EXERCISES PERTAINING TO LECTURE 1

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Exercise 1.1. Let X be a smooth projective variety over a field k. Show that the bounded derived category of the abelian category of coherent sheaves on X is equivalent to the subcategory of the derived category of bounded complexes of $\mathcal{O}_X$-modules whose cohomology sheaves are coherent.

Exercise 1.2. Let $\alpha : X \to X$ be an endomorphism of a smooth projective variety over a field k. Describe the object $P \in D(X \times X)$ such that

$$\mathbf{L}\alpha^* : D(X) \to D(X)$$

is given by Φ^P.

Exercise 1.3. Let X and Y be smooth projective varieties over a field k and let $\Phi : D(X) \to D(Y)$ be a functor defined by $P \in D(X \times Y)$. Show that the functor $D(Y) \to D(X)$

$$\Phi^{P^\vee} \circ S_Y \quad (\text{resp. } S_X \circ \Phi^{P^\vee})$$

is left (resp. right) adjoint to Φ, where $P^\vee$ denotes $\mathcal{R}Hom(P, \mathcal{O}_{X \times Y})$.

Exercise 1.4. With notation as in exercise 1.6, let $k \hookrightarrow \Omega$ be an inclusion of fields, and let P_Ω denote the base change of P to $X_\Omega \times_\Omega Y_\Omega$. Show that Φ^P is an equivalence if and only if

$$\Phi^{P_\Omega} : D(X_\Omega) \to D(Y_\Omega)$$

is an equivalence.

Exercise 1.5. Let n, m be integers. Show that the diagram

$$\begin{array}{ccc} H^0(X, \omega_X^{\otimes n}) \times H^0(X, \omega_X^{\otimes m}) & \xrightarrow{\text{mult}} & H^0(X, \omega_X^{\otimes (n+m)}) \\ \downarrow & & \downarrow \\ \text{Hom}(\text{id}, S_X^n) \times \text{Hom}(\text{id}, S_X^m) & \xrightarrow{\gamma} & \text{Hom}(\text{id}, S_X^{n+m}), \end{array}$$

where γ is the map sending two maps $a : \text{id} \to S_X^n$ and $b : \text{id} \to S_X^m$ to the composition

$$\text{id} \xrightarrow{a} S_X^n \xrightarrow{S_X^n(b)} S_X^{n+m}$$

and the vertical maps are as defined in the lecture.

Exercise 1.6. Let X be a smooth projective curve over a field k.

(i) Show that for any two coherent sheaves $\mathscr{F}, \mathscr{G}$ on X we have

$$\mathrm{Ext}^i(\mathscr{F}, \mathscr{G}) = 0$$

for $i > 1$.

(ii) Using (i) show that for any object $K \in D(X)$ we have

$$K \simeq \oplus_i \mathscr{H}^i(K)[-i].$$

Exercise 1.7. Let $\Phi : D(X) \to D(Y)$ be a derived equivalence between smooth projective varieties such that for every closed point $x \in X$ the complex $\Phi(\kappa(x))$ is, up a shift, a sheaf. Show that then Φ is of the form $\Phi^{\mathscr{F}[n]}$ for a sheaf $\mathscr{F}$ on $X \times Y$ flat over both X and Y.

Exercise 1.8. Let Y be a genus 1 curve over a field k and let d be an integer. Assume that there exists a universal sheaf $\mathscr{P}_d$ on $Y \times \mathrm{Pic}^d_Y$. Show that

$$\Phi^{\mathscr{P}_d} : D(Y) \to D(\mathrm{Pic}^d_Y)$$

is an equivalence. Hint: what is the inverse functor?

Exercise 1.9. Let X be a curve of genus 1 over a field k.

(i) Let $I_\Delta \subset \mathscr{O}_{X \times X}$ be the ideal of the diagonal. Show that $\Phi^{I_\Delta} : D(X) \to D(X)$ is an equivalence.

(ii) Assume we have a closed point $x \in X$ and let let $\kappa(x)$ be the skyscraper sheaf of x. Let $K \in D(X \times X)$ denote the cone of the natural map

$$\kappa(x)^\vee \boxtimes \kappa(x) \to \Delta_{X*}\mathscr{O}_X.$$

Show that $\Phi^K : D(X) \to D(X)$ is an equivalence and represented by a sheaf on $X \times X$.

Remark: This is a special case of the notion of spherical twist [1, Chapter 8].

References

1. D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006. MR 2244106