0(X_{\text{et}}, V_2^* \otimes V_2)$ corresponding to the identity element $V_2 \to V_2$. In other words, $V \to V_2$ admits a section.

For the other direction, suppose $\mathcal{V}$ is not semi–simple in the category of isocrystals. Set

$$\mathcal{U} = \text{Ker}(\pi_1(\mathcal{C}, \omega_x) \longrightarrow G), \tag{4.15.7}$$

so that $\mathcal{U}$ acts non–trivially on $\omega_x(\mathcal{V})$. Since $\mathcal{U}$ is a unipotent group scheme, $\omega_x(\mathcal{V})^{\mathcal{U}}$ is not zero. Denote by $\mathcal{W}$ the associated isocrystal so that there is a non–trivial extension

$$0 \longrightarrow \mathcal{W} \longrightarrow \mathcal{V} \longrightarrow (\mathcal{V}/\mathcal{W}) \longrightarrow 0 \tag{4.15.8}$$

of isocrystals. Since the map $\pi_1(\mathcal{C}, \omega_x) \to \pi_1(\mathcal{C}, \omega_x)^\sigma$ induced by Frobenius preserves $\mathcal{U}$, the $\mathcal{U}$–invariants in $\omega_x(\mathcal{V})$ are preserved by the Frobenius morphism φ_x. It follows that $\mathcal{W} \subset \mathcal{V}$ is preserved by φ and hence (4.15.8) gives a non–trivial extension of unit–root F–isocrystals. This shows that V is not a semi–simple sheaf. □

4.16. As explained in ([To1], 3.5.3), one can also construct a pointed stack $X_{\mathcal{C}_{\text{et}}}$ over $\mathbb{Q}_p$ whose fundamental group is isomorphic to $\pi_1(\mathcal{C}_{\text{et}}, \bar{x})$ and whose cohomology groups compute étale cohomology. The construction of $X_{\mathcal{C}_{\text{et}}}$ can be summarized as follows. Let G^{et} be as above, and let $\mathbb{V}(\mathcal{O}_{G^{\text{et}}})$ be the Ind–object in the category of p–adic local systems on X_{et} corresponding to $\mathcal{O}_{G^{\text{et}}}$ with action given by right translation. The sheaf $\mathbb{V}(\mathcal{O}_{G^{\text{et}}})$ has a natural right G^{et}–action, and the same method used in the construction of $X_{\mathcal{C}}$ gives an algebra

$$\mathbb{R}\Gamma_{\text{et}}(\mathbb{V}(\mathcal{O}_{G^{\text{et}}})) \in \text{Ho}(G^{\text{et}} - \text{Alg}^{\Delta}_{\mathbb{Q}_p}). \tag{4.16.1}$$

We define

$$X_{\mathcal{C}_{\text{et}}} := [\mathbb{R}\text{Spec}_{G^{\text{et}}}\mathbb{R}\Gamma_{\text{et}}((\mathbb{V}(\mathcal{O}_{G^{\text{et}}})))/G^{\text{et}}] \in \text{Ho}(\text{SPr}_*(\mathbb{Q}_p)). \tag{4.16.2}$$

Here $\mathbb{R}\Gamma_{\text{et}}(\mathbb{V}(\mathcal{O}_{G^{\text{et}}}))$ is defined as in ([De2], 5.2.2) (see also ([Ol], section 4). In the case when $\{(\mathcal{V}_i, \varphi_i)\}$ is the set of all unit–root F–isocrystals, the category $\mathcal{C}_{\text{et}}$ is the category of all smooth $\mathbb{Q}_p$–sheaves in which case we write X_{et} for $X_{\mathcal{C}_{\text{et}}}$.

Remark 4.17. The relationship between X_{et} and the Artin–Mazur homotopy type of X ([A-M]) is subtle, and we do not consider the étale homotopy type of X in the sense of (loc. cit.).

4.18. There is a natural functor

$$H : G^{\text{et}} - \text{Alg}^{\Delta}_{\mathbb{Q}_p} \longrightarrow G - \text{Alg}^{\Delta}_{K,F} \tag{4.18.1}$$

which sends A to $A \otimes_{\mathbb{Q}_p} K$ with F–isocrystal structure given by $\text{id} \otimes \sigma$. This functor has a right adjoint ι which sends B to $B^{\varphi=1}$. Since taking the slope zero part is an exact functor, this is a Quillen adjunction, and so there is a derived functor

$$\mathbb{R}\iota : \text{Ho}(G - \text{Alg}^{\Delta}_{K,F}) \longrightarrow \text{Ho}(G^{\text{et}} - \text{Alg}^{\Delta}_{\mathbb{Q}_p}). \tag{4.18.2}$$

Theorem 4.19. *Let $F^0_{\mathcal{C}} \in \mathrm{Ho}(\mathrm{SPr}_*(G^{\mathrm{et}} - \mathrm{Aff}_{\mathbb{Q}_p,F}))$ be as in* (3.62). *Then there is an isomorphism in* $\mathrm{Ho}(\mathrm{SPr}_*(\mathbb{Q}_p))$ *between $X_{\mathcal{C}_{\mathrm{et}}}$ and*

$$[\mathbb{R}\mathrm{Spec}_{G^{\mathrm{et}}}(\mathbb{R}\iota(\mathbb{L}\mathcal{O}_{G^{\mathrm{et}}}(F^0_{\mathcal{C}})))/G^{\mathrm{et}}]. \tag{4.19.1}$$

Proof. Fix an embedding system $Y^\bullet$ as in (2.14), and let $C^\bullet(\mathcal{O}_G)$ denote the cohomology complex of $\mathbb{L}(\mathcal{O}_G)$. There is a natural map $\mathbb{V}(\mathcal{O}_{G^{\mathrm{et}}})|_{Y^\bullet} \to C^\bullet(\mathcal{O}_G)$ which maps $\mathbb{V}(\mathcal{O}_{G^{\mathrm{et}}})|_{Y^\bullet}$ into the Frobenius invariants of $C^\bullet(\mathcal{O}_G)$. This map induces an equivariant map

$$\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G^{\mathrm{et}}})) \longrightarrow \mathbb{R}\iota\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_G)), \tag{4.19.2}$$

which by (3.56) can be represented by a map $\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G^{\mathrm{et}}})) \otimes K \to \mathbb{L}\mathcal{O}_{G^{\mathrm{et}}}(F^0_{\mathcal{C}})$ in $\mathrm{Ho}(G-\mathrm{dga}_F)$. That it is an equivalence can be checked on cohomology groups and hence follows from ([Og2]). □

4.20. Let X_{ur} be the pointed stack obtained by taking $\mathcal{C}$ in the above to be the small Tannakian category containing the underlying isocrystals of unit–root F–isocrystals. The isomorphism in (4.19) induces a morphism of stacks with Frobenius structures $X_{\mathrm{ur}} \to X_{\mathrm{et}} \otimes_{\mathbb{Q}_p} K$ which for every $i \geq 1$ gives a morphism of group schemes with F–isocrystal structure

$$\pi_i(X_{\mathrm{ur}}) \longrightarrow \pi_i(X_{\mathrm{et}}) \otimes_{\mathbb{Q}_p} K \tag{4.20.1}$$

Let F (resp. F_{et}) denote the homotopy fiber of the projection $X_{\mathrm{ur}} \to BG$ (resp. $X_{\mathrm{et}} \to BG^{\mathrm{et}}$). As explained in (2.11) if A (resp. A_{et}) denotes $\mathbb{R}\Gamma(\mathbb{L}(\mathcal{O}_{G^{\mathrm{et}}_K}))$ (resp. $\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G^{\mathrm{et}}})))$ then there is a natural isomorphism $F \simeq \mathbb{R}\mathrm{Spec}(A)$ (resp. $F_{\mathrm{et}} \simeq \mathbb{R}\mathrm{Spec}(A_{\mathrm{et}})$).

From the long exact sequence of homotopy group associated to a fibration we find that $\pi_i(F) \simeq \pi_i(X_{\mathrm{ur}})$ (resp. $\pi_i(F_{\mathrm{et}}) \simeq \pi_i(X_{\mathrm{et}})$) for $i \geq 2$ and that for $i = 1$ there are exact sequences

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(X_{\mathrm{ur}}) \longrightarrow G^{\mathrm{et}} \otimes K \longrightarrow 1 \tag{4.20.2}$$

$$1 \longrightarrow \pi_1(F^{\mathrm{et}}) \longrightarrow \pi_1(X_{\mathrm{et}}) \longrightarrow G^{\mathrm{et}} \longrightarrow 1, \tag{4.20.3}$$

and the sequence (4.20.2) is compatible with the F–isocrystal structures. It follows that $\pi_1(X_{\mathrm{et}}) \otimes K$ with its trivial F–isocrystal structure is isomorphic to the pushout of (4.20.2) via $\pi_1(F) \to \pi_1(F^{\mathrm{et}}) \otimes K$. The following theorem shows that we can recover $\pi_1(F^{\mathrm{et}})$ from $\pi_1(F)$ with its F–isocrystal structure and hence we can also recover $\pi_1(X_{\mathrm{et}}) \otimes K$ from $\pi_1(X_{\mathrm{ur}})$ with its F–isocrystal structure. Since $\mathcal{O}_{\pi_1(X_{\mathrm{et}})} \simeq (\mathcal{O}_{\pi_1(X_{\mathrm{et}})} \otimes K)^{\varphi=1}$ this shows that we can reconstruct $\pi_1(X_{\mathrm{et}})$ from $\pi_1(X_{\mathrm{ur}})$.

Theorem 4.21. *For every $i \geq 1$ the natural map $\mathcal{O}_{\pi_i(F^{\mathrm{et}})} \to \mathcal{O}_{\pi_i(F)}$ induced by* (4.20.1) *identifies $\mathcal{O}_{\pi_i(F^{\mathrm{et}})}$ with the slope zero part of $\mathcal{O}_{\pi_i(F)}$.*

Proof. Let $S^{i,unip}$ denote the affination of S^i in the sense of ([To1], 2.3). By the universal property of $S^i \to S^{i,unip}$ there is a natural isomorphism

$$[S^i, F]_{\mathrm{Ho(SPr(K))}} \simeq [S^{i,unip}, F]_{\mathrm{Ho(SPr(K))}} \simeq [A, K^{S^i}]_{\mathrm{Ho}(\mathrm{Alg}^{\Delta}_K)}, \tag{4.21.1}$$

where K^{S^i} denotes the cosimplicial algebra of cochains of S^i. Applying the normalization functor we obtain an isomorphism

$$\pi_i(F) \simeq [D(A), D(K^{S^i})]_{\mathrm{Ho}(\mathrm{Alg}^{\Delta}_K)}. \tag{4.21.2}$$

The algebra $D(K^{S^i})$ is the unique differential graded algebra which is equal to K in degrees 0 and i are zero elsewhere. From this it follows that $\pi_i(F)$ is isomorphic to the dual of the degree i part of the space of indecomposables of a cofibrant model with decomposable differential for $D(A)$ (that is, I/I^2 where I denotes the ideal of elements of the algebra in degrees > 0).

By (3.61), we can choose a cofibrant model A with decomposable differential such that the slopes appearing in the ind–F–isocrystal underlying A are all greater than or equal to zero. In this case the differential graded algebra $A^{\varphi=1}$ again has decomposable differential and the projection $A \to A^{\varphi=1} \otimes K$ to the slope zero part is a K–algebra homomorphism. The algebra $A^{\varphi=1}$ is also cofibrant in $G^{\text{et}} - \text{dga}_{\mathbb{Q}_p}$. To see this consider a commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & C \\ \big\downarrow & & \big\downarrow \alpha \\ A^{\varphi=1} & \longrightarrow & D \end{array} \tag{4.21.3}$$

with α a trivial fibration. The existence of an arrow $A^{\varphi=1} \to C$ filling in the diagram is equivalent to the existence of an arrow $A^{\varphi=1} \otimes K \to C \otimes K$ in the category $G - \text{dga}_{K,F}$. Now consider the diagram

$$\begin{array}{ccccccc} & & & & & & C \otimes K \\ & & & & & & \big\downarrow \\ A^{\varphi=1} \otimes K & \longrightarrow & A & \longrightarrow & A^{\varphi=1} \otimes K & \longrightarrow & D \otimes K. \end{array} \tag{4.21.4}$$

Since A is cofibrant in $G - \text{dga}_{K,F}$ there exists a lifting $A \to C \otimes K$ which induces the desired lifting $A^{\varphi=1} \to C$.

From this and (4.19) it follows that $A^{\varphi=1} \in G^{\text{et}} - \text{dga}_{\mathbb{Q}_p}$ is a cofibrant model with decomposable differential for $\mathbb{R}\Gamma_{\text{et}}(\mathbb{V}(\mathcal{O}_{G^{\text{et}}}))$. By the same reasoning as above, the ring $\mathcal{O}_{\pi_i(F)}$ is therefore isomorphic to the degree i part of the space of indecomposables of $A^{\varphi=1}$. Since taking Frobenius invariants is an exact functor, the theorem follows. □

4.22. To deduce (1.10) from this, note first that since taking Frobenius invariants is an exact functor the identification in (4.21) induces an isomorphism

$$\text{Lie}(\pi_i(F^{\text{et}})) \simeq \text{Lie}(\pi_i(F))^{\varphi=1}. \tag{4.22.1}$$

This gives (1.10) for $i \geq 2$. For $i = 1$ note that the sequence (4.20.2) induces an exact sequence of Lie algebras with F–isocrystal structure

$$0 \longrightarrow \text{Lie}(\pi_1(F)) \longrightarrow \text{Lie}(\pi_1(X_{\text{ur}})) \longrightarrow \text{Lie}(G^{\text{et}}) \otimes K \longrightarrow 0. \tag{4.22.2}$$

Again because taking Frobenius invariants is an exact functor this induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Lie}(\pi_1(F))^{\varphi=1} & \longrightarrow & \text{Lie}(\pi_1(X_{\text{ur}}))^{\varphi=1} & \longrightarrow & \text{Lie}(G^{\text{et}}) & \longrightarrow & 0 \\ & & \simeq\big\downarrow & & \big\downarrow & & \big\downarrow\simeq & & \\ 0 & \longrightarrow & \text{Lie}(\pi_1(F^{\text{et}})) & \longrightarrow & \text{Lie}(\pi_1(X_{\text{et}})) & \longrightarrow & \text{Lie}(G^{\text{et}}) & \longrightarrow & 0. \end{array} \tag{4.22.3}$$

Since the outside vertical arrows are isomorphisms it follows that the middle vertical arrow is an isomorphism as well.

The formality theorem

4.23. Let k be a finite field with p^a elements, W the ring of Witt vectors of k, and K the field of fractions of W. Denote by σ the automorphism of K induced by the canonical lifting of Frobenius to W. Also, fix an embedding $\iota : K \hookrightarrow \mathbb{C}$.

4.24. Let $(X, x)/k$ be a smooth proper scheme with a point $x \in X(k)$, and let $\{(\mathcal{V}_i, \varphi_i)\}$ be a family of ι–pure F–isocrystals on X/K. Let $\mathcal{C} \subset \mathrm{Isoc}(X/K)$ be the smallest Tannakian subcategory closed under extensions which contains the family $\{\mathcal{V}_i\}$. Let G be the pro–reductive completion of $\pi_1(\mathcal{C}, \omega_x)$, and let $\mathbb{L}(\mathcal{O}_G)$ denote the G–equivariant ind–F–isocrystal discussed in (2.15). Cup product makes $H^*(X/K, \mathbb{L}(\mathcal{O}_G))$ into an object of $G - \mathrm{dga}_K$.

Theorem 4.25. *There is an isomorphism of pointed stacks*

$$X_{\mathcal{C}} \simeq [\mathbb{R}\mathrm{Spec}_G(H^*(X/K, \mathbb{L}(\mathcal{O}_G)))/G]. \tag{4.25.1}$$

Note that (1.12) is an immediate consequence.

The proof of (4.25) will be in several steps (4.26)–(4.32).

4.26. Let $\mathbb{S}$ be as in (4.6), and let V be a K–vector space with an automorphism $\alpha : V \to V$ which is a direct limit of automorphisms of finite–dimensional subspaces of V. For $\lambda \in \mathbb{C}^*$, let $V_{\mathbb{C},\lambda} \subset V_{\mathbb{C}}$ be the generalized eigenspace corresponding to λ, so we have a decomposition $V_{\mathbb{C}} = \oplus_\lambda V_{\mathbb{C},\lambda}$. Define

$$V_{\mathbb{C}}^\dagger := \oplus_{\lambda \in \mathbb{S}} V_{\mathbb{C},\lambda}, \quad V_{\mathbb{C}}^\flat := \oplus_{\lambda \notin \mathbb{S}} V_{\mathbb{C},\lambda} \tag{4.26.1}$$

so that $V_{\mathbb{C}} = V_{\mathbb{C}}^\dagger \oplus V_{\mathbb{C}}^\flat$. The action of $\mathrm{Aut}(\mathbb{C}/K)$ preserves this decomposition, and hence by descent theory there is a natural decomposition $V = V^\dagger \oplus V^\flat$. Note also that this decomposition is functorial in the sense that if $f : V \to W$ is a morphism of K–spaces and $\beta : W \to W$ an automorphism such that $\beta \circ f = f \circ \alpha$, then f decomposes as $(f^\dagger, f^\flat) : V^\dagger \oplus V^\flat \to W^\dagger \oplus W^\flat$. Indeed this can be verified after base change to $\mathbb{C}$ in which case it follows from the statement that f must preserve the decompositions into generalized eigenspaces.

Let G be as in (4.24), and let $\alpha_G : G \to G$ be the automorphism induced by the a–th power of Frobenius.

Lemma 4.27. *The vector space with automorphism $(\mathcal{O}_G, \alpha_G^*)$ is a direct limit of finite dimensional vector spaces with automorphism. Moreover, all the eigenvalues of $\alpha_G^* \otimes_\iota \mathbb{C}$ have absolute value* 1 *and are in $\mathbb{S}$, so in particular $\mathcal{O}_G = \mathcal{O}_G^\dagger$.*

Proof. Let $\pi_i : G \to G_i$ be the quotient corresponding the inclusion of the semi–simple objects of $\langle \mathcal{V}_i \rangle_\otimes$ into $\mathcal{C}$. Then π_i is compatible with Frobenius and since every object semi–simple object of $\mathcal{C}$ lies in $\langle \oplus_{i \in J} \mathcal{V}_i \rangle_\otimes$ for some finite set J it suffices to prove the lemma for G_i. By the same argument used in (4.3) we may even assume that $G = \pi_1(\langle \mathcal{V} \rangle_\otimes, \omega_x)$ for some ι–pure F–isocrystal $(\mathcal{V}, \varphi_{\mathcal{V}})$. Set $V = \omega_x(V)$ and let $\alpha_V : V \to V$ be the automorphism induced by the a–th power of Frobenius. Define L to be the dual of $\Lambda^r V$ (where r is the rank of $\mathcal{V}$), and let $\alpha_L : L \to L$ be the automorphism induced by α_V. Then just as in the proof of (4.4), the inclusion $G \hookrightarrow GL(V)$ induces a surjection of vector spaces with automorphism

$$\mathrm{Sym}^\bullet(V \otimes V^*) \otimes \mathrm{Sym}^\bullet(L \otimes L^*) \longrightarrow \mathcal{O}_G. \tag{4.27.1}$$

Hence to prove the lemma it suffices to show that the automorphism of $(V \otimes V^*) \otimes \mathbb{C}$ (resp. $(L \otimes L^*) \otimes \mathbb{C}$) induced by α_V (resp. α_L) only has eigenvalues in $\mathbb{S}$ with absolute value 1. This follows from the assumption that $(\mathcal{V}, \varphi)$ is ι–pure. □

Proposition 4.28. *Let $M \in G - \mathrm{dga}_K$ be an algebra with an automorphism $\alpha : M \to M$ compatible with the automorphism α_G.*

(i) *The G–equivariant algebra structure on M restricts to a G–equivariant differential graded algebra structure on $M^\dagger$.*

(ii) *The inclusion $M^\dagger \hookrightarrow M$ induces an isomorphism $H^i(M^\dagger) \simeq H^i(M)^\dagger$ for every i.*

Proof. For (i), note that the statement that the G–action on M induces a G–action on $M^\dagger$ is equivalent to the statement that if $\rho : M \to M \otimes \mathcal{O}_G$ denotes the map giving the G–action then $\rho(M^\dagger) \subset M^\dagger \otimes \mathcal{O}_G$. Since α is compatible with α_G, the map ρ induces by functoriality a map $M^\dagger \to (M \otimes \mathcal{O}_G)^\dagger$. On the other hand, the preceding lemma implies that $(M \otimes \mathcal{O}_G)^\dagger = M^\dagger \otimes \mathcal{O}_G$, and therefore the action of G induces an action on $M^\dagger$. That $M^\dagger$ is closed under the product and differential follows from the functoriality of the decomposition $M = M^\dagger \oplus M^\flat$.

As for (ii), note that again by functoriality, we have $d = d^\dagger \oplus d^\flat : M^\dagger \oplus M^\flat \to M^\dagger \oplus M^\flat$. It follows that $H^i(M^\dagger)$ is a direct summand of $H^i(M)$ for all i, and is equal to $H^i(M)^\dagger$. □

4.29. The homomorphism

$$2\log_q |\cdot| : \mathbb{S} \longrightarrow \mathbb{R} \tag{4.29.1}$$

defined in (4.6.2) sends every element $\zeta^{-1} \cdot \zeta^\sigma$ ($\zeta \in \mathbb{S}$, $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$) to zero. It follows that for any K–vector space V with an automorphism α as in (4.26), there is a natural action of the diagonalizable group scheme $D(\mathbb{R})$ on $V^\dagger$. In particular, in the setting of (4.28) the algebra $M^\dagger$ has a natural action of $D(\mathbb{R})$.

In general, if V is a K–vector space then to give an action of $D(\mathbb{R})$ on V is equivalent to giving a decomposition $V = \oplus_{w \in \mathbb{R}} V_w$ ([SGA3], I.4.7.3). We will refer to the set

$$\mathrm{Wt}(V) := \{w \in \mathbb{R} | V_w \neq 0\} \tag{4.29.2}$$

as the *weights* of V, and say that V is *pure of weight* w_0 if $V_w = 0$ for $w \neq w_0$.

In addition this action of $D(\mathbb{R})$ commutes with the action of G. For this it suffices be descent theory to show that the action of G preserves the decomposition

$$V^\dagger \otimes \mathbb{C} = \oplus_{w \in \mathbb{R}} W_w, \quad W_w := \oplus_{\lambda \in \mathbb{S}, 2\log_q |\lambda| = w} V_{\mathbb{C},\lambda}. \tag{4.29.3}$$

This follows from the (4.27) which shows that for any $w \in \mathbb{R}$

$$\oplus_{\lambda \in \mathbb{S}, 2\log_q |\lambda = w|} (V^\dagger \otimes \mathcal{O}_G)_{\mathbb{C},\lambda} = (\oplus_{\lambda \in \mathbb{S}, 2\log_q |\lambda| = w} V_{\mathbb{C},\lambda}) \otimes_{\mathbb{C}} \mathcal{O}_{G_{\mathbb{C}}}. \tag{4.29.4}$$

Thus $M^\dagger$ has a natural structure of an object in $G \times D(\mathbb{R}) - \mathrm{dga}_K$.

Proposition 4.30. *Let $A \in G - \mathrm{dga}_K$ be an algebra and $\alpha : A \to A$ an equivalence compatible with $\alpha_G : G \to G$ such that $H^i(A)^\dagger = H^i(A)$ for all i. Assume there exists an integer N such that $H^i(A) = 0$ for all $i \geq N$. Then there exists a pair (M, ρ), where $M \in G \times D(\mathbb{R}) - \mathrm{dga}_K$ and $\rho : M \to A$ is an equivalence in $G - \mathrm{dga}_K$ such that for all i the isomorphism $H^i(M) \to H^i(A)$ is compatible with the $D(\mathbb{R})$–actions.*

Proof. We construct inductively using the method of (3.31) an algebra $M_k \in G \times D(\mathbb{R}) - \mathrm{dga}_K$ and a morphism $\rho_k : M_k \to A$ in $G - \mathrm{dga}_K$ which is a k–stage minimal model such that the induced map $H^*(M_k) \to H^*(A)$ is compatible with the $D(\mathbb{R})$–actions.

First let $\rho_k : M_k \to A$ be a k–stage minimal model where $k > N$. By the $G - \mathrm{dga}_K$–version of (3.41) (see (3.43)) applied to the diagram

(4.30.1)
$$\begin{array}{ccc} & & M_k \\ & & \downarrow{\scriptstyle \rho_k} \\ & & A \\ & & \downarrow{\scriptstyle \alpha} \\ M_k & \xrightarrow{\rho_k} & A \end{array}$$

there exists an automorphism $\beta : M_k \to M_k$ such that the induced diagram

(4.30.2)
$$\begin{array}{ccc} H^*(M_k) & \xrightarrow{\rho_k} & H^*(A) \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \alpha} \\ H^*(M_k) & \xrightarrow{\rho_k} & H^*(A) \end{array}$$

commutes. Now consider the map $\rho_k^\dagger : M_k^\dagger \to A$. By (4.28 (ii)), the induced map $H^i(M_k^\dagger) \to H^i(A)$ is an isomorphism for $i \leq k$ and an injection for $i = k+1$. Replacing $M_k^\dagger$ by a k–stage minimal model for $M_k^\dagger$ in $G \times D(\mathbb{R}) - \mathrm{dga}_K$ we see that we can find a pair (M_k, ρ_k) with $M_k \in G \times D(\mathbb{R}) - \mathrm{dga}_K$ and $\rho_k : M_k \to A$ a k–stage minimal model such that the map $H^*(M_k) \to H^*(A)$ is compatible with the $D(\mathbb{R})$–actions. The pair (M_{k+1}, ρ_{k+1}) is now constructed as follows. Since $H^i(A) = 0$ for $i > N$ and $k > N$, we just have to kill of the group $H^{k+2}(M_k)$. Choose a $G - D(\mathbb{R})$–equivariant section $s : H^{k+2}(M_k) \to Z^{k+2}_{M_k}$ and set $M^2_{k+1} := M_k \otimes_s \Lambda(H^{k+2}(M_k))$, where $H^{k+2}(M_k)$ is placed in degree $k+1$. Choosing any map $\tau : H^{k+2}(M_k) \to A^{k+1}$ such that the diagram

(4.30.3)
$$\begin{array}{ccc} H^{k+2}(M_k) & \xrightarrow{s} & Z^{k+2}_{M_k} \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \rho_k} \\ A^{k+1} & \xrightarrow{d} & A^{k+2} \end{array}$$

commutes, we obtain an object $M^2_{k+1} \in G \times D(\mathbb{R}) - \mathrm{dga}_K$ and a map $\rho^2_{k+1} : M^2_{k+1} \to A$ such that the induced map on cohomology is compatible with the $D(\mathbb{R})$–actions. Repeating this construction to get $(M^i_{k+1}, \rho^i_{k+1})$ inductively and finally passing to the limit, we get the desired pair (M_{k+1}, ρ_{k+1}). Passing to the limit in k now yields (M, ρ). □

Lemma 4.31. *Let $A \in \mathrm{Ho}(G \times D(\mathbb{R}) - \mathrm{dga}_K)$ be an algebra such that for each $i \geq 0$ the $D(\mathbb{R})$–representation $H^i(A)$ is pure of weight i. Then A is isomorphic in $\mathrm{Ho}(G \times D(\mathbb{R}) - \mathrm{dga}_K)$ to $H^*(A)$.*

Proof. The key to the proof is a close look at the construction of the cofibrant model $M \to A$ given in (3.31 (ii)). We claim that as a $G \times D(\mathbb{R})$–equivariant algebra (that is, after forgetting the differential), M is isomorphic to an algebra of the form

$$\Lambda[\bigoplus_{i=1}^{\infty} V_i], \tag{4.31.1}$$

where each V_i is placed in degree i, such that if $C_i \subset V_i$ denote the closed elements in V_i then there is a decomposition $V_i = C_i \oplus N_i$, where all the weights of N_i are greater than i and $\mathrm{Wt}(C_i) = \{i\}$.

Before explaining why M has this structure, let us note how (4.31) follows. Define a map $q : M \rightarrow H^*(M)$ by sending all N_i to 0 and $c \in C_i$ to the class of c. If $(c, n) \in C_i \oplus N_i$ and $(c', n') \in C_{i'} \oplus N_{i'}$, then by considering the weights we see that $(c, n) \wedge (c', n')$ is equal to $c \wedge c'$ plus an element of $N_{i+i'}$. Hence q is compatible with the algebra structure. Furthermore, if $(c, n) \in C_i \oplus N_i$ is a boundary, say equal to $d\lambda$, then decomposing λ into weight components we see that c is also a boundary. From this it follows that q is also compatible with the differential and is an equivalence.

To prove that M has the form described above, we show that each k–stage minimal model $M_k \rightarrow A$ has the form

$$\Lambda(\bigoplus_{i=1}^{k} V_i), \tag{4.31.2}$$

where V_i admits a decomposition $V_i = C_i \oplus N_i$ as above. To see this proceed by induction on k. So suppose the result is true for $M_k \rightarrow A$. Recall that M_{k+1} was constructed from M_k as follows. First choose a complement $C_{k+1} \subset H^{k+1}(A)$ to $H^{k+1}(M_k)$, and choose a lifting $\ell : C_{k+1} \rightarrow Z_A^{k+1}$ compatible with the $G \times D(\mathbb{R})$–actions. Then set $M_k^1 := M_k \otimes \Lambda(C_{k+1})$ and let $M_k^1 \rightarrow A$ be the map induced by ℓ. Next let $K_{k+1}^1 \subset H^{k+2}(M_k^1)$ be the kernel of $H^{k+2}(M_k^1) \rightarrow H^{k+2}(A)$. Note that the weights of K_{k+1}^1 are all greater than or equal to $k+2$. Set $M_k^2 := M_k^1 \otimes \Lambda(K_{k+1}^1)$ with differential the map $K_{k+1} \rightarrow M_k^1$ obtained from a lifting $K_{k+1}^1 \rightarrow Z_{M_k^1}^{k+2}$ of the inclusion into $H^{k+2}(M_k^1)$. Observe that the underlying algebra of M_k^2 is $M_k \otimes \Lambda(C_{k+1} \oplus K_{k+1}^1)$. Repeating this construction inductively we obtain $M_k^j \rightarrow A$ for which the underlying $G \times D(\mathbb{R})$–equivariant algebra of M_k^j is of the form

$$M_k \otimes \Lambda(C_{k+1} \oplus \bigoplus_{t=1}^{j} K_{k+1}^t). \tag{4.31.3}$$

Since $M_{k+1} = \varinjlim M_k^j$, the underlying algebra of M_{k+1} is of the form

$$M_k \otimes \Lambda(C_{k+1} \oplus \bigoplus_{t=1}^{\infty} K_{k+1}^t). \tag{4.31.4}$$

Setting $N_{k+1} = \bigoplus K_{k+1}^j$ it follows that M_{k+1} also has the desired form. □

4.32 (Proof of (4.25)). Let $\alpha_G : G \rightarrow G$ (resp. α) be the automorphism of G (resp. $\mathbb{R}\Gamma(\mathbb{L}(\mathcal{O}_G))$) induced by the a–th power of Frobenius on X. Choosing a cofibrant representative for $\mathbb{R}\Gamma(\mathbb{L}(\mathcal{O}_G))$ in $\mathrm{Ho}(G - \mathrm{dga}_F)$ we obtain a representative (M, α) for the pair $(\mathbb{R}\Gamma(\mathbb{L}(\mathcal{O}_G)), \alpha)$ which by ([Ke], 5.6.2) satisfies the assumptions of (4.30). We thus obtain a model $A \in \mathrm{Ho}(G \times D(\mathbb{R}) - \mathrm{dga}_K)$ for $\mathbb{R}\Gamma(\mathbb{L}(\mathcal{O}_G))$ satisfying the assumptions of (4.31) (again by ([Ke], 5.6.2)). From this (4.25) follows.

Remark 4.33. We do not know if the decomposition in (4.25) can be chosen in a functorial manner. For example, if $f : Y \to X$ is a morphism of smooth proper varieties over k, and if $\{(\mathcal{V}_i, \varphi_i)\}$ is a family of F–isocrystals on X as in (4.24), then there is a natural induced morphism of pointed stacks with F–isocrystal structure

$$f^* : Y_{f^*\mathcal{C}} \longrightarrow X_{\mathcal{C}}, \tag{4.33.1}$$

where $f^*\mathcal{C}$ denotes the smallest Tannakian subcategory of $\mathrm{Isoc}(Y/K)$ closed under extensions and containing all the $f^*\mathcal{V}_i$. We do not know if this morphism is determined by morphisms on cohomology groups as in ([KPT]).

Formality of étale homotopy type

Let $Y/\mathbb{F}_q$ be a smooth proper scheme over a finite field with a point $y \in Y(\mathbb{F}_q)$, and let $(X, x)/k$ be its base change to some algebraic closure $\mathbb{F}_q \to k$. Let $\mathcal{C}_{\mathrm{et}}$ be the smallest Tannakian subcategory of the category of smooth $\mathbb{Q}_p$–sheaves on X closed under extensions and containing all restrictions of smooth $\mathbb{Q}_p$–sheaves on Y. Denote by G^{et} the pro–reductive completion of the group $\pi_1(\mathcal{C}_{\mathrm{et}}, x)$, and let $\mathbb{V}(\mathcal{O}_{G^{\mathrm{et}}})$ be the sheaf defined in (4.16).

Theorem 4.34. *There is an isomorphism in* $\mathrm{Ho}(G^{\mathrm{et}}_K - \mathrm{dga}_K)$

$$\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G^{\mathrm{et}}})) \otimes_{\mathbb{Q}_p} K \simeq H^*(\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G^{\mathrm{et}}}))) \otimes_{\mathbb{Q}_p} K. \tag{4.34.1}$$

Proof. Let K_q be the field of fractions of the ring of Witt vectors of $\mathbb{F}_q$, and let $\mathcal{C} \subset \mathrm{Isoc}(Y/K_q)$ be the smallest Tannakian subcategory closed under extensions and containing the underlying isocrystals of all unit–root F–isocrystals on Y/K_q. Let K denote the field of fractions of the ring of Witt vectors of k, and let $\widetilde{\mathcal{C}} \subset \mathrm{Isoc}(X/K)$ denote the smallest Tannakian subcategory closed under extensions and containing the base changes to X/K of the isocrystals in $\mathcal{C}$. If $\{(\mathcal{V}_i, \varphi_i)\}$ denotes the collection of unit–root F–isocrystals on Y/K, then $\widetilde{\mathcal{C}}$ is the smallest Tannakian subcategory of $\mathrm{Isoc}(X/K)$ closed under extensions and containing all the $\mathcal{V}_i|_{X/K}$.

Lemma 4.35. *The natural functor* $\mathcal{C} \otimes_{K_q} K \to \widetilde{\mathcal{C}}$ *is an equivalence of categories.*

Proof. If $\mathcal{V} \in \mathcal{C}$, then

$$H^*(Y/K_q, \mathcal{V}) \otimes_{K_q} K \simeq H^*(X/K, \mathcal{V} \otimes_{K_q} K). \tag{4.35.1}$$

This implies that the functor is fully faithful and its essential image is closed under extensions. It follows that the essential image is a Tannakian subcategory of $\mathcal{C}$ closed under extensions and containing all the $\mathcal{V}_i$. By definition of $\mathcal{C}$ this implies that the functor is essentially surjective. □

Let G be the pro–reductive completion of $\pi_1(\mathcal{C}, y)$, and let $\mathbb{L}(\mathcal{O}_G)$ be the G–equivariant ind–isocrystal on Y/K_q defined in (2.16). By (4.25), there is an isomorphism in $\mathrm{Ho}(G - \mathrm{dga}_K)$

$$\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_G)) \simeq H^*(Y/K_q, \mathbb{L}(\mathcal{O}_G)). \tag{4.35.2}$$

On the other hand, the proof of (4.19) shows that there is a natural isomorphism $G_K \simeq G^{\mathrm{et}}_K$ and a cofibrant model $A \in G_K - \mathrm{dga}_K$ for $\mathbb{R}\Gamma_{\mathrm{cris}}(\mathbb{L}(\mathcal{O}_G))$ such that $\mathbb{R}\Gamma_{\mathrm{et}}(\mathbb{V}(\mathcal{O}_{G^{\mathrm{et}}})) \otimes_{\mathbb{Q}_p} K$ is isomorphic to a sub–algebra $A_{\mathrm{et}} \subset A$ which admits a retraction $A \to A_{\mathrm{et}}$. If $t : A \to H^*(A)$ is an equivalence inducing the identity on cohomology, then the composite

$$A_{\mathrm{et}} \hookrightarrow A \to H^*(A) \to H^*(A_{\mathrm{et}}) \tag{4.35.3}$$

induces an equivalence $A_{\mathrm{et}} \to H^*(A_{\mathrm{et}})$. □

Formality theorem over $\mathbb{C}$ via reduction mod p

4.36. Let $X/\mathbb{C}$ be a smooth proper variety, and let X_{an} denote the associated analytic space. The *rational homotopy type* of X_{an} is an object $A^X \in \mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}})$ defined for example in ([DGMS], §2). The algebra $A^X \otimes \mathbb{R}$ is isomorphic in $\mathrm{Ho}(\mathrm{dga}_{\mathbb{R}})$ to the differential graded algebra of real C^∞–forms on X_{an}. We now give a proof based on "reduction mod p" techniques rather than Hodge theory of the following "formality theorem":

Theorem 4.37 ([DGMS] §6, [Su] 12.1)**.** *The rational homotopy type A^X of a smooth proper scheme $X/\mathbb{C}$ is formal. That is, in* $\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}})$ *there is an isomorphism $A^X \simeq H^*(A^X)$.*

Proof. As shown in ([Su], §12), the algebra A^X is formal if and only if the algebra $A^X \otimes \mathbb{C}$ is formal. On the other hand, if $\Omega^\bullet_X$ denotes the algebraic de Rham complex on X and if $\mathcal{E}^*_{X_{\mathrm{an}}}$ denotes the complex of $\mathbb{C}$–valued C^∞–forms, then the natural map $\Omega^\bullet_X|_{X_{\mathrm{an}}} \to \mathcal{E}^*_{X_{\mathrm{an}}}$ induces an isomorphism

$$\mathbb{R}\Gamma(\Omega^\bullet_X) \longrightarrow A^X \otimes \mathbb{C} \tag{4.37.1}$$

in $\mathrm{Ho}(\mathrm{dga}_{\mathbb{C}})$. Here we view $\Omega^\bullet_X$ as an object in the category of sheaves of differential graded algebras on X_{et} which by the same reasoning as in (2.14) has a natural model category structure such that we have a derived global section functor $\mathbb{R}\Gamma$ to $\mathrm{Ho}(\mathrm{dga}_{\mathbb{C}})$.

Let us first consider the special case when X admits a model over a number field L. By "spreading out", we can in this case find a smooth proper scheme Y over the ring of integers $\mathcal{O}_K$ of a complete local field K and an embedding $K \hookrightarrow \mathbb{C}$ such that X is isomorphic to $Y \otimes_{\mathcal{O}_K} \mathbb{C}$. Furthermore, we may assume that $\mathcal{O}_K$ is equal to the ring of Witt vectors of its residue field k. In this case, the usual comparison between de Rham and crystalline cohomology induces an isomorphism of differential graded algebras

$$\mathbb{R}\Gamma(\Omega^\bullet_{Y_K/K}) \simeq \mathbb{R}\Gamma_{\mathrm{cris}}(Y_0, \mathcal{O}_{Y_0/K}), \tag{4.37.2}$$

where Y_0 denotes the reduction of Y modulo the maximal ideal of $\mathcal{O}_K$ and we have written $\mathbb{R}\Gamma_{\mathrm{cris}}(Y_0, \mathcal{O}_{Y_0/K})$ for the object of dga_K obtained from the construction (2.13) by taking $\mathcal{C}$ to be the category of unipotent isocrystals (that is, $\mathbb{R}\Gamma_{\mathrm{cris}}(Y_0, \mathcal{O}_{Y_0/K})$ is the differential graded algebra corresponding to the affine stack $X_{\mathcal{C}}$ obtained from the category of unipotent isocrystals). Therefore, in the case when we have a model over a number field, (4.37) follows from (4.25) since

$$\mathbb{R}\Gamma_{\mathrm{cris}}(Y_0, \mathcal{O}_{Y_0/K}) \otimes_K \mathbb{C} \simeq \mathbb{R}\Gamma(\Omega^\bullet_{Y_K/K}) \otimes_K \mathbb{C} \simeq A^X \otimes_{\mathbb{Q}} \mathbb{C}. \tag{4.37.3}$$

The general case of (4.37) can be reduced to the case when we have a model over a number field. To see this, choose first an algebra $A \subset \mathbb{C}$ of finite type over $\mathbb{C}$ and a smooth model X_A/A for X. Set $T = \mathrm{Spec}(A)$. By shrinking T if necessary, we can also assume that T is smooth over $\mathbb{Q}$. The inclusion $A \hookrightarrow \mathbb{C}$ induces a point $P_0 \in T_{\mathbb{C}}(\mathbb{C})$. Let $X_{T_{\mathbb{C}}}$ denote the base change of X_A. The induced morphism of analytic spaces $X_{T_{\mathbb{C}},\mathrm{an}} \to T_{\mathbb{C},\mathrm{an}}$ is in the C^∞–category locally trivial, and hence since the rational homotopy type is a homotopy invariant, it suffices to verify (4.37) for fibers of $X_{T_{\mathbb{C}}}$ over a dense (in the analytic topology) set of points of $T_{\mathbb{C}}(\mathbb{C})$. Since the set $T(\overline{\mathbb{Q}})$ has this property, we are therefore reduced to the case when X admits a model over a number field. □

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