BOREL-MOORE HOMOLOGY, RIEMANN-ROCH TRANSFORMATIONS, AND LOCAL TERMS

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1. Introduction

The purpose of this paper is to develop various properties of étale Borel-Moore homology and study its relationship with intersection theory. The paper roughly breaks into three parts as follows.

Let k be an algebraically closed field, and let ℓ be a prime number invertible in k.

Cycle class maps and intersection products (§2).

1.1. For a finite type separated Deligne-Mumford stack X/k and integer i, the *i-th ℓ-adic Borel-Moore homology group of X*, denoted $H_i(X)$, is defined to be $H^{-i}(X, \Omega_X)$, where $\Omega_X \in D_c^b(X, \mathbb{Q}_\ell)$ is the ℓ-adic dualizing complex of X. These groups were considered already by Laumon in [17] (and, we have been informed, by Grothendieck in unpublished work), where he showed they enjoyed a number of good properties. In particular, there is a cycle class map

$$\mathrm{cl}_X^s : A_s(X) \to H_{2s}(X)(-s),$$

where $A_s(X)$ denotes the group of s-cycles on X modulo rational equivalence. In the case when X is smooth of dimension d we have $\Omega_X = \mathbb{Q}_\ell(d)[2d]$ and this map reduces to the usual cycle class map $A_s(X) \to H^{2d-2s}(X, \mathbb{Q}_\ell(d-s))$.

Using Gabber's localized cycle classes, we show in section 2.32 that for any cartesian square

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g'"] & V \arrow[d, "g"] \\ X \arrow[r, "f"] & Y, \end{tikzcd} \tag{1.1.1}$$

where f is a regular imbedding of codimension c, there is a homological Gysin homomorphism

$$f^!_{\mathrm{hom}} : H_i(V) \to H_{i-2c}(W)(c).$$

On the other hand we can also consider Fulton's refined intersection product

$$f^! : A_s(V) \to A_{s-c}(W).$$

A slightly weakened form of one of our main results about Borel-Moore homology is the following:

Theorem 1.2 (Special case of 2.34)**.** *For a cartesian square (1.1.1) of quasi-projective schemes with f a regular imbedding of codimension c, the diagram*

$$\begin{tikzcd} A_s(V) \arrow[r, "f^!"] \arrow[d, "\mathrm{cl}_V"] & A_{s-c}(W) \arrow[d, "\mathrm{cl}_W"] \\ H_{2s}(V)(-s) \arrow[r, "f^!_{\mathrm{hom}}"] & H_{2(s-c)}(W)(c-s) \end{tikzcd}$$

commutes for all s.

Étale bivariant operations (§3 and §4).

1.3. As noted by Fulton and MacPherson in [11, §7.4], for any morphism $f : X \to Y$ of separated Deligne-Mumford stacks of finite type over k one can consider $H^{2i}(X, f^!\mathbb{Q}_\ell(i))$, which plays the role in étale cohomology of the bivariant groups $A^i(f : X \to Y)$ in intersection theory. This point of view is supported by the fact that there is a natural pairing

$$H^{2i}(X, f^!\mathbb{Q}_\ell(i)) \times H_{2s}(Y)(-s) \to H_{2(s-i)}(X)(-(s-i)),$$

similar to the (almost tautological) pairing

$$A^i(f : X \to Y) \times A_s(Y) \to A_{s-i}(X).$$

Motivated by this we show how to define a map

$$\tau^X_Y : K(f\text{-perfect complexes}) \to \widehat{H}^*(X \to Y)$$

from the Grothendieck group of f-perfect complexes (see for example 3.10) to the graded $\mathbb{Q}_\ell$-vector space

$$\widehat{H}^*(X \to Y) := \oplus_i H^{2i}(X, f^!\mathbb{Q}_\ell(i)).$$

This should be viewed as an étale version of the Fulton-MacPherson construction (see [10, p. 366]) which gives a map

$$\tau^{FM} : K(f\text{-perfect complexes}) \to A^*(f : X \to Y).$$

1.4. After giving the definition of τ_Y^X and establishing its basic properties, we prove the compatibility of τ_Y^X with the Riemann-Roch transformation in the following sense. If Y is smooth of dimension d, then there is a canonical isomorphism $f^!\mathbb{Q}_\ell(i) \simeq \Omega_X(i-d)[-2d]$, and therefore an isomorphism

$$\widehat{H}^j(X \to Y) \simeq \oplus_i H_{2(d-j)}(X)(-(d-j)).$$

This should be viewed as capping with the fundamental class of Y as shown by the following (note that in this case any complex of coherent sheaves on X is f-perfect):

Theorem 1.5 (Theorem 4.17 in the text)**.** *Let $\tau_X : K_0(X) \to A_*(X)_{\mathbb{Q}}$ be the Riemann-Roch transformation. Then the diagram*

$$\begin{tikzcd} K_0(X) \arrow[r, "\tau_X"] \arrow[d, "\mathrm{td}(f^*T_Y)\cdot\tau_Y^X"] & A_*(X)_{\mathbb{Q}} \arrow[d, "\mathrm{cl}_X"] \\ \oplus_i \widehat{H}^i(X \to Y) \arrow[r, "\simeq"] & \oplus_i H_{2(d-i)}(X)(-(d-i)) \end{tikzcd}$$

commutes.

More generally we prove a compatibility of the transformation τ_Y^X with the Fulton-MacPherson construction τ^{FM} in 4.24.

Applications to local terms §5.

1.6. One of our original motivations for pursuing this work is its applications to local terms, and in particular to understanding local terms for correspondences in the smooth case. Let k be an algebraically closed field, and let $c = (c_1, c_2) : C \to X \times X$ be a morphism of quasi-projective k-schemes with X smooth of dimension d. Following SGA 5 [12], an *action of c on $\mathbb{Q}_\ell$* is a map $u : c_1^*\mathbb{Q}_\ell \to c_2^!\mathbb{Q}_\ell$. Since X is smooth of dimension d, this is equivalent to a global section

$$u \in H^0(C, \Omega_C(-d)[-2d]) = H_{2d}(C)(-d).$$

In particular, for $Z \in A_d(C)_{\mathbb{Q}}$ we get an induced action $u_Z : c_1^*\mathbb{Q}_\ell \to c_2^!\mathbb{Q}_\ell$ from $\mathrm{cl}(Z) \in H_{2d}(C)(-d)$.

Associated to an action $u : c_1^*\mathbb{Q}_\ell \to c_2^!\mathbb{Q}_\ell$ there is an associated class

$$\mathrm{Tr}_c(u) \in H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)}) = H_0(C)$$

defined as in [12, III, §4]. For any proper component $Z \subset \mathrm{Fix}(c)$ the *local term* of u at Z is defined as $\mathrm{lt}_Z(u) := \int_Z \mathrm{Tr}_c(u) \in \mathbb{Q}_\ell$.

Since X/k is smooth of dimension d the diagonal morphism $\Delta : X \to X \times X$ is a regular imbedding of codimension d and so we have a map $\Delta^! : A_d(C)_{\mathbb{Q}} \to A_0(\mathrm{Fix}(c))_{\mathbb{Q}}$. Our main result on local terms is the following:

Theorem 1.7. *Let $Z \in A_d(C)_{\mathbb{Q}}$ be a cycle with associated action $u_Z : c_1^*\mathbb{Q}_\ell \to c_2^!\mathbb{Q}_\ell$. Then*

$$\mathrm{Tr}_c(u_Z) = \mathrm{cl}(\Delta^! Z) \in H_0(\mathrm{Fix}(c)).$$

In particular, the local terms of u_Z are in $\mathbb{Q}$ and independent of ℓ, and in $\mathbb{Z}$ if $Z \in A_d(C)$.

In an appendix we collect together some basic facts about algebraic stacks that can be realized as the quotient of a quasi-projective scheme by an action of a finite group. In following work it will be important to consider not just quasi-projective schemes, but also Deligne-Mumford stacks which can be presented as the quotient of a quasi-projective scheme by a finite group action.

1.8. **Terminology and notation.** Throughout we work over a field k of characteristic p and ℓ denotes a prime not equal to p.

We work over a coefficient ring Λ assumed to be a Gorenstein local ring of dimension 0 and finite residue field of characteristic ℓ (the primary example we will consider is $\Lambda = \mathbb{Z}/(\ell^n)$ for some $n \geq 1$). For a separated Deligne-Mumford stack X of finite type over k we write $D^b_{ctf}(X, \Lambda)$ for the derived category of bounded complexes of Λ-modules which are of finite tor-dimension and have constructible cohomology sheaves. We will also consider $\Lambda = \overline{\mathbb{Q}}_\ell$, $\mathbb{Z}_\ell$, or $\mathbb{Q}_\ell$, in which case $D^b_{ctf}(X, \Lambda) = D^b_c(X, \Lambda)$ denotes the usual derived category of constructible Λ-modules as defined in [5, 1.1.2]. If the coefficient ring Λ is understood we sometimes write $D^b_{ctf}(X)$ instead of $D^b_{ctf}(X, \Lambda)$. The reader so inclined can work throughout with $\Lambda = \overline{\mathbb{Q}}_\ell$.

If $f : X \to Y$ is a morphism of finite type and separated Deligne-Mumford stacks over k, we write

$$f^*, f^! : D^b_{ctf}(X, \Lambda) \to D^b_{ctf}(X, \Lambda)$$

for the resulting operations on the derived category as in [12], [18], and [19]. If for every geometric point $\bar{x} \to X$ with image $\bar{y} \to Y$ the kernel of the map of stabilizer groups

$$\underline{\mathrm{Aut}}_X(\bar{x}) \to \underline{\mathrm{Aut}}_Y(\bar{y})$$

has order invertible in Λ then we also have operations

$$f_!, f_* : D^b_{ctf}(X, \Lambda) \to D^b_{ctf}(X, \Lambda).$$

If we wish to consider the non-derived operations we write $R^0 f_!$, $R^0 f_*$, etc.

We write $\Omega_X \in D^b_{ctf}(X, \Lambda)$ for the dualizing complex of X, and

$$D : D^b_{ctf}(X, \Lambda) \to D^b_{ctf}(X, \Lambda), \quad K \mapsto \mathscr{R}Hom(K, \Omega_X)$$

for the Verdier duality functor.

Following [10] we use the terminology *closed imbedding* for the notion called a *closed immersion* in [13].

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2. Borel-Moore homology

2.1. **Basic definitions.** In this section we review the basic definitions and results on Borel-Moore homology from [17] (with some slight generalization to Deligne-Mumford stacks). Throughout this section we work over an algebraically closed field k. We denote by Λ a coefficient ring as in 1.8.

Definition 2.2. Let X/k be a finite type separated Deligne-Mumford stack. For an integer i the *i-th Borel-Moore homology of X*, denoted $H_i(X)$, is defined to be $H^{-i}(X, \Omega_X)$.

Here are some basic properties of these groups.

2.3. If $f : X \to Y$ is a morphism of separated finite type Deligne-Mumford stacks over k, then we have a map $f_!\Omega_X \to \Omega_Y$ arising from the adjunction $f_!f^! \to \mathrm{id}$ and the canonical isomorphism $f^!\Omega_Y \simeq \Omega_X$. In particular, if f is proper and representable (which implies that $f_* \simeq f_!$ by [18, 5.2.1] and a similar argument in the case of ℓ-adic coefficients) we get a *pushforward map*

$$f_*^{\mathrm{hom}} : H_i(X) \to H_i(Y).$$

2.4. If $f : E \to X$ is a vector bundle of rank d, then we have canonical isomorphism $\Omega_E \simeq f^*\Omega_X(d)[2d]$, which induces an isomorphism $f_*\Omega_E \simeq \Omega_X(d)[2d]$. We therefore obtain an isomorphism

$$H_i(X) \simeq H_{i+2d}(E)(-d).$$

Lemma 2.5. *Let X be a finite type separated Deligne-Mumford stack over k of dimension d with structure morphism $f : X \to \mathrm{Spec}(k)$, and assume that the stabilizer groups of X have order invertible in Λ (for example if $\Lambda = \mathbb{Q}_\ell$).*

(i) If $\pi : X \to \overline{X}$ is the coarse moduli space of X, then the adjunction map $\Lambda_{\overline{X}} \to R\pi_*\Lambda_X$ is an isomorphism.

(ii) There is a canonical isomorphism $\tau_{\geq 2d} f_!\Lambda \to \oplus_{I_d}\Lambda(-d)[-2d]$, where I_d denotes the set of irreducible components of X of dimension d.

Proof. For (i) note that by [20, 5.1], we have $R\pi_!\Lambda_X \simeq R\pi_*\Lambda_X$. By base change [18, 5.5.6] and [19, 12.5.3], this implies that for every geometric point $\bar{x} \to \overline{X}$ the stalk $(R^q\pi_*\Lambda_X)_{\bar{x}}$ is isomorphic to $H^q(X_{\bar{x}}, \Lambda)$, where $X_{\bar{x}}$ denotes the fiber product $X \times_{\overline{X}} \bar{x}$. The condition that for every algebraically closed field Ω the map $X(\Omega) \to \overline{X}(\Omega)$ identifies the isomorphism classes $|X(\Omega)|$ with $\overline{X}(\Omega)$ (which is part of the definition of coarse moduli space) implies that the fibers of π are connected, and therefore we get that the adjunction map $\Lambda_{\overline{X}} \to R^0\pi_*\Lambda_X$ is an isomorphism. Furthermore, to verify that the sheaf $R^q\pi_*\Lambda_X$ is zero for $q > 0$, it suffices to verify that its stalk vanish at geometric points $\bar{x} : \mathrm{Spec}(k) \to \overline{X}$, since $R^q\pi_*\Lambda_X$ is a constructible sheaf. Now for such a point, the maximal reduced closed substack $X_{\bar{x},\mathrm{red}} \subset X_{\bar{x}}$ is isomorphic to $BG_{\bar{x}}$, where G_x is the stabilizer group of a lifting of $\bar{x}$ to a point $x \in X(k)$. For such a point $\bar{x}$ the stalk $(R^q\pi_*\Lambda_X)_{\bar{x}}$ is therefore isomorphic to the group cohomology $H^q(G_x, \Lambda)$ which is zero since the order of G_x is assumed invertible in k. This proves (i).

For (ii) we proceed by induction on the dimension d of X. If $d = 0$ the result follows from (i). For the inductive step, note that if $j : U \hookrightarrow X$ is a dense open substack with complement

$i : Z \hookrightarrow X$ of dimension $< d$, then from (ii) applied to Z (using the inductive hypothesis), the exact sequence

$$0 \to j_!\Lambda_U \to \Lambda_X \to i_*\Lambda_Z \to 0$$

and the resulting long exact sequence

$$\cdots \to H^q_c(U, \Lambda) \to H^q_c(X, \Lambda) \to H^q_c(Z, \Lambda) \to \cdots,$$

we conclude that the map $H^{2d}_c(U, \Lambda_U) \to H^{2d}_c(X, \Lambda_X)$ is an isomorphism. From this it follows that it suffices to consider the case when X and $\overline{X}$ are smooth, and $\overline{X}$ is a scheme. In this case we get the isomorphism in (ii) from the usual trace map [1, XVIII, 3.2.5] for each of the connected components of $\overline{X}$. □

Remark 2.6. Statement (ii) implies that $H^q_c(X, \Lambda) = 0$ for $q > 2d$.

2.7. In particular, if X is an irreducible separated Deligne-Mumford stack of finite type over k with stabilizer groups of order invertible in Λ and structure morphism $f : X \to \mathrm{Spec}(k)$, then we have $\tau_{\geq 2d} f_!\Lambda \simeq \Lambda(-d)[-2d]$, so we get a map

$$f_!\Lambda \to \Lambda(-d)[-2d]$$

which by adjunction defines a map

$$\Lambda \to f^!\Lambda(-d)[-2d] \simeq \Omega_X(-d)[-2d].$$

We denote the corresponding cohomology class in $H_{2d}(X)(-d)$ by $[X]$, and call it the *fundamental class of* X.

2.8. If X/k is smooth of relative dimension d, then $\Omega_X \simeq \Lambda(d)[2d]$, and therefore

$$H_i(X) \simeq H^{-i}(X, \Lambda(d)[2d]) \simeq H^{2d-i}(X, \Lambda(d)).$$

2.9. Because of the Tate twist involved in defining cycle classes, it is convenient to introduce the notation

$$\widetilde{H}_i(X) := H_{2i}(X)(-i),$$

and

$$\widetilde{H}_.(X) := \oplus_i \widetilde{H}_i(X).$$

2.10. Borel-Moore homology is the natural target for cycle class maps, as we now explain. For basics on intersection theory on Deligne-Mumford stacks we refer to [24].

Let X/k be a finite type separated Deligne-Mumford stack over k with stabilizer groups of order invertible in Λ. For an integer s, let $Z_s(X)$ denote the free abelian group generated by the irreducible closed s-dimensional substacks of X. We have a map

$$\mathrm{cl}^s_X : Z_s(X) \to \widetilde{H}_s(X)$$

sending the class of an irreducible closed substack $i : V \hookrightarrow X$ to the pushforward $i^{\mathrm{hom}}_*[V]$ of the fundamental class of V.

This cycle map passes to rational equivalence and defines a map of graded groups

$$\mathrm{cl}_X : A_.(X) \to \widetilde{H}_.(X).$$

In the case of schemes, this is [17, 6.3.1], the same proof gives the result for Deligne-Mumford stacks.

Furthermore, if $f : X \to Y$ is a proper representable morphism then the resulting diagram

$$\begin{tikzcd} A.(X) \arrow[r, "\mathrm{cl}_X"] \arrow[d, "f_*"] & \widetilde{H}.(X) \arrow[d, "f_*^{\mathrm{hom}}"] \\ A.(Y) \arrow[r, "\mathrm{cl}_Y"] & \widetilde{H}.(Y) \end{tikzcd}$$

commutes, where the left vertical arrow is the proper pushforward map for cycles. In the case of schemes this is [17, 6.1], and once again the same proof gives the result for Deligne-Mumford stacks. Similarly, if $f : X \to Y$ is a proper, but not necessarily representable morphism, and if Λ has characteristic 0 then the diagram

$$\begin{tikzcd} A.(X)_{\mathbb{Q}} \arrow[r, "\mathrm{cl}_X"] \arrow[d, "f_*"] & \widetilde{H}.(X) \arrow[d, "f_*^{\mathrm{hom}}"] \\ A.(Y)_{\mathbb{Q}} \arrow[r, "\mathrm{cl}_Y"] & \widetilde{H}.(Y) \end{tikzcd}$$

commutes, where $A.(X)_{\mathbb{Q}}$ and $A.(Y)_{\mathbb{Q}}$ denote the rational Chow groups and the proper pushforward is defined as in [24, 3.7].

2.11. **Localized cycle maps.** In this section we review some things from [6, Cycle, §2.1] and [9].

2.12. Let X be a Deligne-Mumford stack, and let $i : D \subset X$ be an effective Cartier divisor with complement $j : U \hookrightarrow X$. Consider the set of pairs (L, σ), where L is a line bundle on X and $\sigma : j^*L \to \mathscr{O}_U$ is an isomorphism of line bundles on U. We say that two pairs (L, σ) and (L', σ') are equivalent if there exists an isomorphism $\rho : L \to L'$ such that the diagram

$$\begin{tikzcd} j^*L \arrow[rr, "j^*\rho"] \arrow[dr, "\sigma"'] & & j^*L' \arrow[dl, "\sigma'"] \\ & \mathscr{O}_U & \end{tikzcd}$$

commutes. Note that since D is an effective Cartier divisor, the map $\mathscr{O}_X \to j_*\mathscr{O}_U$ is injective and the isomorphism ρ is unique if it exists. Denote by $\mathbb{T}_D$ the set of isomorphism classes of pairs (L, σ). Tensor product defines on $\mathbb{T}_D$ the structure of an abelian group.

2.13. From the distinguished triangle of complexes of sheaves on the étale site of X

$$i_*i^!\mathbb{G}_{m,X} \to \mathbb{G}_{m,X} \to Rj_*\mathbb{G}_{m,U} \to i_*i^!\mathbb{G}_{m,X}[1]$$

we get an isomorphism

$$R^0j_*\mathbb{G}_{m,U}/\mathbb{G}_{m,X} \to \mathscr{H}^1_D(\mathbb{G}_{m,X}).$$

Let $\mathscr{T}_D$ denote the sheaf on the étale site of X which to any étale morphism $V \to X$ associates the group of isomorphism classes of line bundles on V with trivialization on $V_U := V \times_X U$. The fact that $\mathscr{T}_D$ is a sheaf follows from the above observation that there are no nontrivial automorphisms of a pair (L, σ) consisting of a line bundle L on V and a trivialization σ over V_U. We have a map

$$R^0j_*\mathbb{G}_{m,U} \to \mathscr{T}_D$$

sending a local section $u \in (R^0 j_* \mathbb{G}_{m,U})(V) = \Gamma(V_U, \mathscr{O}^*_{V_U})$ to the trivial line bundle $\mathscr{O}_V$ with the trivialization over V_U given by multiplication by u. This map induces an isomorphism of sheaves of abelian groups

$$R^0 j_* \mathbb{G}_{m,U}/\mathbb{G}_{m,X} \to \mathscr{T}_D.$$

We therefore get an isomorphism

$$\mathbb{T}_D = H^0(X, \mathscr{T}_D) \simeq H^0(X, \mathscr{H}^1_D(\mathbb{G}_{m,X})).$$

On the other hand, since $\mathscr{H}^0_D(\mathbb{G}_{m,X}) = 0$ the map

$$H^1_D(X, \mathbb{G}_{m,X}) \to H^0(X, \mathscr{H}^1_D(\mathbb{G}_{m,X}))$$

is an isomorphism, so we get an isomorphism

$$\mathbb{T}_D \simeq H^1_D(X, \mathbb{G}_{m,X}).$$

2.14. Now let ℓ be a prime invertible on X, and for an integer $n \geq 1$ consider the Kummer sequence

$$1 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbb{G}_{m,X} \xrightarrow{u \mapsto u^{\ell^n}} \mathbb{G}_{m,X} \longrightarrow 1.$$

Applying the functor $H^*_D(X, -)$ we obtain a map

$$\mathbb{T}_D \to H^2_D(X, \mu_{\ell^n}).$$

This map is compatible with the maps

$$H^2_D(X, \mu_{\ell^{n+1}}) \to H^2_D(X, \mu_{\ell^n})$$

induced by the map

$$\mu_{\ell^{n+1}} \to \mu_{\ell^n}, \quad v \mapsto v^\ell$$

and therefore we get a map

$$\mathbb{T}_D \to H^2_D(X, \mathbb{Z}_\ell(1)).$$

Following [6, Cycle, 2.1.1], we define the *localized Chern class of D*, denoted $\mathrm{cl}^{\mathrm{loc}}_X(D)$, to be the image of the pair

$$(\mathscr{O}_X(D), \text{trivialization over } U \text{ induced by natural inclusion } \mathscr{O}_X \hookrightarrow \mathscr{O}_X(D))$$

under this map. For any coefficient ring Λ we also write $\mathrm{cl}^{\mathrm{loc}}_X(D)$ for the image of this class in $H^2_D(X, \Lambda(1))$ under the canonical map

$$H^2_D(X, \mathbb{Z}_\ell(1)) \to H^2_D(X, \Lambda(1)).$$

Remark 2.15. We will often think of $\mathrm{cl}^{\mathrm{loc}}_X(D)$ as a morphism $\Lambda_D \to i^!\Lambda(1)[2]$, or dually as a morphism

$$i^*\Omega_X \to \Omega_D(1)[2],$$

or again by adjunction as a map

$$\Omega_X \to i_*\Omega_D(1)[2].$$

We write $\mathrm{cl}^{\mathrm{loc}}_X(D)$ for any of these maps.

Remark 2.16. The above construction of the localized Chern class generalizes to Artin stacks. The only modification needed is to replace the étale site in the above by the lisse-étale site.

2.17. Following Gabber and Fujiwara [9, §1], we extend this definition of the localized Chern class to higher codimension substacks as follows.

Let $i : Y \hookrightarrow X$ be a regular imbedding of pure codimension c. Consider the blowup $\pi : \widetilde{X} \to X$ of X along Y, and let $i' : E \hookrightarrow \widetilde{X}$ be the exceptional divisor, so we get

$$\mathrm{cl}^{\mathrm{loc}}_{\widetilde{X}}(E) : \Lambda_{\widetilde{X}} \to i'_* i'^! \Lambda(1)[2].$$

Let $\pi_E : E \to Y$ be the restriction of π to Y. We have a distinguished triangle

$$\Lambda_X \to \pi_* \Lambda_{\widetilde{X}} \to \tau_{\geq 1} i_* \pi_{E*} \Lambda_E \to \Lambda_X[1],$$

and since $E \to Y$ is a projective bundle we have $\tau_{\geq 1}\pi_{E*}\Lambda_E \simeq \oplus_{i=1}^{c-1} \Lambda_Y(-i)[-2i]$. Applying $i^!$ to this distinguished triangle we therefore get a distinguished triangle

$$i^! \Lambda_X \to i^! \pi_* \Lambda_{\widetilde{X}} \to \oplus_{i=1}^{c-1} \Lambda_Y(-i)[-2i] \to i^! \Lambda_X[1].$$

By base change we have $i^! \pi_* \Lambda_{\widetilde{X}} \simeq \pi_{E*} i'^! \Lambda_{\widetilde{X}}$, so we can rewrite this as a distinguished triangle

$$i^! \Lambda_X \to \pi_{E*} i'^! \Lambda_{\widetilde{X}} \to \oplus_{i=1}^{c-1} \Lambda_Y(-i)[-2i] \to i^! \Lambda_X[1]. \tag{2.17.1}$$

This triangle is split. Namely, the first localized Chern class $c_1 := -\mathrm{cl}^{\mathrm{loc}}_{\widetilde{X}}(E) \in H^2_E(\widetilde{X}, \Lambda(1))$ defines a map for each i

$$c_1^{\otimes i} : \Lambda_Y(-i)[-2i] \to \pi_* i'^! \Lambda$$

which defines a splitting of (2.17.1) giving a decomposition

$$\pi_{E*} i'^! \Lambda_{\widetilde{X}} \simeq i^! \Lambda_X \oplus (\oplus_{i=1}^{c-1} \Lambda_Y(-i)[-2i]).$$

In particular we get an isomorphism

$$\alpha : \left(\oplus_{i=1}^{c-1} H^{2(c-i)}(Y, \Lambda(c-i))\right) \oplus H^{2c}_Y(X, \Lambda(c)) \to H^{2c}_E(\widetilde{X}, \Lambda(c)).$$

Under this isomorphism we can write $-c_1^{\otimes c}$ as

$$\left(\sum_{i=1}^{c-1} \delta_i c_1^{\otimes (c-i)}\right) + \delta_c,$$

and we define

$$\mathrm{cl}^{\mathrm{loc}}_X(Y) := \delta_c \in H^{2c}_Y(X, \Lambda(c)).$$

As in the case of a divisor, we often think of this class as a map

$$\Lambda \to i^! \Lambda(c)[2c]$$

or by duality as a map

$$\Omega_X \to i_* \Omega_Y(c)[2c].$$

In the following 2.18–2.30 we develop some of the basic properties of these localized Chern classes. In the case of schemes these results are already known by [21, §2.5].

Proposition 2.18. *Consider a commutative diagram of Deligne-Mumford stacks*

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[rr, bend left, "s"] & Y \arrow[r, hook, "j"] & X, \end{tikzcd}$$

where i, j, and s are regular imbeddings of pure codimensions a, b, and $a+b$ respectively. Then the diagram

$$\begin{tikzcd} \Lambda \arrow[r, "\mathrm{cl}^{\mathrm{loc}}_Y(X)"] \arrow[drr, "\mathrm{cl}^{\mathrm{loc}}_X(Z)"'] & i^!\Lambda(a)[2a] \arrow[r, "i^!\mathrm{cl}^{\mathrm{loc}}_X(Y)"] & i^!j^!\Lambda(a+b)[2(a+b)] \arrow[d, "\simeq"] \\ & & s^!\Lambda(a+b)[2(a+b)] \end{tikzcd}$$

commutes.

Proof. This is the same as in [9, 1.2.1]. □

Proposition 2.19. *Let $i : Y \hookrightarrow X$ be a regular imbedding of codimension c, and suppose X is smooth of dimension d, and Y is smooth of dimension $d-c$. Then via the natural isomorphisms $\Omega_X \simeq \Lambda_X(d)[2d]$ and $\Omega_Y \simeq \Lambda_Y(d-c)[2-c]$, the map $\mathrm{cl}^{\mathrm{loc}}_X(Y) : \Omega_X \to i_*\Omega_Y(c)[2c]$ is identified with the map*

$$\Lambda_X(d)[2d] \to i_*\Lambda_Y(d)[2d]$$

induced by the restriction map $\Lambda_X \to i_*\Lambda_Y$.

Proof. The assertion can be verified étale locally on X, so it suffices to consider the case when $X = \mathbb{A}^d$ with coordinates $x_1, \ldots, x_d$ and Y is given by the variables $x_{d-c+1} = x_{d-c+2} = \cdots = x_d = 0$. Proceeding by induction on c using 2.18 we are further reduced to case when $c = 1$. In this case the result follows from [6, Cycle, 2.1.5]. □

Proposition 2.20. *Consider a commutative diagram of separated finite type Deligne-Mumford stacks over k*

$$\begin{tikzcd} & P \arrow[d, "f"] \\ Y \arrow[ur, hook, "s"] \arrow[r, hook, "i"] & X, \end{tikzcd}$$

where i is a regular imbedding of codimension c, f is smooth and representable of relative dimension d, and s is a regular imbedding of codimension $c+d$. Then the diagram

$$\begin{tikzcd} \Lambda \arrow[r, "\mathrm{cl}^{\mathrm{loc}}_X(Y)"] \arrow[drr, "\mathrm{cl}^{\mathrm{loc}}_P(Y)"'] & i^!\Lambda(c)[2c] \arrow[r, "\simeq"] & s^!f^!\Lambda(c)[2c] \arrow[d, "f^!\simeq f^*(d)[2d]"] \\ & & s^!\Lambda(c+d)[2(c+d)] \end{tikzcd}$$

commutes. Here the isomorphism $f^! \simeq f^*(d)[2d]$ is defined as in [18, 4.6.2] *and* [19, 9.1.2].

Proof. Consider first the case when i is the identity map $X \to X$, in which case s is a section of f. In this case we need to show that the composite map

(2.20.1)
$$\Lambda \xrightarrow{\mathrm{cl}^{\mathrm{loc}}_P(X)} s^!\Lambda(d)[2d] \xrightarrow{\simeq} s^!f^!\Lambda \simeq \Lambda$$

is the identity map. This is an étale local question on X and P so it suffices to consider the case when $P = \mathbb{A}^d_X$, and s is the zero section. Furthermore, by functoriality of the local

Chern classes [9, 1.1.3] and consideration of the cartesian square

$$\begin{tikzcd} \mathbb{A}^d_X \arrow[r] \arrow[d] & \mathbb{A}^d_k \arrow[d] \\ X \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

it suffices to consider the case when $X = \mathrm{Spec}(k)$. In this case the class $\mathrm{cl}^{\mathrm{loc}}_X(P) \in H^{2d}_0(\mathbb{A}^d, \Lambda(d))$ has the property that its image in $H^{2d}(\mathbb{P}^d, \Lambda(d))$ is the cohomology class of a point. This same property also characterizes the isomorphism $\Lambda(d)[2d] \simeq f^!\Lambda$, from which the result follows.

For the general case, consider the diagram

$$\begin{tikzcd} Y \arrow[r, hook, "s'"] \arrow[rr, bend left, "s"] \arrow[dr, equal] & P_Y \arrow[r, hook, "\tilde{i}"] \arrow[d, "f'"] & P \arrow[d, "f"] \\ & Y \arrow[r, hook, "i"] & X, \end{tikzcd}$$

where the square is cartesian. We then obtain a diagram

$$\begin{tikzcd} & s^!\Lambda(c+d)[2(c+d)] \arrow[r, "\simeq"] \arrow[d, "\simeq"] & s^!f^!\Lambda(c)[2c] \arrow[r, "\simeq"] & i^!\Lambda(c)[2c] \\ \Lambda \arrow[ur, "\mathrm{cl}^{\mathrm{loc}}_P(Y)"] \arrow[dr, "\mathrm{cl}^{\mathrm{loc}}_{P_Y}(Y)"'] \arrow[drrr, bend right, "\mathrm{id}"'] & s'^!\tilde{i}^!\Lambda(c+d)[2(c+d)] \arrow[r, "\text{base change}"] & s'^!f'^*i^!\Lambda(c+d)[2(c+d)] \arrow[r, "\simeq"] & s'^!f'^!i^!\Lambda(c)[2c] \arrow[u, "\simeq"'] \\ & s'^!\Lambda(d)[2d] \arrow[r, "\text{base change}"] \arrow[u, "\mathrm{cl}^{\mathrm{loc}}_P(P_Y)"'] & s'^!f'^*\Lambda(d)[2d] \arrow[r, "\simeq"] \arrow[u, "\mathrm{cl}^{\mathrm{loc}}_X(Y)"'] & s'^!f'^!\Lambda \simeq \Lambda. \arrow[u, bend right=60, "\mathrm{cl}^{\mathrm{loc}}_X(Y)"'] \end{tikzcd}$$

The lower inner square commutes by the same argument as in the case of schemes [21, 2.5.7], and all the other small inner diagrams have already been shown to commute. From this the result follows. ∎

Corollary 2.21. *Let $f : P \to X$ be a smooth representable morphism of relative dimension d, and let $s : X \to P$ be a section. Then the map $\mathrm{cl}_P(X) : \Omega_P \to s_*\Omega_X(d)[2d]$ is equal to the map obtained from the isomorphism $\Omega_P \simeq f^*\Omega_X(d)[2d]$ and the adjunction map*

$$f^*\Omega_X(d)[2d] \to s_*s^*f^*\Omega_X(d)[2d] \simeq s_*\Omega_X(d)[2d].$$

Proof. This follows from dualizing the map (2.20.1). ∎

2.22. We can generalize the definition of localized cycle classes for regularly embedded substacks to local complete intersection morphisms as follows.

First recall (following Fulton [10, B.7.6]) that a morphism of Deligne-Mumford stacks $f : Y \to X$ is called a *local complete intersection morphism of codimension c* if there exists a factorization of f

(2.22.1)
$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[dr, "f"'] & P \arrow[d, "p"] \\ & X, \end{tikzcd}$$

where p is smooth of relative dimension d and i is a regular imbedding of codimension $c+d$. If $f : Y \rightarrow X$ is a local complete intersection morphism of codimension c, then for any factorization

$$Y \xhookrightarrow{i} P \xrightarrow{p} X$$

of f with p smooth of relative dimension n and i a closed imbedding, the morphism i is a regular imbedding of codimension $n+c$. Indeed to verify this, we may work étale locally on X and P so the result follows from the corresponding result for schemes [10, B.7.6].

Remark 2.23. In the case of Deligne-Mumford stacks the existence of global factorizations (2.22.1) and the basic properties of such factorizations is more subtle than in the case of schemes, especially when f is not representable. In this paper we will only consider Deligne-Mumford stacks which are quasi-projective finite group quotients in the sense of A.1, and we develop the basic facts about factorizations (2.22.1) in appendix A.

Remark 2.24. This definition of a local complete intersection morphism is different than the one in [7, IV.19.3.6], where local complete intersection morphisms are assumed flat but not necessarily admitting a global factorization through a smooth morphism.

2.25. Let $f : Y \rightarrow X$ be a local complete intersection morphism of codimension c between quasi-projective finite group quotients, and fix a factorization (2.22.1) with p smooth of relative dimension d and P also a quasi-projective finite group quotient. We then get a morphism

$$\Lambda_Y \xrightarrow{\mathrm{cl}_P^{\mathrm{loc}}(Y)} i^!\Lambda_P(c+d)[2(c+d)] \xrightarrow{p^!\simeq p^*(d)[2d]} i^!p^!\Lambda(c)[2c] \xrightarrow{\simeq} f^!\Lambda_X(c)[2c].$$

Lemma 2.26. *The composite map*

$$\mathrm{cl}_X^{\mathrm{loc}}(Y) : \Lambda_Y \rightarrow f^!\Lambda_X(c)[2c]$$

is independent of the choice of factorization (2.22.1).

Proof. In the case of schemes this is [21, 2.5.5]. We treat the case of stacks similarly.

Consider two factorizations

$$Y \xhookrightarrow{i_j} P_j \xrightarrow{p_j} X \qquad\qquad j = 1, 2$$

with p_j smooth of relative dimension d_j. We wish to show that the resulting two maps $\Lambda_Y \to f^!\Lambda_X(c)[2c]$ are the same. By A.8 there exists a commutative diagram of factorizations

$$\begin{tikzcd}
 & & P_1 \arrow[dr, "p_1"] & \\
Y \arrow[urr, hook, "i_1"] \arrow[r, hook, "i"] \arrow[drr, hook, "i_1"'] & P \arrow[ur, "q_1"] \arrow[rr, "p"] \arrow[dr, "q_2"'] & & X \\
 & & P_2 \arrow[ur, "p_2"'] &
\end{tikzcd}$$

with q_1 and q_2 smooth. To prove that the two factorizations define the same map, it suffices to show that they define the same map as the map defined by the factorization

$$Y \overset{i}{\hookrightarrow} P \xrightarrow{p} X,$$

so it suffices to consider the case when there exists a smooth morphism $q : P_1 \to P_2$ of relative dimension $d_1 - d_2$ filling in the diagram

$$\begin{tikzcd}
 & P_1 \arrow[d, "q"] \arrow[dr, "p_1"] & \\
Y \arrow[ur, hook, "i_1"] \arrow[r, hook, "i_2"] & P_2 \arrow[r, "p_2"] & X.
\end{tikzcd}$$

In this case it suffices to show that the diagram

$$\begin{tikzcd}
\Lambda_Y \arrow[r, "\mathrm{cl}^{\mathrm{loc}}_{P_2}(Y)"] \arrow[ddr, bend right, "\mathrm{cl}^{\mathrm{loc}}_{P_1}(Y)"'] & i_2^!\Lambda_{P_2}(c+d_2)[2(c+d_2)] \arrow[r, "\simeq"] \arrow[d, "\simeq"] & i_2^!p_2^!\Lambda_X(c)[2c] \arrow[dd, equal] \\
 & i_1^!q^!\Lambda_{P_2}(c+d_2)[2(c+d_2)] \arrow[d, "\simeq"] & \\
 & i_1^!\Lambda_{P_1}(c+d_1)[2(c+d_1)] \arrow[r, "\simeq"] & i_1^!p_1^!\Lambda_X(c)[2c]
\end{tikzcd}$$

commutes, which holds since the left inner diagram commutes by 2.20 and the right inner diagram clearly commutes. □

Remark 2.27. As in the case of regular imbeddings, if $f : Y \to X$ is a local complete intersection morphism of codimension c of quasi-projective finite group quotients, then we also write $\mathrm{cl}^{\mathrm{loc}}_X(Y)$ for the map

$$\Omega_X \to f_*\Omega_Y(c)[2c]$$

obtained by duality.

2.28. **Top localized Chern class of a vector bundle.**

2.29. Let X be an Artin stack, and let $\mathscr{E}$ be a locally free sheaf of finite rank on X of rank r. Let $p : E \to X$ denote the relative spectrum

$$\underline{\mathrm{Spec}}_X(\mathrm{Sym}^{\cdot}\mathscr{E})$$

and let $s : X \hookrightarrow E$ be the zero section. We have a canonical isomorphism $p^!\Lambda_X \simeq \Lambda_E(r)[2r]$, and therefore also an isomorphism $s^!\Lambda_E \simeq \Lambda_X(-r)[-2r]$. In particular, we get an isomorphism

$$\kappa : H^0(X, \Lambda_X) \simeq H^{2r}(X, s^!\Lambda_E(r)). \tag{2.29.1}$$

We define $c_r^{\mathrm{loc}}(\mathscr{E}^\vee) \in H^{2r}(X, s^!\Lambda_E(r))$ to be the image of $1 \in H^0(X, \Lambda_X)$ under this isomorphism. The class $c_r^{\mathrm{loc}}(\mathscr{E}^\vee)$ is called the *top localized Chern class of* $\mathscr{E}^\vee$.

Lemma 2.30. *Let X be a quasi-projective finite group quotient. Then the class $c_r^{\mathrm{loc}}(\mathscr{E}^\vee)$ agrees with the localized cycle class $\mathrm{cl}_E^{\mathrm{loc}}(X)$ of the codimension r regular imbedding $s : X \hookrightarrow E$.*

Proof. This is a special case of 2.20 taking i to be the identity map $X \to X$. □

2.31. **Comparison of intersection products.**

2.32. Consider a cartesian diagram of quasi-projective finite group quotients over k

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g'"] & V \arrow[d, "g"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

where f is a local complete intersection morphism of codimension c. The localized cycle class

$$\mathrm{cl}_Y^{\mathrm{loc}}(X) : \Omega_Y \to f_*\Omega_X(c)[2c]$$

gives upon applying $g^!$ a morphism

$$g^!\mathrm{cl}_Y^{\mathrm{loc}}(X) : \Omega_V \to g^!f_*\Omega_X(c)[2c] \simeq f'_*\Omega_W(c)[2c],$$

where the second isomorphism is the base change isomorphism $g^!f_* \simeq f'_*g'^!$. For any integer i we therefore get a map

$$f^!_{\mathrm{hom}} : H_i(V) \to H_{i-2c}(W)(c), \tag{2.32.1}$$

and in particular maps

$$f^!_{\mathrm{hom}} : \widetilde{H}_i(V) \to \widetilde{H}_{i-c}(W),$$

which we call the *homological Gysin homomorphisms* (see 2.9 for the definition of $\widetilde{H}_i$).

Remark 2.33. As the notation suggests, the map 2.32.1 depends on more data than just the morphism $W \to V$, even when this morphism is a regular imbedding itself. For example, in the case when f is a closed embedding and g factors through X we have $W = V$. It is natural to consider replacing W by the fiber product in the sense of derived algebraic geometry, but we have not pursued this.

Theorem 2.34. *With notation as in 2.32, the diagram*

$$\begin{tikzcd} A_i(V) \arrow[r, "f^!"] \arrow[d, "\mathrm{cl}_V"] & A_{i-c}(W) \arrow[d, "\mathrm{cl}_W"] \\ \widetilde{H}_i(V) \arrow[r, "f^!_{\mathrm{hom}}"] & \widetilde{H}_{i-c}(W) \end{tikzcd}$$

commutes, where

$$f^! : A_i(V) \to A_{i-c}(W)$$

is the localized intersection product defined in [24, 3.10] *(and in the case of schemes* [10, §6.6]*).*

The proof of 2.34 occupies the remainder of this section.

2.35. First we reduce to the case when f is a regular imbedding. For this choose a factorization

$$X \overset{i}{\hookrightarrow} P \xrightarrow{p} Y,$$

with i a regular imbedding of codimension $c+d$ and p smooth of relative dimension d. By the definition of $\mathrm{cl}^{\mathrm{loc}}_Y(X)$ we then have a commutative diagram

(2.35.1)
$$\begin{tikzcd}
\Omega_Y \arrow[r, "\mathrm{cl}^{\mathrm{loc}}_Y(P)"] \arrow[rr, bend left, "\mathrm{cl}^{\mathrm{loc}}_Y(X)"] & p_*\Omega_P(-d)[-2d] \arrow[r, "\mathrm{cl}^{\mathrm{loc}}_P(X)"] & p_*i_*\Omega_X(c)[2c] \simeq f_*\Omega_X(c)[2c].
\end{tikzcd}$$

Consider the commutative diagram

$$\begin{tikzcd}
W \arrow[r, hook, "i'"] \arrow[d, "g'"] \arrow[rr, bend left, "f'"] & U \arrow[r, "p'"] \arrow[d, "g''"] & V \arrow[d, "g"] \\
X \arrow[r, hook, "i"] \arrow[rr, bend right, "f"'] & P \arrow[r, "p"] & Y,
\end{tikzcd}$$

where the squares are cartesian. Applying $g^!$ to (2.35.1) we get a commutative diagram

$$\begin{tikzcd}
\Omega_V \arrow[r, "g''^!\mathrm{cl}^{\mathrm{loc}}_P(X)"] \arrow[rr, bend left, "g^!\mathrm{cl}^{\mathrm{loc}}_Y(X)"] & p'_*\Omega_U(-d)[-2d] \arrow[r, "g^!\mathrm{cl}^{\mathrm{loc}}_Y(P)"] & f'_*\Omega_W(c)[2c].
\end{tikzcd}$$

Passing to cohomology we deduce that the diagram

$$\begin{tikzcd}
\widetilde{H}.(V) \arrow[r, "p^!_{\mathrm{hom}}"] \arrow[rr, bend right, "f^!_{\mathrm{hom}}"'] & \widetilde{H}.(U) \arrow[r, "i^!_{\mathrm{hom}}"] & \widetilde{H}.(W)
\end{tikzcd}$$

commutes. From this, [10, 6.5], and consideration of the diagram

$$\begin{tikzcd}
A.(V) \arrow[r, "p^!"] \arrow[d, "\mathrm{cl}_V"] \arrow[rr, bend left, "f^!"] & A.(U) \arrow[r, "i^!"] \arrow[d, "\mathrm{cl}_U"] & A.(W) \arrow[d, "\mathrm{cl}_W"] \\
\widetilde{H}.(V) \arrow[r, "p^!_{\mathrm{hom}}"] \arrow[rr, bend right, "f^!_{\mathrm{hom}}"'] & \widetilde{H}.(U) \arrow[r, "i^!_{\mathrm{hom}}"] & \widetilde{H}.(W)
\end{tikzcd}$$

we conclude that to prove (2.34) it suffices to consider the two cases when f is smooth and when f is a regular imbedding.

2.36. Consider first the case when f is smooth of relative dimension d. In this case $f^!_{\text{hom}}$ is induced by the canonical isomorphism $f^*\Omega_Y \simeq \Omega_X(-d)[-2d]$ induced by duality from the isomorphism $f^!\Lambda_Y \simeq \Lambda_X(d)[2d]$ in [18, 4.6.2] and [19, 9.1.2]. This isomorphism is functorial in the sense that the map obtained by applying $g^!$ to this map is simply the corresponding isomorphism

$$f'^*\Omega_V \to \Omega_W(-d)[-2d]$$

for $f' : W \to V$. So to prove (2.34) it suffices to show that the cycle class map commutes with smooth pullback which is shown in [17, 6.1].

It remains to prove (2.34) for a regular imbedding.

2.37. Let W be a quasi-projective finite group quotient, and let $p : E \to W$ be a smooth morphism of relative dimension r with a section $i : W \to E$. Let $f : C \hookrightarrow E$ be a closed substack containing the zero section $i(W)$, so we have a commutative diagram

$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[d, equal] & C \arrow[d, hook, "f"] \arrow[dr, "q"] & \\ W \arrow[r, hook, "i"] & E \arrow[r, "p"] & W, \end{tikzcd}$$

where the square is cartesian. We have

$$\Omega_E \simeq p^*\Omega_W(r)[2r],$$

and therefore the adjunction map

$$p^*\Omega_W(r)[2r] \to i_*i^*p^*\Omega_W(r)[2r] \simeq i_*\Omega_W(r)[2r]$$

defines for every k a map

$$\theta : \widetilde{H}_k(E) \to \widetilde{H}_{k-r}(W).$$

Lemma 2.38. *Assume C is irreducible of dimension k. Then the class $i^!_{\text{hom}}([C]) \in \widetilde{H}_{k-r}(W)$ is equal to the image under θ of the cycle class $\mathrm{cl}_E(C) \in \widetilde{H}_k(E)$, where $[C]$ denotes the fundamental class of C as defined in 2.7.*

Proof. Note first that for any complex K on W the base change map

$$f^!i_*K \to j_*K$$

is an isomorphism, and the adjunction map $f_!f^!i_*K \to i_*K$ is equal to the composite

$$f_!f^!i_*K \xrightarrow{\text{base change}} f_!j_*K \xrightarrow{f_! \simeq f_*} f_*j_*K \xrightarrow{\simeq} i_*K.$$

Applying $f_*f^!$ to the adjunction map $\Omega_E \to i_*i^*\Omega_E$ and using 2.21, we get a commutative diagram

$$\begin{tikzcd}
f_*\Lambda(k)[2k] \arrow[d, "{[C]}"] & & \\
f_*\Omega_C \arrow[rr, "f^!\mathrm{cl}_E^{\mathrm{loc}}(W)"] \arrow[d, "f_*^{\mathrm{hom}}"] & & f_*j_*\Omega_W(r)[2r] \arrow[d, "\simeq"] \\
\Omega_E \arrow[r, "\mathrm{id}\to i_*i^*"] \arrow[rr, bend right, "\mathrm{cl}_E^{\mathrm{loc}}(W)"'] & i_*i^*\Omega_E \arrow[r, "\simeq"] & i_*\Omega_W(r)[2r].
\end{tikzcd}$$

This implies that the class $i^!_{\mathrm{hom}}([C])$ is the image of $\mathrm{cl}_E(C)$ under the sequence of maps

$$\begin{array}{rcl}
\widetilde{H}_k(E) = H^{-2k}(E, \Omega_E)(-k) & \overset{\mathrm{id}\to i_*i^*}{\to} & H^{-2k}(W, i^*\Omega_E)(-k) \\
& \simeq & H^{-2k}(W, \Omega_W(r)[2r])(-k) \\
& \simeq & H^{-2(k-r)}(W, \Omega_W)(r-k) \\
& \simeq & \widetilde{H}_{k-r}(W),
\end{array}$$

which is the morphism θ. $\square$

Remark 2.39. In the case when $p : E \to W$ is the total space of a vector bundle of rank r the map θ is the inverse of the isomorphism

$$\widetilde{H}_k(E) \simeq \widetilde{H}_{k-r}(W) \tag{2.39.1}$$

given by 2.4. In this case, 2.38 can be restated as saying that the class $i^!_{\mathrm{hom}}([C]) \in \widetilde{H}_{k-r}(W)$ corresponds under (2.39.1) to the cycle class $\mathrm{cl}_E(C) \in \widetilde{H}_k(E)$.

2.40. Now consider a commutative diagram of quasi-projective finite group quotients

$$\begin{tikzcd}
W \arrow[r, hook, "f'"] \arrow[d, "g'"] & V \arrow[d, "g"] \\
X \arrow[r, hook, "f"] & Y,
\end{tikzcd}$$

where f is a regular imbedding of codimension c. Let $z : Z \to V$ be a proper morphism of quasi-projective finite group quotients, and let W_Z denote the pullback of W to Z so we have a commutative diagram with cartesian squares

$$\begin{tikzcd}
W_Z \arrow[r, hook, "f''"] \arrow[d, "z'"] & Z \arrow[d, "z"] \\
W \arrow[r, hook, "f'"] \arrow[d, "g'"] & V \arrow[d, "g"] \\
X \arrow[r, hook, "f"] & Y.
\end{tikzcd}$$

Lemma 2.41. *The diagrams*

(2.41.1)
$$\begin{tikzcd} \widetilde{H}.(Z) \arrow[r, "z_*^{\mathrm{hom}}"] \arrow[d, "f^!_{\mathrm{hom}}"] & \widetilde{H}.(V) \arrow[d, "f^!_{\mathrm{hom}}"] \\ \widetilde{H}.(W_Z) \arrow[r, "z_*^{\prime,\mathrm{hom}}"] & \widetilde{H}.(W) \end{tikzcd}$$

and

(2.41.2)
$$\begin{tikzcd} A.(Z) \arrow[r, "z_*"] \arrow[d, "f^!"] & A.(V) \arrow[d, "f^!"] \\ A.(W_Z) \arrow[r, "z'_*"] & A.(W) \end{tikzcd}$$

commute.

Proof. The commutativity of 2.41.2 is [10, 6.2 (a)].

To see that 2.41.1 commutes, it suffices to note that the diagram

$$\begin{tikzcd} z_*\Omega_Z \arrow[r, "z_*"] \arrow[d, "\simeq"] & \Omega_V \arrow[d, "\simeq"] & \\ z_*z^!g^!\Omega_Y \arrow[r, "z_*z^!\to\mathrm{id}"] \arrow[d, "\mathrm{cl}_Y^{\mathrm{loc}}(X)"'] & g^!\Omega_Y \arrow[d, "\mathrm{cl}_Y^{\mathrm{loc}}(X)"] & \\ z_*z^!g^!f_*\Omega_X \arrow[r, "z_*z^!\to\mathrm{id}"] \arrow[d, "bc_g"] \arrow[dd, bend right=60, "bc_{zg}"'] & g^!f_*\Omega_X \arrow[d, "bc_g"] & \\ z_*z^!f'_*g'^!\Omega_X \arrow[r, "z_*z^!\to\mathrm{id}"] \arrow[d, "bc_z"] & f'_*g'^!\Omega_X \arrow[r, "\simeq"] & f'_*\Omega_W \\ z_*f''_*z'^!g'^!\Omega_X \arrow[r, "\simeq"] & f'_*z'_*z'^!g'^!\Omega_X \arrow[r, "\simeq"] \arrow[u, "z'_*z'^!\to\mathrm{id}"] & f'_*z'_*\Omega_{W_Z} \arrow[u, "z'_*z'^!\to\mathrm{id}"'] \end{tikzcd}$$

commutes, where for $? = g, z, zg$ we write $bc_?$ for the corresponding base change morphism. □

2.42. From this it follows that to prove 2.34, it suffices to show that for a cartesian square (where we modify our notation to match that of [10, §6.1])

(2.42.1)
$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[d, "g"] & V \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y, \end{tikzcd}$$

with i a regular imbedding of codimension d and V irreducible of dimension k, the class $i^!_{\mathrm{hom}}\mathrm{cl}_V([V]) \in \widetilde{H}_{k-d}(W)$ is equal to $\mathrm{cl}_W(X \cdot V)$, where $[V] \in A_k(V)$ denotes the fundamental class of V and $X \cdot V \in A_{k-d}(W)$ is the intersection product as defined in [10, §6.1]. This we will do by deforming to the normal cone.

Remark 2.43. With even more restrictive hypotheses this special case follows from [16, 2.1.2].

2.44. Let $\mathbb{A}^1$ denote the affine line over k with coordinate t, and let $M \to \mathbb{A}^1$ be the blowup of $Y \times \mathbb{A}^1$ along the substack $X \times \{0\}$. If B denotes the blowup of Y along X, then there is a natural closed imbedding $B \hookrightarrow M$ of B into the fiber over 0, and we define $\mathscr{Y}$ to be the complement of B in M, so we have a morphism

$$\pi : \mathscr{Y} \to Y \times \mathbb{A}^1.$$

Locally $\mathscr{Y}$ can be described as follows. Let $\mathrm{Spec}(R) \to Y$ be an étale morphism from an affine scheme such that the ideal of X over R is generated by a regular sequence $(f_1, \ldots, f_d)$. Then $\pi^{-1}(\mathrm{Spec}(R) \times \mathbb{A}^1)$ is isomorphic to the scheme

$$\mathrm{Spec}(R[t][z_1, \ldots, z_d]/(tz_i = f_i)_{i=1}^d).$$

This local description has several consequences:

(i) The morphism π is an isomorphism over $Y \times \mathbb{G}_m$.

(ii) The scheme-theoretic closure in $\mathscr{Y}$ of the imbedding

$$X \times \mathbb{G}_m \xrightarrow{i \times \mathrm{id}} Y \times \mathbb{G}_m \xhookrightarrow{\text{(i)}} \mathscr{Y}$$

maps isomorphically under π to $X \times \mathbb{A}^1$, so we obtain a commutative diagram

$$\begin{tikzcd} & \mathscr{Y} \arrow[d, "\pi"] \\ X \times \mathbb{A}^1 \arrow[ur, hook, "\tilde{i}"] \arrow[r, hook, "i\times\mathrm{id}"] & Y \times \mathbb{A}^1, \end{tikzcd}$$

where $\tilde{i}$ is also a regular imbedding of codimension d.

(iii) $\mathscr{Y}$ is flat over $\mathbb{A}^1$.

(iv) The fiber of $\mathscr{Y}$ over the point $t = 0$ in $\mathbb{A}^1$ is isomorphic to the normal cone $C_X Y$ of X in Y, mapping to X via π, with the fiber of $\tilde{i}$ over $t = 0$ being the zero section of $C_X Y$.

Remark 2.45. The preceding construction will be used again later in the paper. We refer to the family $\mathscr{Y} \to \mathbb{A}^1$ as the *standard degeneration of* Y *with respect to* X.

2.46. Consider the stack over $V \times \mathbb{A}^1$ given by $\mathscr{Y} \times_Y V$, and note that since π is an isomorphism over $Y \times \mathbb{G}_m$ there is a natural imbedding $V \times \mathbb{G}_m \hookrightarrow \mathscr{Y} \times_Y V$ over the inclusion $V \times \mathbb{G}_m \subset V \times \mathbb{A}^1$. Let $\widetilde{V} \subset \mathscr{Y} \times_Y V$ denote the scheme-theoretic closure of $V \times \mathbb{G}_m$.

Locally the stack $\widetilde{V}$ can be described as follows. Suppose $Y = \mathrm{Spec}(R)$ with X defined by a regular sequence $(f_1, \ldots, f_d)$, and that $V = \mathrm{Spec}(A)$ with f defined by a ring homomorphism $\varphi : R \to A$. Then $\widetilde{V}$ is the subscheme of

$$\mathscr{Y} \times_Y V \simeq \mathrm{Spec}(A[t][z_1, \ldots, z_d]/(tz_i = \varphi(f_i))_{i=1}^d)$$

defined by the spectrum of the image of the ring homomorphism

$$A[t][z_1, \ldots, z_d]/(tz_i = \varphi(f_i))_{i=1}^d \to A[t^{\pm}], \quad z_i \mapsto \varphi(f_i)/t.$$

This image is by definition the Rees algebra of the ideal of W in V, as discussed for example in [8, §6.5]. In particular the following hold:

(1) $\widetilde{V}$ is flat over $\mathbb{A}^1$ with fiber over $t=0$ equal to the normal cone C_VW of W in V.
(2) The morphism $\tilde{f}:\widetilde{V}\to\mathscr{Y}$ restricts over the fiber $t=0$ to the natural morphism of normal cones.
(3) The diagram

$$\begin{tikzcd} W\times\mathbb{A}^1 \arrow[r, hook, "\tilde{j}"] \arrow[d, "g\times\mathrm{id}"] & \widetilde{V} \arrow[d, "\tilde{f}"] \\ X\times\mathbb{A}^1 \arrow[r, hook, "\tilde{i}"] & \mathscr{Y} \end{tikzcd} \tag{2.46.1}$$

is cartesian.

2.47. For any $a\in\mathbb{A}^1(k)$, we get by taking the fiber over a of (2.46.1) a cartesian diagram

$$\begin{tikzcd} W \arrow[r, hook, "\tilde{j}_a"] \arrow[d, "g"] & \widetilde{V}_a \arrow[d, "\tilde{f}_a"] \\ X \arrow[r, hook, "\tilde{i}_a"] & \mathscr{Y}_a. \end{tikzcd}$$

For $a\neq 0$, this cartesian diagram is isomorphic (via the projection π) to the starting diagram (2.42.1), and for $a=0$ we get the cartesian diagram

$$\begin{tikzcd} W \arrow[r, hook, "j_0"] \arrow[d, "g"] & C_VW \arrow[d] \\ X \arrow[r, hook, "i_0"] & C_YX, \end{tikzcd}$$

where the horizontal arrows are the zero sections. Note that this diagram can be extended to a commutative diagram

$$\begin{tikzcd} W \arrow[r, hook, "j_0"] \arrow[d, "g"] & C_VW \arrow[r, "h_V"] \arrow[d, "\tilde{f}_0"] & W \arrow[d, "g"] \\ X \arrow[r, hook, "i_0"] & C_YX \arrow[r, "h"] & X, \end{tikzcd}$$

where $h:C_YX\to X$ is a vector bundle. Let $\chi:N\to W$ denote the fiber product $C_YX\times_XW$, so we have a commutative diagram

$$\begin{tikzcd} W \arrow[r, hook, "j_0"] \arrow[d, equal] & C_VW \arrow[d, hook] \arrow[dr, "h_V"] & \\ W \arrow[r, hook, "i'_0"] & N \arrow[r, "\chi"] & W. \end{tikzcd}$$

By definition, the intersection product $X\cdot V$ is the unique class in $A_{k-d}(W)$ whose pullback along χ to $A_k(N)$ is the class of C_VW (which is purely k-dimensional by [10, Appendix B, 6.6]). By [17, 6.1] the diagram

$$\begin{tikzcd} A_{k-d}(W) \arrow[r, "\chi^*"] \arrow[d, "\mathrm{cl}_W"] & A_k(N) \arrow[d, "\mathrm{cl}_N"] \\ \widetilde{H}_{k-d}(W) \arrow[r, "(2.39.1)"] & \widetilde{H}_k(N) \end{tikzcd}$$

commutes. From this and 2.39 it follows that

$$\mathrm{cl}_W(X \cdot V) = i'^{!}_{0,\mathrm{hom}}(\mathrm{cl}_{C_V W}([C_V W]))$$

in $\widetilde{H}_{k-d}(W)$.

Let $q : N \to C_Y X$ be the projection. By functoriality of the localized cycle class, the diagram

$$\begin{tikzcd} q^!\Omega_{C_Y X} \arrow[r, "\mathrm{cl}^{\mathrm{loc}}_{C_Y X}(X)"] \arrow[d, "\simeq"] & q^! i_{0*}\Omega_X(d)[2d] \arrow[dr, "\text{base change}"] & \\ \Omega_N \arrow[r, "\mathrm{cl}^{\mathrm{loc}}_N(W)"] & i'_{0*}\Omega_W(d)[2d] & i'_{0*}g^!\Omega_X \arrow[l, "\simeq"'] \end{tikzcd}$$

commutes. This implies that

$$i'^{!}_{0,\mathrm{hom}}(\mathrm{cl}_{C_V W}([C_V W])) = i^{!}_{0,\mathrm{hom}}(\mathrm{cl}_{C_V W}([C_V W])).$$

To complete the proof of 2.34, we are therefore reduced to proving the equality

$$i^{!}_{0,\mathrm{hom}}(\mathrm{cl}_{C_V W}([C_V W])) = i^{!}_{\mathrm{hom}}(\mathrm{cl}_V([V])). \tag{2.47.1}$$

For this consider again the cartesian diagram (2.46.1), and note that we have a canonical isomorphism

$$\Omega_{W\times\mathbb{A}^1} \simeq \mathrm{pr}_1^*\Omega_W(1)[2]$$

which induces an isomorphism (this is a special case of 2.4)

$$\widetilde{H}_{k-d}(W) \simeq \widetilde{H}_{k+1-d}(W \times \mathbb{A}^1). \tag{2.47.2}$$

To show the equality (2.47.1) we show that under this isomorphism, both classes

$$i^{!}_{0,\mathrm{hom}}(\mathrm{cl}_{C_V W}([C_V W])) \ \text{ and } \ i^{!}_{\mathrm{hom}}(\mathrm{cl}_V([V]))$$

map to

$$\tilde{i}^{!}_{\mathrm{hom}}(\mathrm{cl}_{\widetilde{V}}([\widetilde{V}])).$$

2.48. For $a \in \mathbb{A}^1(k)$, let σ_a denote the map

$$X \hookrightarrow X \times \mathbb{A}^1.$$

The map σ_a is a regular closed imbedding, and the induced map

$$\sigma^{!}_{a,\mathrm{hom}} : \widetilde{H}_{k+1-d}(W \times \mathbb{A}^1) \to \widetilde{H}_{k-d}(W)$$

is the inverse of the isomorphism (2.47.2) by construction. In particular, the map $\sigma^{!}_{a,\mathrm{hom}}$ is independent of the choice of a.

Let $\lambda_a : \mathscr{Y}_a \hookrightarrow \mathscr{Y}$ be the inclusion of the fiber over a. By associativity of the homological Gysin homomorphisms [9, 1.2.1] we have

$$\sigma^!_{a,\mathrm{hom}} \circ \tilde{i}^!_{\mathrm{hom}}(\mathrm{cl}_{\widetilde{V}}([\widetilde{V}])) = \tilde{i}^!_{a,\mathrm{hom}}\lambda^!_a(\mathrm{cl}_{\widetilde{V}}([\widetilde{V}])).$$

To prove the equality (2.47.1), it therefore suffices to show that for any $a \in \mathbb{A}^1(k)$ we have

$$\lambda^!_a(\mathrm{cl}_{\widetilde{V}}([\widetilde{V}])) = \mathrm{cl}_{\widetilde{V}_a}([\widetilde{V}_a]).$$

This follows from the following lemma:

Lemma 2.49. *Let X be a finite type separated Deligne-Mumford stack over k of pure dimension d, and let $f : X \to \mathbb{A}^1$ be a flat morphism (note that this implies that all fibers are of pure dimension $d-1$). Let $s : \mathrm{Spec}(k) \hookrightarrow \mathbb{A}^1$ be the zero section, and let $i : X_s \hookrightarrow X$ be the inclusion of the fiber (which is a regular imbedding of codimension 1). Then the diagram*

$$\begin{tikzcd} \Lambda_X(d)[2d] \arrow[r, "\mathrm{id}\to i_*i^*"] \arrow[d, "{[X]}"] & i_*\Lambda_{X_s}(d)[2d] \arrow[d, "{[X_s]}"] \\ \Omega_X \arrow[r, "\mathrm{cl}^{\mathrm{loc}}_X(X_s)"] & i_*\Omega_{X_s}(1)[2] \end{tikzcd}$$

commutes.

Proof. By [1, XVIII, 3.1.7], we have $\Omega_{X_s} \in D^{\geq -2(d-1)}(X_s)$, and therefore the validity of the lemma is equivalent to the equality of two global sections of $\mathscr{H}^{-2(d-1)}(\Omega_{X_s}(-(d-1)))$. In particular the assertion is étale local on X so we may assume that X is a scheme.

To prove the lemma, it suffices to show that the map

$$H^{-2d}(X, \Omega_X(-d)) \to H^{-2(d-1)}(X_s, \Omega_{X_s})(-d+1) \tag{2.49.1}$$

defined by the class of $X_s \subset X$ sends the fundamental class $[X]$ to $[X_s]$. For this we may replace X by a neighborhood of a generic point in the closed fiber, and hence we may assume that X and X_s are irreducible.

The map (2.49.1) is dual to a map

$$H^{2(d-1)}_c(X_s, \Lambda)(d-1) \to H^{2d}_c(X, \Lambda)(d) \tag{2.49.2}$$

which can be described as follows. Let $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ be the inclusion, so we have a distinguished triangle

$$\Lambda_{\mathbb{A}^1} \to Rj_*\Lambda_{\mathbb{G}_m} \to s_*\Lambda(1)[-1] \to \Lambda_{\mathbb{A}^1}[1]. \tag{2.49.3}$$

Tensoring this sequence with $f_!\Lambda(d)$ and taking cohomology we get a boundary map

$$H^{2(d-1)}_c(X_s, \Lambda(d-1)) \to H^{2d}_c(X, \Lambda)(d)$$

which is the map (2.49.2) (this follows from the construction and [6, Cycle, 2.3.8]).

By [20, 4.1 (ii)], the trace map

$$H^{2(d-1)}_c(X_s, \Lambda)(d-1) \to \Lambda$$

is induced by taking the restriction to the fiber of a relative trace map

$$\mathrm{tr}_f : f_!\Lambda(d-1) \to \Lambda[-2(d-1)].$$

On the other hand, by [20, 4.1 (iii)] the trace map $H^{2d}_c(X, \Lambda(d)) \to \Lambda$ is equal to the composite map

$$H^{2d}_c(\mathbb{A}^1, f_!\Lambda(d)) \xrightarrow{\mathrm{tr}_f} H^2_c(\mathbb{A}^1, \Lambda(1)) \xrightarrow{\mathrm{tr}_{\mathbb{A}^1}} \Lambda.$$

This reduces the problem to showing that the image $\alpha \in H^2_c(\mathbb{A}^1, \Lambda(1))$ of $1 \in \Lambda$ under the boundary map coming from (2.49.3) maps to $1 \in \Lambda$ under the trace map $\mathrm{tr}_{\mathbb{A}^1}$, which is immediate. □

This completes the proof of 2.34. □

3. Localized Chern character

Throughout this section we work over an algebraically closed field k, and with coefficient ring $\Lambda = \mathbb{Q}_\ell$-coefficients for a prime ℓ invertible in k.

3.1. **Definition of localized Chern classes.**

3.2. Let $f : X \to Y$ be a morphism of finite type separated Deligne-Mumford stacks. Define

$$\widehat{H}^i(f : X \to Y) := H^{2i}(X, f^!\Lambda(i)),$$

and let $\widehat{H}^{\cdot}(f : X \to Y)$ denote the graded Λ-module $\oplus_i \widehat{H}^i(f : X \to Y)$.

3.3. The construction of localized Chern classes that we use in this paper is the one of Iversen [15]. We review it here for the convenience of the reader. While loc. cit. is written in the context of vector bundles on topological spaces, the techniques and results there generalize immediate to the case of Deligne-Mumford stacks and étale cohomology (as noted at the end of the introduction of loc. cit.).

Consider first a closed imbedding $i : X \hookrightarrow M$ of finite type separated Deligne-Mumford stacks over k. Let $K^{\cdot}$ be a bounded complex of locally free sheaves of finite rank on M, which is acyclic over the open substack $M - X$. Assume further that $M - X$ is dense in M and that M is connected. We define localized Chern classes

$$\mathrm{ch}^X_M(K)^s \in \widehat{H}^s(i : X \to M)$$

as follows.

3.4. Let ν_i denote the rank of K^i. The assumption that $M - X$ is dense in M implies that there exist nonnegative integers λ_i satisfying

$$\lambda_i + \lambda_{i+1} = \nu_i$$

for all i (namely let λ_i be the generic rank of the image of $K^{i-1} \to K^i$). Let Fl_ν/M denote the relative scheme classifying flags in $K := \oplus_i K^i$ of nationality ν and let F denote the flag given by

$$F_i := \oplus_{t \leq i} K^t.$$

Let

$$T \hookrightarrow \mathrm{Fl}_\nu$$

denote the closed substack classifying flags $D.$ in K with the property that

$$F_{i-1} \subset D_i \subset F_{i+1}$$

for all i, and let $p : T \to M$ be the projection.

3.5. Let $E.$ denote the tautological flag over T. On T there is a complex $C^\cdot$, which we refer to as the *canonical complex*, given by setting $C^i := E_i/p^*F_{i-1}$ and differential given by the map

$$E_i/p^*F_{i-1} \to E_{i+1}/p^*F_i$$

induced by the inclusion $E_i \to E_{i+1}$. Let $T_\Psi \subset T$ denote complement in T of the support of the homology sheaves of $C^\cdot$, and let

$$j_\Psi : T_\Psi \hookrightarrow T$$

be the inclusion. By [15, 2.3] the map

$$H^s(T, \Lambda) \to H^s(T_\Psi, \Lambda)$$

is surjective for every s. In particular, if $i : W \hookrightarrow T$ denotes the inclusion of the complement of T_Ψ, then for every s we have an exact sequence

$$0 \to H^{2s}(W, i^!\Lambda(s)) \to H^{2s}(T, \Lambda(s)) \to H^{2s}(T_\Psi, \Lambda(s)) \to 0.$$

The graded pieces of the usual Chern character $\mathrm{ch}^s(C^\cdot) \in H^{2s}(T, \Lambda(s))$ (see [12, VII, §3]) of the complex $C^\cdot$ map to zero in $H^{2s}(T_\Psi, \Lambda(s))$ since the complex $C^\cdot$ is acyclic over T_Ψ. This class therefore defines a class

$$\gamma_s \in H^{2s}(W, i^!\Lambda(s)).$$

3.6. Let $\delta_i : K^i \to K^{i+1}$ denote the differential on $K^\cdot$ and let

$$s_i(\delta) \subset K$$

denote the graph in $K = (\oplus_{t\leq i} K^t) \oplus (\oplus_{t>i} K^t)$ of the map

$$\oplus_{t\leq i} K^t \to \oplus_{t>i} K^t, \quad (\dots, k_{i-1}, k_i) \mapsto (\delta(k_i), 0, 0, \dots).$$

The $s_i(\delta)$ form a flag in K, and define a morphism

$$s.(\delta) : M \to T$$

such that $s.(\delta)^*C^\cdot \simeq K^\cdot$. In particular, $s.(\delta)^{-1}(W) \subset X$ so by base change we get a morphism

$$H^{2s}(W, i^!\Lambda(s)) \to \widehat{H}^s(X \hookrightarrow M).$$

We define

$$\mathrm{ch}^X_M(K^\cdot)_s \in \widehat{H}^s(X \hookrightarrow M)$$

to be the image of γ_s under this map, and write

$$\mathrm{ch}^X_M(K^\cdot) \in \widehat{H}^\cdot(X \hookrightarrow M)$$

for the sum $\sum_s \mathrm{ch}^X_M(K^\cdot)_s$.

Proposition 3.7. *(i) If $K^\cdot$ is acyclic then* $\mathrm{ch}^X_M(K^\cdot) = 0$.

(ii) $\mathrm{ch}^X_M(K^\cdot[1]) = -\mathrm{ch}^X_M(K^\cdot)$.

(iii) If

$$0 \to K^\cdot \to L^\cdot \to N^\cdot \to 0$$

is an exact sequence of complexes of vector bundles on M with support in X, then

$$\mathrm{ch}^X_M(L^\cdot) = \mathrm{ch}^X_M(K^\cdot) + \mathrm{ch}^X_M(N^\cdot).$$

(iv) If $f : E^\cdot \to K^\cdot$ is a morphism of bounded complexes of vector bundles on M with support in X, we have

$$\mathrm{ch}_M^X(\mathrm{Cone}(f)) = \mathrm{ch}_M^X(K^\cdot) - \mathrm{ch}_M^X(E^\cdot).$$

If f is a quasi-isomorphism then $\mathrm{ch}_M^X(K^\cdot) = \mathrm{ch}_M^X(E^\cdot)$.

(v) Given a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "f"] & M' \arrow[d, "g"] \\ X \arrow[r, hook] & M \end{tikzcd}$$

we have

$$g^*\mathrm{ch}_M^X(K^\cdot) = \mathrm{ch}_{M'}^{X'}(g^*K^\cdot).$$

Proof. Statements (i), (ii), and (v) are immediate from the construction. Statement (iii) is [15, 4.2]. The first part of statement (iv) follows from (ii) and (iii) applied to the exact sequence

$$0 \to K^\cdot \to \mathrm{Cone}(f) \to E^\cdot[1] \to 0.$$

The second part of (iv) is [15, 4.3]. □

Remark 3.8. The preceding construction can also be applied with $X \hookrightarrow M$ a closed imbedding of Artin stacks, replacing the étale site by the lisse-étale site and 3.7 holds also in this more general context with the same proof.

Remark 3.9. In the above we assumed that M was connected. To extend the definition to disconnected M apply the construction to each connected component of M.

Definition 3.10. If $f : X \to Y$ is a morphism of quasi-projective finite group quotients over k and $F^\cdot \in D^b_{\mathrm{coh}}(X)$, then F is called *f-perfect* if for some factorization of f

$$X \xhookrightarrow{i} M \xrightarrow{p} Y, \tag{3.10.1}$$

where i is a closed imbedding of X into a quasi-projective finite group quotient M and p is smooth, the complex $i_*F^\cdot \in D^b_{\mathrm{coh}}(M)$ is isomorphic in $D_{\mathrm{coh}}(M)$ to a bounded complex of locally free sheaves of finite rank on M.

3.11. The existence of a factorization (3.10.1) is shown in A.6. Moreover, by A.7 every coherent sheaf on M is a quotient of a locally free sheaf of finite rank. From this and [2, III.4.4] it follows that a complex $F^\cdot \in D_{\mathrm{coh}}(X)$ is f-perfect if and only if it is f-perfect in the sense of [2, III.4.1]. In particular, this condition is independent of the choice of factorization (3.10.1), is smooth local on X and Y, and if $F^\cdot$ is f-perfect then for every factorization (3.10.1) the complex $i_*F^\cdot$ can be represented by a bounded complex of locally free sheaves of finite rank.

Furthermore, the subcategory $D^b_{f-\mathrm{perf}}(X) \subset D^b_{\mathrm{coh}}(X)$ of f-perfect complexes is a triangulated subcategory, and we can define the Grothendieck group $K(f : X \to Y)$ of f-perfect complexes. By definition this is the Grothendieck group of $D^b_{f-\mathrm{perf}}(X)$.

It follows from 3.7 that for a closed imbedding $X \hookrightarrow M$ of quasi-projective finite group quotients over k and with dense complement, the localized Chern character defines a morphism

$$\mathrm{ch}_M^X : K(X \hookrightarrow M) \to \widehat{H}^\cdot(X \hookrightarrow M).$$

This map is obtained by representing a class $K(X \hookrightarrow M)$ by a complex $K^\cdot \in D^b_{f-\mathrm{perf}}(X)$, then representing the complex $i_*K^\cdot$ by a bounded complex of locally free sheaves on M, and finally applying the preceding construction to this resolution.

Remark 3.12. A special case of the preceding is when M is smooth over k. In this case $K(X \hookrightarrow M)$ is the Grothendieck group of coherent sheaves $K_0(X)$ on X, and we get a homomorphism

$$\mathrm{ch}^X_M : K_0(X) \to \widehat{H}^\cdot(X \hookrightarrow M).$$

This map can be generalized also to closed imbeddings $X \hookrightarrow M$ of Artin stacks of finite type over k, where M has the resolution property as discussed in [22]. Recall that this means that every coherent sheaf on M is a quotient of a locally free sheaf on M. For example, by [22, 2.1] if M is a quotient $[U/G]$ of a finite type scheme U by the action of an affine finite type k-group scheme G and U has an ample family of G-linearized line bundles, then M has the resolution property.

3.13. **Some calculations of localized Chern classes.**

3.14. Let $s : X \hookrightarrow Y$ be a regular closed imbedding of separated finite type Deligne-Mumford stacks over k. The key tool for our computations will be the standard degeneration of Y with respect to X, constructed in 2.44 and denoted $q : \mathscr{Y} \to Y \times \mathbb{A}^1$.

Recall that this degeneration is obtained as follows. Let $\mathscr{I} \subset \mathscr{O}_Y$ be the ideal defining $s : X \hookrightarrow Y$, and consider the affine morphism

$$\underline{\mathrm{Spec}}_Y((\mathrm{Sym}^\cdot_{\mathscr{O}_Y}\mathscr{I})[t]) \to Y.$$

We have two inclusions

$$\rho_1, \rho_2 : \mathscr{I} \hookrightarrow \mathrm{Sym}^\cdot_{\mathscr{O}_Y}\mathscr{I},$$

where ρ_1 is defined by the inclusion $\mathscr{I} \hookrightarrow \mathscr{O}_Y$ and ρ_2 is the inclusion of $\mathscr{I}$ as the degree 1 elements with respect to the natural grading. Let $\mathscr{J} \subset \mathrm{Sym}^\cdot_{\mathscr{O}_Y}(\mathscr{I})[t]$ be the ideal generated by the image of $\rho_1 - t\rho_2$, and let $\mathscr{Y}$ be the corresponding closed substack. There is a projection

$$q : \mathscr{Y} \to Y \times \mathbb{A}^1.$$

In the local setting when we have a regular sequence $x_1, \ldots, x_r \in \mathscr{O}_Y$ defining the ideal $\mathscr{I}$ we have

$$\mathscr{Y} = \underline{\mathrm{Spec}}_X(\mathscr{O}_Y[t][z_1, \ldots, z_r]/(tz_i - x_i)_{i=1}^r).$$

From this local description we see the following:

(i) For a point $a \in \mathbb{G}_m(k) \subset \mathbb{A}^1(k)$ the fiber $\mathscr{Y}_a$ projects isomorphically to Y.
(ii) The fiber $\mathscr{Y}_0$ of $\mathscr{Y}$ over $X \times \{0\}$ is the total space $N := \underline{\mathrm{Spec}}_X(\mathrm{Sym}^\cdot \mathscr{I}/\mathscr{I}^2)$ of the normal bundle of X in Y.
(iii) There is a natural inclusion $\tilde{s} : X \times \mathbb{A}^1 \hookrightarrow \mathscr{Y}$, whose restriction to $\mathscr{Y}_a$ (for $a \in \mathbb{G}_m(k)$) is the inclusion s, and whose fiber over 0 is the zero section of $N \to X$.

3.15. We will also consider a variant of this construction. Let X/k be a finite type separated Deligne-Mumford stack. An *effective pseudo-divisor on* X is a pair $(\mathscr{L}, \alpha)$, where $\mathscr{L}$ is an invertible sheaf and $\alpha : \mathscr{L} \to \mathscr{O}_X$ is a morphism of $\mathscr{O}_X$-modules. The image of α is an ideal in $\mathscr{O}_X$ and the corresponding closed subscheme $Z \subset X$ is called the *support of* $(\mathscr{L}, \alpha)$. Then over $X \times \mathbb{A}^1$ we get an ideal sheaf $\mathscr{J} \subset (\mathrm{Sym}^\cdot_{\mathscr{O}_X}\mathscr{L})[t]$ by taking the ideal generated locally by elements of the form $tm - \alpha(m)$, for local sections $m \in \mathscr{L}$. More concretely, if $X = \mathrm{Spec}(R)$, $\mathscr{L}$ is trivial, and the map α is given by an element $f \in R$, then the ideal sheaf $\mathscr{J} \subset R[t, z]$ is the ideal generated by $tz - f$. We therefore get a degeneration

$$\mathscr{X} \to X \times \mathbb{A}^1.$$

The fiber over $a \in \mathbb{A}^1(k)$ is isomorphic to X if $a \neq 0$, and the fiber over 0 is equal to the total space of the line bundle $i^*\mathscr{L}^\vee$, where $i : Z \hookrightarrow X$ is the closed substack defined by $\alpha(\mathscr{L})$. We refer to $\mathscr{X}$ as the *standard degeneration of* X *with respect to* $(\mathscr{L}, \alpha)$.

Note that in the case when $\alpha : \mathscr{L} \to \mathscr{O}_X$ is the inclusion of the ideal sheaf of an effective Cartier divisor $Z \subset X$, this variant construction agrees with the standard degeneration of X with respect to Z.

Calculation 3.16. *Let* X *be a quasi-projective finite group quotient over* k*, and let* $\mathscr{L}$ *be an invertible sheaf on* X*. Let*

$$\mathbb{L} := \underline{\mathrm{Spec}}_X(\mathrm{Sym}^{\cdot}\mathscr{L}) \xrightarrow{p} X$$

be the total space of $\mathscr{L}^\vee$*, and let* $s : X \to \mathbb{L}$ *be the zero section. Then the image of* $\mathrm{ch}^X_{\mathbb{L}}(\mathscr{O}_X)_q$ *under the isomorphism*

$$\rho : \widehat{H}^q(X \hookrightarrow \mathbb{L}) \to H^{2(q-1)}(X, \Lambda(q-1)) \tag{3.16.1}$$

induced by the canonical isomorphism $p^!\Lambda_X \simeq \Lambda_{\mathbb{L}}(1)[2]$ *is equal to* $(\mathrm{td}(\mathscr{L}^\vee)^{-1})_{q-1}$.

Remark 3.17. Here $\mathrm{td}(\mathscr{L}^\vee)$ denotes the Todd class of the invertible sheaf $\mathscr{L}^\vee$ [10, 3.2.4].

Proof. Consider the exact sequence on $\mathbb{L}$

$$0 \to p^*\mathscr{L} \to \mathscr{O}_{\mathbb{L}} \to s_*\mathscr{O}_X \to 0.$$

Therefore in the notation of 3.4, we have $K = p^*\mathscr{L} \oplus \mathscr{O}_{\mathbb{L}}$,

$$\lambda_i = \begin{cases} 1 & i = 0 \\ 0 & i \neq 0. \end{cases}$$

The flag $F_\cdot$ is the flag with $F_{-2} = 0$, $F_{-1} = p^*\mathscr{L}$, and $F_0 = K$. The flag variety Fl_ν is simply $\mathbb{P} := \mathbb{P}(p^*\mathscr{L} \oplus \mathscr{O}_{\mathbb{L}})$, and $T = \mathrm{Fl}_\nu$ since the condition on compatiblity with $F_\cdot$ automatically holds.

The canonical complex on $\mathbb{P}(p^*\mathscr{L} \oplus \mathscr{O}_{\mathbb{L}})$ can be described as follows. Let $\mathscr{M} \subset (p^*\mathscr{L} \oplus \mathscr{O}_{\mathbb{L}})|_{\mathbb{P}}$ be the universal subbundle. Then the complex $C^\cdot$ is the complex

$$\mathscr{M} \to \mathscr{O}_{\mathbb{P}}$$

induced by the second projection, with $\mathscr{M}$ sitting in degree -1. Note that this complex is acyclic away from the section $s(\delta) : \mathbb{L} \to \mathbb{P}(p^*\mathscr{L} \oplus \mathscr{O}_{\mathbb{L}})$ corresponding to

$$\mathrm{id} \oplus 0 : p^*\mathscr{L} \to p^*\mathscr{L} \oplus \mathscr{O}_{\mathbb{L}},$$

so T_Ψ is in this case isomorphic to $\mathrm{Spec}(\mathrm{Sym}^{\cdot}(p^*\mathscr{L}^\vee))$.

Also the graph of the inclusion $p^*\mathscr{L} \hookrightarrow \mathscr{O}_{\mathbb{L}}$ defines a morphism $q : \mathbb{L} \to \mathbb{P}(p^*\mathscr{L} \oplus \mathscr{O}_{\mathbb{L}})$ and we have a cartesian square

$$\begin{tikzcd} X \arrow[r, hook, "s"] \arrow[d, hook, "s"] & \mathbb{L} \arrow[d, "q"] \\ \mathbb{L} \arrow[r, "s(\delta)"] & \mathbb{P}(p^*\mathscr{L} \oplus \mathscr{O}_{\mathbb{L}}). \end{tikzcd}$$

Now to calculate the Chern classes, note that we have

$$\mathrm{ch}(C^{\cdot}) = -\mathrm{ch}(\mathscr{M}) + 1 = 1 - e^{c_1(\mathscr{M})} = c_1(\mathscr{M}^\vee)\left(\frac{c_1(\mathscr{M}^\vee)}{1 - e^{-c_1(\mathscr{M}^\vee)}}\right)^{-1} = c_1(\mathscr{M}^\vee)\mathrm{td}(\mathscr{M}^\vee)^{-1}.$$

in $\oplus_i H^{2i}(\mathbb{P}(p^*\mathscr{L} \oplus \mathscr{O}_{\mathbb{L}}), \Lambda(i))$. From this it follows that if $c_1^{\mathrm{loc}}(p^*\mathscr{L}^\vee) \in H^2(X, s^!\Lambda(1))$ denotes Deligne's localized Chern class (see 2.14), then

$$\mathrm{ch}^X_{\mathbb{L}}(\mathscr{O}_X) = c_1^{\mathrm{loc}}(p^*\mathscr{L}^\vee) \cdot \mathrm{td}(\mathscr{L}^\vee)^{-1}.$$

Since multiplication by $c_1^{\mathrm{loc}}(p^*\mathscr{L}^\vee)$ is the inverse to the isomorphism 3.16.1 (by 2.30) this completes the proof of 3.16. □

3.18. One advantage of the notion of effective pseudo-divisor is that it is functorial. Namely if X is an algebraic stack and $\alpha : \mathscr{L} \to \mathscr{O}_X$ is an effective pseudo-divisor, then for any morphism $f : Y \to X$ we get an effective pseudo-divisor $f^*\alpha : f^*\mathscr{L} \to \mathscr{O}_Y$ by pullback. There is in fact a universal effective pseudo-divisor. Let $[\mathbb{A}^1/\mathbb{G}_m]$ denote the stack-quotient of $\mathbb{A}^1$ by the usual multiplication action of $\mathbb{G}_m$. Then the origin in $\mathbb{A}^1$ is invariant and defines an effective Cartier divisor in $[\mathbb{A}^1/\mathbb{G}_m]$. Let $\mathscr{L}^u \to \mathscr{O}_{[\mathbb{A}^1/\mathbb{G}_m]}$ be the corresponding effective pseudo-divisor. Then for any algebraic stack X, pullback of $\mathscr{L}^u \to \mathscr{O}_{[\mathbb{A}^1/\mathbb{G}_m]}$ defines an equivalence of groupoids between effective pseudo-divisors on X and morphisms $X \to [\mathbb{A}^1/\mathbb{G}_m]$.

One application of this is the following. Let X be an algebraic stack and let $\alpha : \mathscr{L} \to \mathscr{O}_X$ be an effective pseudo-divisor with support $i : Z \hookrightarrow X$. This defines a cartesian diagram

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\ B\mathbb{G}_m \arrow[r, hook] & {[\mathbb{A}^1/\mathbb{G}_m]}, \end{tikzcd}$$

and we define the localized Chern class $c_1^{\mathrm{loc}}(\mathscr{L}, \alpha) \in H^2_Z(X, \Lambda(1))$ to be the pullback along f of the localized Chern class of $\alpha : \mathscr{L}^u \to \mathscr{O}_{[\mathbb{A}^1/\mathbb{G}_m]}$ in $H^2_{B\mathbb{G}_m}([\mathbb{A}^1/\mathbb{G}_m], \Lambda(1))$.

Calculation 3.19. *Let X be an algebraic stack, and let $\alpha : \mathscr{L} \to \mathscr{O}_X$ be an effective pseudo-divisor with support $i : Z \hookrightarrow X$. Let $\mathscr{E}^{\cdot}$ denote the complex $\mathscr{L} \to \mathscr{O}_X$ (with $\mathscr{L}$ placed in degree -1). Then*

$$\mathrm{ch}^Z_X(\mathscr{E}^{\cdot}) = c_1^{\mathrm{loc}}(\mathscr{L}, \alpha) \cdot \mathrm{td}(\mathscr{L}^\vee)^{-1} \tag{3.19.1}$$

in $\widehat{H}^{\cdot}(i : Z \hookrightarrow X)$.

Proof. By functoriality of both sides, it suffices to consider the universal case $X = [\mathbb{A}^1/\mathbb{G}_m]$ and $\alpha^u : \mathscr{L}^u \to \mathscr{O}_{[\mathbb{A}^1/\mathbb{G}_m]}$. In this case the map

$$\widehat{H}^{\cdot}(B\mathbb{G}_m \to [\mathbb{A}^1/\mathbb{G}_m]) \to \oplus_{i \geq 0} H^{2i}([\mathbb{A}^1/\mathbb{G}_m], \Lambda(i))$$

is injective. By definition, the left side of 3.19.1 maps to $1-\mathrm{ch}(\mathscr{L}^u)$ in $\oplus_{i\geq 0}H^{2i}([\mathbb{A}^1/\mathbb{G}_m], \Lambda(i))$, and the right side maps to

$$c_1(\mathscr{L}^{u\vee})\cdot \mathrm{td}(\mathscr{L}^{u\vee})^{-1} = c_1(\mathscr{L}^{u\vee})\cdot \frac{1-e^{c_1(\mathscr{L}^u)}}{c_1(\mathscr{L}^{u\vee})} = 1-\mathrm{ch}(\mathscr{L}^u).$$

□

Calculation 3.20. *Let $\mathscr{E}\to\mathscr{E}''$ be a surjective map of locally free sheaves of finite rank on a quasi-projective finite group quotient X/k, and set*

$$E'' = \underline{\mathrm{Spec}}_X(\mathrm{Sym}^{\cdot}\mathscr{E}''),\quad E = \underline{\mathrm{Spec}}_X(\mathrm{Sym}^{\cdot}\mathscr{E})$$

so we have a commutative diagram of zero sections

$$\begin{tikzcd} X \arrow[r, hook, "s''"] \arrow[rr, hook, bend left, "s"] & E'' \arrow[r, hook, "t"] & E. \end{tikzcd}$$

Then we have

$$\mathrm{ch}^X_E(\mathscr{O}_X) = \mathrm{ch}^X_{E''}(\mathscr{O}_X)\cdot \mathrm{ch}^{E''}_E(\mathscr{O}_{E''}).$$

Proof. It suffices to prove the formula after making a base change $X'\to X$, where X' is an affine bundle over X (locally on X isomorphic to $\mathbb{A}^r_X$ for some r). Let X'/X be the scheme representing the functor which to any scheme $g: T\to X$ associates the set of sections of the map $g^*\mathscr{E}\to g^*\mathscr{E}''$. Then X' is a torsor under $\mathscr{H}om(\mathscr{E}'', \mathscr{E}')$, where $\mathscr{E}'$ denotes the kernel $\mathscr{E}' := \mathrm{Ker}(\mathscr{E}\to\mathscr{E}'')$. In particular, $X'\to X$ is an affine bundle over X. It therefore suffices to prove our formula after base change to X', over which we have $\mathscr{E} = \mathscr{E}'\oplus\mathscr{E}''$. Now in this split case, setting $E' := \underline{\mathrm{Spec}}_X(\mathrm{Sym}^{\cdot}\mathscr{E}')$ we have a cartesian square

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & E' \arrow[d, hook] \\ E'' \arrow[r, hook] & E, \end{tikzcd}$$

and the formula follows from [15, 1.8]. □

Calculation 3.21. *Let $\mathscr{E}$ be a locally free sheaf of finite rank r on a quasi-projective finite group quotient X, and let $E = \underline{\mathrm{Spec}}_X(\mathrm{Sym}^{\cdot}\mathscr{E})$ with zero section $s: X\hookrightarrow E$. Then under the isomorphism*

(3.21.1) $$\rho : \widehat{H}^q(X\hookrightarrow E)\to H^{2(q-r)}(X, \Lambda(s-r))$$

the image of $\mathrm{ch}^X_E(\mathscr{O}_X)_q$ is equal to $\mathrm{td}(\mathscr{E}^\vee)^{-1}_{q-r}$.

Proof. We proceed by induction on r, the case $r=1$ being 3.16.

For the inductive step we proceed as in the proof of 2.30. Note that if $q: X'\to X$ is a projective bundle, then it suffices to prove the formula after base change to X'. Taking $X' = \mathbb{P}(\mathscr{E})$, we may therefore assume that there exists an exact sequence of locally free sheaves

$$0\to\mathscr{E}'\to\mathscr{E}\to\mathscr{E}''\to 0$$

where $\mathscr{E}'$ and $\mathscr{E}''$ have strictly smaller ranks. Looking at total spaces, we get a commutative diagram

$$\begin{tikzcd}
X \arrow[r, hook, "s''"] \arrow[rr, hook, bend left, "s"] \arrow[dr, equal] & E'' \arrow[r, hook, "t"] \arrow[d, "a"] & E \arrow[d, "b"] \\
& X \arrow[r, hook, "s'"] \arrow[dr, equal] & E' \arrow[d] \\
& & X,
\end{tikzcd}$$

where $E' = \underline{\mathrm{Spec}}_X(\mathrm{Sym}^{\cdot}\mathscr{E}')$ and $E'' = \underline{\mathrm{Spec}}_X(\mathrm{Sym}^{\cdot}\mathscr{E}'')$.

By 3.20, we have

$$\mathrm{ch}_E^X(\mathscr{O}_X) = \mathrm{ch}_{E''}^X(\mathscr{O}_X) \cdot \mathrm{ch}_E^{E''}(\mathscr{O}_{E'}) = \mathrm{ch}_{E''}^X(\mathscr{O}_X) \cdot b^*\mathrm{ch}_{E'}^X(\mathscr{O}_X).$$

By induction, this equals

$$(c_{r''}^{\mathrm{loc}}(\mathscr{E}''^{\vee})\mathrm{td}(\mathscr{E}''^{\vee})^{-1}) \cdot (c_{r'}^{\mathrm{loc}}(\mathscr{E}'^{\vee})\mathrm{td}(\mathscr{E}'^{\vee})^{-1}),$$

which by multiplicativity of the top localized Chern classes (as in the proof of 2.30) and Todd classes [10, top of p. 57] is equal to $c_r^{\mathrm{loc}}(\mathscr{E}^{\vee})\mathrm{td}(\mathscr{E}^{\vee})^{-1}$. $\square$

Calculation 3.22. *Let $g : Y \to X$ be a smooth representable morphism of quasi-projective finite group quotients of relative dimension r, and let $s : X \to Y$ be a section. Then under the isomorphism*

$$\rho : \widehat{H}^{\cdot}(s : X \hookrightarrow Y) \to H^{2(\cdot - r)}(X, \Lambda(\cdot - r))$$

the class of $\mathrm{ch}_Y^X(\mathscr{O}_X)$ maps to $\mathrm{td}(s^*T_{Y/X})^{-1}$.

Proof. Let

$$\mathscr{Y} \to Y \times \mathbb{A}^1$$

be the standard degeneration of Y with respect to $s : X \hookrightarrow Y$ (see 3.14), and observe that by the local description of $\mathscr{Y}$ in 3.14, the map $\tilde{g} : \mathscr{Y} \to X \times \mathbb{A}^1$ is smooth of relative dimension r. Let $\tilde{s} : X \times \mathbb{A}^1 \hookrightarrow \mathscr{Y}$ be the natural inclusion, whose restriction to $\mathscr{Y}_a$ (for $a \in \mathbb{G}_m(k)$) is the inclusion s, and whose fiber over 0 is the zero section of $N \to X$. Here N denotes $\mathrm{Spec}_X \mathrm{Sym}^{\cdot}(\mathscr{I}/\mathscr{I}^2)$, where $\mathscr{I}$ denotes the ideal of X in Y.

Consider the classes

$$\tilde{\gamma}_i := \mathrm{ch}_{\mathscr{Y}}^{X \times \mathbb{A}^1}(\mathscr{O}_{X \times \mathbb{A}^1})_i \in H^{2i}(X \times \mathbb{A}^1, \tilde{s}^!\Lambda(i)),$$

and its image under the isomorphism

$$\tilde{\rho} : H^{2i}(X \times \mathbb{A}^1, \tilde{s}^!\Lambda(i)) \to H^{2(i-r)}(X \times \mathbb{A}^1, \Lambda(i - r)).$$

For a point $a \in \mathbb{A}^1(k)$, let $s_a : X \hookrightarrow \mathscr{Y}_a$ denote the restriction of $\tilde{s}$. Then for every $a \in \mathbb{A}^1(k)$ the restriction map induced by base change

$$\mathrm{res}_a : H^{2i}(X \times \mathbb{A}^1, \tilde{s}^!\Lambda(i)) \to H^{2i}(X, s_a^!\Lambda(i))$$

is an isomorphism and the diagram

$$\begin{tikzcd} H^{2i}(X\times\mathbb{A}^1, \tilde{s}^!\Lambda(i)) \arrow[r, "\mathrm{res}_a"] \arrow[d, "\tilde{\rho}"] & H^{2i}(X, s_a^!\Lambda(i)) \arrow[d, "\rho_a"] \\ H^{2(i-r)}(X\times\mathbb{A}^1, \Lambda(i-r)) \arrow[r, "\mathrm{res}_a"] & H^{2(i-r)}(X, \Lambda(i-r)) \end{tikzcd}$$

commutes. This follows from the compatibility of the isomorphism $\tilde{g}^!\Lambda_{X\times\mathbb{A}^1}\simeq\Lambda_{\mathscr{Y}}(-r)[-2r]$ with base change, which in turn follows from compatibility of the trace map with base change [1, XVIII, 2.9 (Var 2)] (and in the case of stacks [20, 4.1 (ii)]). Also observe that the maps

$$\mathrm{res}_a : H^{2(i-r)}(X\times\mathbb{A}^1, \Lambda(i-r)) \to H^{2(i-r)}(X, \Lambda(i-r))$$

are independent of a since the inverse is given by the pullback map

$$\mathrm{pr}_1^* : H^{2(i-r)}(X, \Lambda(i-r)) \to H^{2(i-r)}(X\times\mathbb{A}^1, \Lambda(i-r)).$$

It follows that we have

$$\begin{aligned} \rho_1(\mathrm{ch}_Y^X(\mathscr{O}_X)) &= \mathrm{res}_1\tilde{\rho}(\mathrm{ch}_{\mathscr{Y}}^{X\times\mathbb{A}^1}(\mathscr{O}_{X\times\mathbb{A}^1})) \\ &= \mathrm{res}_0\tilde{\rho}(\mathrm{ch}_{\mathscr{Y}}^{X\times\mathbb{A}^1}(\mathscr{O}_{X\times\mathbb{A}^1})) \\ &= \rho_0(\mathrm{ch}_N^X(\mathscr{O}_X)) \\ &= \mathrm{td}((\mathscr{I}/\mathscr{I}^2)^\vee)^{-1} \\ &= \mathrm{td}(s^*T_{Y/X})^{-1}, \end{aligned}$$

where the second to last equality is by 3.21. □

Calculation 3.23. *Consider a commutative diagram of quasi-projective finite group quotients*

$$\begin{tikzcd} X \arrow[r, hook, "s"] & Y \arrow[r, hook, "i"] \arrow[dr, "a"'] & M' \arrow[d, "b"] \\ & & M \end{tikzcd}$$

where s and i are closed imbeddings and a and b are smooth. Then for every s-perfect complex K on X, we have

$$\mathrm{ch}_{M'}^X(K) = \mathrm{ch}_Y^X(K)\cdot\mathrm{ch}_{M'}^Y(\mathscr{O}_Y). \tag{3.23.1}$$

Remark 3.24. Note that since a and b are assumed smooth, K is s-perfect if and only if K is is-perfect.

Proof of 3.23. Let $\mathscr{M}'\to M'\times\mathbb{A}^1$ be the standard degeneration of M' with respect to $i: Y\hookrightarrow M'$, so we have a commutative diagram

$$\begin{tikzcd} X\times\mathbb{A}^1 \arrow[r, hook] & Y\times\mathbb{A}^1 \arrow[r, hook] \arrow[dr, "\tilde{a}"'] & \mathscr{M}' \arrow[d, "\tilde{b}"] \\ & & M\times\mathbb{A}^1, \end{tikzcd}$$

where $\tilde{a} = a\times\mathrm{id}_{\mathbb{A}^1}$ and $\tilde{b}$ is smooth. As in the proof of 3.22, it suffices to prove that the formula (3.23.1) holds in the fiber over $0\in\mathbb{A}^1(k)$. Now the fiber over 0 of $\mathscr{M}'$ is equal to the normal cone N of Y in M' with the inclusion of the zero section $Y\hookrightarrow N$. It therefore suffices to show the formula (3.23.1) in the case when $M = Y$, and $M' = \underline{\mathrm{Spec}}_Y(\mathrm{Sym}^\cdot\mathscr{E})$ for some locally free sheaf of finite rank $\mathscr{E}$ on Y. In this case we have a commutative diagram with cartesian squares

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & Y \arrow[d, hook, "0"] \\ P \arrow[r, hook] \arrow[d, "\mathrm{pr}_1"] & M' \arrow[d] \\ X \arrow[r, hook] & Y, \end{tikzcd}$$

where $P := X \times_Y M'$, and the formula (3.23.1) follows from [15, 1.8]. $\square$

Calculation 3.25. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & Y \arrow[d, "g"] \\ & M \end{tikzcd}$$

of quasi-projective finite group quotients over k, where i is a closed imbedding and f and g are smooth. Let r denote the codimension of X in Y, so we have an isomorphism $i^!\Lambda_Y \simeq \Lambda_X(-r)[-2r]$ which induces an isomorphism

$$\lambda_q : \widehat{H}^q(X \to Y) \simeq H^{2(q-r)}(X, \Lambda(s-r)).$$

Then the image of $\mathrm{ch}_Y^X(\mathscr{O}_X)$ under the resulting isomorphism $\lambda : \widehat{H}^{\cdot}(X \hookrightarrow Y) \simeq \widehat{H}^{\cdot - r}(X \to X)$ is $\mathrm{td}(\mathscr{N})^{-1}$, where $\mathscr{N}$ is the normal bundle of X in Y.

Proof. Let $\mathscr{Y} \to Y \times \mathbb{A}^1$ be the standard degeneration of Y with respect to X, so we have a commutative diagram

$$\begin{tikzcd} X \times \mathbb{A}^1 \arrow[r, hook, "\tilde{i}"] \arrow[dr, "f \times \mathrm{id}"'] & \mathscr{Y} \arrow[d, "\tilde{g}"] \\ & M \times \mathbb{A}^1. \end{tikzcd}$$

The fiber over $0 \in \mathbb{A}^1(k)$ of this diagram is equal to

$$\begin{tikzcd} X \arrow[r, hook, "s"] \arrow[dr] & N \arrow[d] \\ & M, \end{tikzcd}$$

where $N = \underline{\mathrm{Spec}}_X(\mathrm{Sym}^{\cdot}\mathscr{N}^{\vee})$ is the total space of the normal bundle of X in Y. As in the proof of 3.22, it then suffices to show the result in the case when $Y = N$, where the result follows from 3.21. $\square$

Calculation 3.26. *Consider a commutative diagram of quasi-projective finite group quotients*

$$\begin{tikzcd} & M' \arrow[d, "g"] \\ X \arrow[ur, hook, "t"] \arrow[r, hook, "i"] & M, \end{tikzcd}$$

where g is smooth of relative dimension r (a locally constant function on M'), and i and t are closed imbeddings. Then for any i-perfect complex K on X, the image of $\mathrm{ch}^X_{M'}(K)$ under the isomorphism

$$\lambda : \widehat{H}^{\cdot}(X \hookrightarrow M') \xrightarrow{\simeq} \widehat{H}^{\cdot - r}(X \hookrightarrow M) \tag{3.26.1}$$

is equal to $\mathrm{ch}^X_M(K) \cdot \mathrm{td}(t^* T_{M'/M})^{-1}$.

Proof. First we reduce to the case when g is quasi-projective. For this write $M = [U/G]$ where U is a quasi-projective k-scheme and G is a finite group acting on U. Let M'_U (resp. X_U) denote the base change of M' (resp. X) to U. Then the pullback maps

$$\widehat{H}^{\cdot}(X \hookrightarrow M') \to \widehat{H}^{\cdot}(X_U \hookrightarrow M'_U) \ \text{ and } \ \widehat{H}^{\cdot}(X \hookrightarrow M) \to \widehat{H}^{\cdot}(X_U \hookrightarrow U)$$

are injective. By functoriality of the localized Chern classes and Todd class it therefore suffices to prove the result after replacing M by U which reduces the proof to the case when M, and hence also X, is a scheme. In this case the imbedding $X \hookrightarrow M'$ factors through the maximal schematic open substack of M'. Replacing M' by this open subset we may therefore further assume that M' is a quasi-projective scheme, and in particular that g is quasi-projective.

Next we reduce the proof to the case when M' is $\mathbb{P}^d_M$ for some d. For this consider a factorization

$$\begin{tikzcd} M' \arrow[r, hook, "a"] \arrow[d, "g"'] & P \arrow[dl, "h"'] \\ M, & \end{tikzcd}$$

where a is a closed imbedding, and h is also smooth (we will apply this with P an open subset of $\mathbb{P}^n_M$ for some n). We then have a commutative diagram with cartesian squares

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & Y \arrow[r, hook] \arrow[d, hook] & M' \arrow[d, hook] \\ & P_X \arrow[r, hook] \arrow[d] & P \arrow[d] \\ & X \arrow[r, hook] & M, \end{tikzcd}$$

where Y denotes the fiber product $X \times_M M'$. Now if the proposition holds for the diagram

$$\begin{tikzcd} & P \arrow[d] \\ X \arrow[ur, hook] \arrow[r, hook] & M, \end{tikzcd}$$

then we have

$$\lambda_P(\mathrm{ch}^X_P(K)) = \mathrm{ch}^X_M(K) \cdot \mathrm{td}(T_{P/M}|_X)^{-1}, \tag{3.26.2}$$

where we write $\lambda_P : \widehat{H}^{\cdot}(X \hookrightarrow P) \to \widehat{H}^{\cdot}(X \hookrightarrow M)$ for the natural isomorphism (which involves a shift of the grading). By 3.23 applied to

$$\begin{tikzcd} X \arrow[r, hook] & M' \arrow[r, hook] \arrow[dr] & P \arrow[d] \\ & & M, \end{tikzcd}$$

we have

$$\mathrm{ch}^X_P(K) = \mathrm{ch}^X_{M'}(K) \cdot \mathrm{ch}^{M'}_P(\mathscr{O}_{M'})$$

in $\widehat{H}^{\cdot}(X \hookrightarrow P)$, which by 3.25 maps to $\lambda(\mathrm{ch}^X_{M'}(K)) \cdot \mathrm{td}(\mathscr{N}_{M'/P}|_X)^{-1}$ in $\widehat{H}^{\cdot}(X \hookrightarrow M)$. Combining this with (3.26.2) we find that

$$\lambda(\mathrm{ch}^X_{M'}(K)) \cdot \mathrm{td}(\mathscr{N}_{M'/P}|_X)^{-1} = \mathrm{ch}^X_M(K) \cdot \mathrm{td}(T_{P/M}|_X)^{-1}. \tag{3.26.3}$$

Now from the commutative diagram

$$\begin{tikzcd} M' \arrow[r, hook] \arrow[dr] & P \arrow[d] \\ & M \end{tikzcd}$$

we obtain an exact sequence

$$0 \to T_{M'/M}|_X \to T_{P/M}|_X \to \mathscr{N}_{M'/P}|_X \to 0.$$

By additivity of the Todd class we get that

$$\mathrm{td}(T_{P/M}|_X)^{-1} \cdot \mathrm{td}(\mathscr{N}_{M'/P}|_X) = \mathrm{td}(T_{M'/M})^{-1},$$

and combining this with (3.26.3) we get

$$\lambda(\mathrm{ch}^X_{M'}(K)) = \mathrm{ch}^X_M(K) \cdot \mathrm{td}(t^*T_{M'/M})^{-1}$$

as desired.

It therefore suffices to prove the statement in 3.26 in the case when M' is an open substack of $\mathbb{P}^n_M$ for some n. In this case we have

$$\mathrm{ch}^X_{M'}(K) = \mathrm{ch}^X_{\mathbb{P}^n_M}(K), \quad t^*T_{M'/M} = T_{\mathbb{P}^n_M/M}|_X,$$

so it suffices to verify 3.26 in the case when $M' = \mathbb{P}^n_M$, which we assume for the rest of the proof.

Lemma 3.27. *The pushforward map*

$$\widehat{H}^{\cdot}(X \hookrightarrow \mathbb{P}^n_M) \to \widehat{H}^{\cdot}(\mathbb{P}^n_X \hookrightarrow \mathbb{P}^n_M)$$

is injective.

Proof. Let M denote $\tilde{i}^!\Lambda$. It suffices to show that the morphism $s^!M \to p_{X*}M$ induced by the adjunction $s_*s^! \to \mathrm{id}$ is a split monomorphism. This is true for any M of the form $p^!_XG$, for G

on X (in particular, for $\tilde{i}^!\Lambda \simeq p_X^!\Lambda(-n)[-2n]$). Indeed if $M = p_X^! G$ we have a commutative diagram

$$\begin{tikzcd} G = s^! M \arrow[r] \arrow[rr, bend right, "\mathrm{id}_G"'] & p_{X*}M = p_{X*}p_X^! G \arrow[r, "p_{X*}p_X^! \to \mathrm{id}"] & G. \end{tikzcd}$$

□

Since the pullback map

$$\widehat{H}^{\cdot}(X \hookrightarrow M) \to \widehat{H}^{\cdot}(\mathbb{P}^n_X \hookrightarrow \mathbb{P}^n_M)$$

is injective, to verify the equality in 3.26 it suffices to show that $\mathrm{ch}^X_{\mathbb{P}^n_M}(K)$ and $\mathrm{ch}^X_M(K) \cdot \mathrm{td}(T_{\mathbb{P}^n/M})^{-1}$ have the same image in $\widehat{H}^{\cdot}(\mathbb{P}^n_X \hookrightarrow \mathbb{P}^n_M)$. We therefore have to understand the relationship between the composite map

$$\widehat{H}^{\cdot}(X \hookrightarrow \mathbb{P}^n_M) \xrightarrow{(3.26.1)} \widehat{H}^{\cdot-n}(X \hookrightarrow M) \xrightarrow{p^*} \widehat{H}^{\cdot-r}(\mathbb{P}^n_X \hookrightarrow \mathbb{P}^n_M)$$

and the pushforward map

$$\widehat{H}^{\cdot}(X \hookrightarrow \mathbb{P}^n_M) \to \widehat{H}^{\cdot}(\mathbb{P}^n_X \hookrightarrow \mathbb{P}^n_M).$$

Let $(L, s_0, \dots, s_n)$ be the line bundle with sections on X defining the imbedding $s : X \hookrightarrow \mathbb{P}^n_X$, and let $\mathscr{J}_s \subset \mathscr{O}_{\mathbb{P}^n_X}$ denote the ideal sheaf defining s, so we have an exact sequence

$$0 \to \mathscr{J}_s \to \mathscr{O}_{\mathbb{P}^n_X} \to s_*\mathscr{O}_X \to 0.$$

Tensoring this sequence with $\mathscr{O}_{\mathbb{P}^n_X}(1)$ and applying R^0p_{X*} we get an exact sequence

$$0 \to R^0p_{X*}\mathscr{J}_s(1) \to R^0p_{X*}\mathscr{O}_{\mathbb{P}^n_X}(1) \to L.$$

Under the canonical isomorphism $\mathscr{O}_X^{n+1} \to R^0p_{X*}\mathscr{O}_{\mathbb{P}^n_X}(1)$, the map on the right is the map defining by the sections $(s_0, \dots, s_n)$ and in particular is surjective. Let $\mathscr{E}$ denote the sheaf $R^0p_{X*}\mathscr{J}_s(1)$, so $\mathscr{E}$ is a locally free sheaf of rank n on X which sits in an exact sequence

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{O}_X^{n+1} \xrightarrow{(s_0,\dots,s_n)} L \longrightarrow 0.$$

Observe that the formation of this sheaf $\mathscr{E}$ commutes with base change $X' \to X$. Also if

$$0 \to \mathscr{K} \to \mathscr{O}^{n+1}_{\mathbb{P}^n_X} \to \mathscr{O}_{\mathbb{P}^n_X}(1) \to 0$$

denotes the universal exact sequence over $\mathbb{P}^n_X$, then we get an isomorphism

$$\mathscr{E} \simeq s^*\mathscr{K}.$$

Since $T_{\mathbb{P}^n_X/X} \simeq \mathscr{K}^\vee(1)$, this yields an isomorphism

(3.27.1) $$\mathscr{E}^\vee \otimes L \simeq s^*T_{\mathbb{P}^n_X/X}.$$

Lemma 3.28. *The adjoint map $p_X^*\mathscr{E}(-1) \to \mathscr{J}_s$ is surjective.*

Proof. It suffices to consider the case when $X = \mathrm{Spec}(k)$ is a point. In this case the section s is a k-point of $\mathbb{P}^n_k$, say $s = [s_0 : \cdots : s_n]$. Then

$$\mathscr{E} \subset \oplus_{i=0}^n k \cdot X_i$$

is the subspace spanned by the linear forms $s_j X_i - s_i X_j$, which are precisely the equations defininig s. $\square$

Lemma 3.29. *The diagram*

$$\begin{tikzcd} \widehat{H}^{\cdot -n}(X \hookrightarrow M) \arrow[r, "p^*"] & \widehat{H}^{\cdot -n}(\mathbb{P}^n_X \hookrightarrow \mathbb{P}^n_M) \arrow[d, "\cdot c_n(p_X^*\mathscr{E}^\vee(1))"] \\ \widehat{H}^{\cdot}(X \hookrightarrow \mathbb{P}^n_M) \arrow[u, "(3.26.1)"] \arrow[r, "s_*"] & \widehat{H}^{\cdot}(\mathbb{P}^n_X \hookrightarrow \mathbb{P}^n_M) \end{tikzcd}$$

commutes.

Proof. The composite map

$$\widehat{H}^{\cdot -n}(X \hookrightarrow M) \xrightarrow{(3.26.1)^{-1}} \widehat{H}^{\cdot}(X \hookrightarrow \mathbb{P}^n_M) \xrightarrow{s_*} \widehat{H}^{\cdot}(\mathbb{P}^n_X \hookrightarrow \mathbb{P}^n_M) \tag{3.29.1}$$

is obtained as follows. The isomorphism $p_X^!\Lambda \simeq \Lambda_{\mathbb{P}^n_X}(n)[2n]$ defines a map

$$s_*\Lambda_X = s_*s^!p_X^!\Lambda \to p_X^!\Lambda \simeq \Lambda_{\mathbb{P}^n_X}(n)[2n]. \tag{3.29.2}$$

Tensoring this map with $p_X^*i^!\Lambda_M \simeq \tilde{i}^!\Lambda_{\mathbb{P}^n_M}$ we obtain a morphism

$$s_*i^!\Lambda_M \to \tilde{i}^!\Lambda_{\mathbb{P}^n_M}(n)[2n]$$

which upon taking cohomology gives the map (3.29.1). To prove the lemma, it therefore suffices to show that the map $\Lambda_X \to p_{X*}\Lambda_{\mathbb{P}^n_X}(n)[2n]$ obtained by applying p_{X*} to (3.29.2) is equal to the map defined by $c_n(p_X^*\mathscr{E}^\vee(1))$.

To see this it suffices to consider the universal case

$$\begin{tikzcd} \mathbb{P}^n_k \times \mathbb{P}^n_k \arrow[d, "\mathrm{pr}_1"] \\ \mathbb{P}^n_k, \arrow[u, bend left, hook, "\Delta"] \end{tikzcd}$$

where Δ denotes the diagonal morphism. The map $\Lambda_{\mathbb{P}^n} \to \mathrm{pr}_{1*}\Lambda_{\mathbb{P}^n\times\mathbb{P}^n}(n)[2n]$ is determined by the corresponding global class in $H^{2n}(\mathbb{P}^n \times \mathbb{P}^n, \Lambda(n))$ which by construction is simply the cohomology class of the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$. We show that this cohomology class is equal to $c_n(\mathrm{pr}_1^*\mathscr{E}^\vee(1))$.

We calculate this Chern class as follows. The sheaf $\mathscr{E}$ is the kernel of the universal surjection $\mathscr{O}_{\mathbb{P}^n}^{n+1} \to \mathscr{O}_{\mathbb{P}^n}(1)$. We therefore have an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1) \to \mathscr{O}_{\mathbb{P}^n}^{n+1} \to \mathscr{E}^\vee \to 0.$$

Pulling back this sequence along pr_1, and tensoring with $\mathrm{pr}_2^*\mathscr{O}_{\mathbb{P}^n}(1)$ we get an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^n\times\mathbb{P}^n}(-1,1) \to \mathscr{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,1)^{n+1} \to \mathrm{pr}_1^*\mathscr{E}^\vee(0,1) \to 0,$$

where for integers i and j we write $\mathscr{O}_{\mathbb{P}^n\times\mathbb{P}^n}(i,j)$ for $\mathrm{pr}_1^*\mathscr{O}_{\mathbb{P}^n}(i)\otimes\mathrm{pr}_2^*\mathscr{O}_{\mathbb{P}^n}(j)$. Using the multiplicativity of the total Chern classes we get that in (where α_i is the first Chern class of $\mathrm{pr}_i^*\mathscr{O}_{\mathbb{P}^n}(1)$)

$$\oplus_i H^{2i}(\mathbb{P}^n\times\mathbb{P}^n,\Lambda(i)) = \Lambda[\alpha_1,\alpha_2]/(\alpha_1^{n+1},\alpha_2^{n+1})$$

we have

$$c.(\mathrm{pr}_1^*\mathscr{E}^\vee(0,1))\cdot(1-(\alpha_1-\alpha_2)) = (1+\alpha_2)^{n+1}.$$

By part (ii) of the following elementary result 3.30, it follows that

$$c_n(\mathrm{pr}_1^*\mathscr{E}^\vee(0,1)) = \sum_{i=0}^n \alpha_1^i\alpha_2^{n-i},$$

which indeed is the cohomology class of the diagonal. □

Sub-Lemma 3.30. *Fix a positive integer n, and another integer i in $[0,n]$.*

(i)

$$\sum_{j=i}^n \binom{j}{i} = \binom{n+1}{i+1}.$$

(ii) The coefficient of $\alpha^i t^{n-i}$ in the expansion in $\mathbb{Z}[[t,\alpha]]$ of

$$(1+t)^{n+1}(1+(\alpha-t)+(\alpha-t)^2+(\alpha-t)^3+\cdots)$$

is equal to 1.

Proof. For (i) we proceed by induction on n. The case $n=1$ is immediate. For the inductive step, note that the result is clearly true for $i=n$. It therefore suffices to show that the result holds for $i<n$ assuming the result for $n-1$. Breaking up the sum on the left as

$$\left(\sum_{j=i}^{n-1}\binom{j}{i}\right)+\binom{n}{i}$$

and using the induction, we get that the sum is equal to

$$\binom{n}{i+1}+\binom{n}{i} = \binom{n+1}{i+1}.$$

To prove (ii), it suffices to prove the result after replacing $\mathbb{Z}$ by $\mathbb{R}$. Also note that if we write $z=(\alpha-t)$ and use (i), then we get

$$t^n+\alpha t^{n-1}+\alpha^2t^{n-2}+\cdots+\alpha^n = \sum_{i=0}^n\binom{n+1}{i+1}z^it^{n-i}.$$

Set

$$F(t,z) := \frac{(1+t)^{n+1}}{(1-z)}.$$

Using Taylor's theorem, to prove (ii) it then suffices to show that the value of $\frac{\partial^n F}{\partial z^i\partial t^{n-i}}$ at $t=z=0$ is equal to $(n+1)!/(i+1)$ which is an immediate calculation. □

To complete the proof of 3.26, we now compute the class of $s_*\mathscr{O}_X$ in $\widehat{H}^\cdot(\mathbb{P}^n_X \hookrightarrow \mathbb{P}^n_M)$. For this note that by 3.28 we have an exact sequence

$$p_X^*\mathscr{E}(-1) \to \mathscr{O}_{\mathbb{P}^n_X} \to s_*\mathscr{O}_X \to 0.$$

Taking the associated Koszul complex we get a resolution on $\mathbb{P}^n_X$ of the form

$$0 \to \bigwedge^n(p_X^*\mathscr{E}(-1)) \to \bigwedge^{n-1}(p_X^*\mathscr{E}(-1)) \to \cdots \to (p_X^*\mathscr{E}(-1)) \to \mathscr{O}_{\mathbb{P}^n_X} \to s_*\mathscr{O}_X \to 0.$$

Tensoring this complex with p_X^*K, we find that in $\widehat{H}^\cdot(\mathbb{P}^n_X \hookrightarrow \mathbb{P}^n_M)$ we have

$$\mathrm{ch}^{\mathbb{P}^n_X}_{\mathbb{P}^n_M}(s_*K) = (p^*\mathrm{ch}^X_M(K)) \cdot \sum_{i=0}^n (-1)^i(\mathrm{ch}(\bigwedge^i(p_X^*\mathscr{E}(-1)))),$$

where $\mathrm{ch}(\bigwedge^i(p_X^*\mathscr{E}(-1)))$ denotes the usual unlocalized Chern character of $\bigwedge^i(p_X^*\mathscr{E}(-1))$. Combining this with the basic formula [10, 3.2.5]

$$\sum_{i=0}^n (-1)^i(\mathrm{ch}(\bigwedge^i(p_X^*\mathscr{E}(-1)))) = c_n(p_X^*\mathscr{E}^\vee(1))\mathrm{td}(p_X^*\mathscr{E}^\vee(1))^{-1},$$

we get that in $\widehat{H}^\cdot(\mathbb{P}^n_X \hookrightarrow \mathbb{P}^n_M)$ we have

$$\begin{aligned} s_*\mathrm{ch}^X_{\mathbb{P}^n_M}(K) &= \mathrm{ch}^{\mathbb{P}^n_X}_{\mathbb{P}^n_M}(s_*K) \\ &= (p^*\mathrm{ch}^X_M(K)) \cdot c_n(p_X^*\mathscr{E}^\vee(1)) \cdot \mathrm{td}(p_X^*\mathscr{E}^\vee(1))^{-1} \\ &= p_X^*(\mathrm{ch}^X_M(K) \cdot \mathrm{td}(\mathscr{E}^\vee \otimes L)^{-1}) \cdot c_n(p_X^*\mathscr{E}^\vee(1)) \\ &= p_X^*(\mathrm{ch}^X_M(K) \cdot \mathrm{td}(s^*T_{\mathbb{P}^n_X/X})^{-1}) \cdot c_n(p_X^*\mathscr{E}^\vee(1)), \end{aligned}$$

where the last equality uses (3.27.1). This implies 3.26 in light of 3.27 and 3.29. □

4. The transformation τ^X_Y and Riemann-Roch

Throughout this section we work over an algebraically closed field k, and $\mathbb{Q}_\ell$-coefficients for a prime ℓ invertible in k.

4.1. **Definition of τ^X_Y.**

4.2. Let $f : X \to Y$ be a morphism of quasi-projective finite group quotients over k , and choose a factorization

(4.2.1)
$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & M \arrow[d, "g"] \\ & Y, \end{tikzcd}$$

with i a closed imbedding, M a quasi-projective finite group quotient, and g smooth. We then have $g^!\Lambda_Y(i) \simeq \Lambda_M(i+r)[2r]$, where r is the relative dimension of M over Y (a locally constant function on M). Therefore there is a canonical isomorphism

(4.2.2)
$$\widehat{H}^\cdot(f : X \to Y) \simeq \widehat{H}^{\cdot+r}(X \hookrightarrow M).$$

Define

$$\tau^X_Y : K(f\text{-perfect complexes on } X) \to \widehat{H}^\cdot(f : X \to Y)$$

by sending the class of a complex K to the image of

$$\mathrm{ch}_M^X(K) \cdot \mathrm{td}(i^*T_{M/Y}) \in \widehat{H}^{\cdot}(X \hookrightarrow M)$$

under the isomorphism (4.2.2).

Proposition 4.3. *The map τ_Y^X is independent of the choice of factorization of f.*

Proof. Given two factorizations ($i = 1, 2$)

$$\begin{tikzcd} X \arrow[r, hook, "j_i"] \arrow[dr, "f"'] & M_i \arrow[d, "g_i"] \\ & Y, \end{tikzcd}$$

we can form a commutative diagram as in A.8. Using this it follows that it suffices to prove that for a commutative diagram

$$\begin{tikzcd} & M' \arrow[d, "h"] \arrow[dd, bend left=60, "g'"] \\ X \arrow[ur, hook, "j'"] \arrow[r, hook, "j"] \arrow[dr, "f"'] & M \arrow[d, "g_i"] \\ & Y, \end{tikzcd}$$

with g, g', and h smooth and j and j' closed imbeddings, the images in $\widehat{H}^{\cdot}(X \to Y)$ of

$$\mathrm{ch}_M^X(K) \cdot \mathrm{td}(j^*T_{M/Y}), \quad \text{and} \quad \mathrm{ch}_{M'}^X(K) \cdot \mathrm{td}(j'^*T_{M'/Y})$$

are equal. By 3.26 the image of $\mathrm{ch}_{M'}^X(K)$ is equal to the image of $\mathrm{ch}_M^X(K)$ multiplied by $\mathrm{td}(T_{M'/M})^{-1}$. The independence of factorization therefore follows from the equality

$$\mathrm{td}(j'^*T_{M'/Y}) = \mathrm{td}(j^*T_{M/Y}) \cdot \mathrm{td}(j'^*T_{M'/M}),$$

which follows from the additivity of the Todd class applied to the exact sequence

$$0 \to j'^*T_{M'/M} \to j'^*T_{M'/Y} \to j^*T_{M/Y} \to 0.$$

□

4.4. **Some remarks on equivariant Grothendieck groups.**

4.5. Let X be a quasi-projective finite group quotient over our ground field k, let $K^0(X)$ (resp. $K_0(X)$) denote the Grothendieck group of locally free sheaves of finite rank (resp. coherent sheaves) on X. So $K^0(X)$ is a ring, and $K_0(X)$ is a module over $K^0(X)$. Let $X = [U/G]$ be a presentation of X with U a quasi-projective k-scheme and G a finite group acting on X, and let $g : X \to BG$ be the corresponding map. Let $\alpha \in K^0(X)$ denote the class obtained by pulling back the locally free sheaf on BG defined by the regular representation $\mathscr{O}_G$ of G. Set

$$\overline{K}^0(X) := K^0(X)/(\alpha - |G|), \quad \overline{K}_0(X) := K_0(X) \otimes_{K^0(X)} \overline{K}^0(X).$$

4.6. By [25, 2.2] there is a Riemann-Roch transformation (a morphism of $K^0(X)$-modules)

$$\tau_X : K_0(X) \to A_*(X)_{\mathbb{Q}}$$

which by [25, 2.5] induces an isomorphism

$$\bar{\tau}_X : \overline{K}_0(X)_{\mathbb{Q}} \simeq A_*(X)_{\mathbb{Q}}.$$

Note that this implies in particular that the quotient $K_0(X)_{\mathbb{Q}} \to \overline{K}_0(X)_{\mathbb{Q}}$ is independent of the choice of presentation of X.

Remark 4.7. The fact that the kernel of $\tau_X : K_0(X)_{\mathbb{Q}} \to A_*(X)_{\mathbb{Q}}$ is generated by $\alpha - |G|$ can be seen directly as follows. It suffices to show that if $\xi \in K_0(X)_{\mathbb{Q}}$ and $\tau_X(\xi) = 0$, then $\alpha \cdot \xi = 0$ in $K_0(X)_{\mathbb{Q}}$ (for then $\xi = \frac{1}{|G|}\xi \cdot (\alpha - |G|)$). Consider the projection $\pi : U \to X$ which exhibits U as a G-torsor over X. The composite morphism

$$A_*(X)_{\mathbb{Q}} \xrightarrow{\pi^*} A_*(U)_{\mathbb{Q}} \xrightarrow{\pi_*} A_*(X)_{\mathbb{Q}}$$

is equal to multiplication by $|G|$. In particular, the pullback map $A_*(X)_{\mathbb{Q}} \to A_*(U)_{\mathbb{Q}}$ is injective. Therefore if $\xi_U \in K_0(U)_{\mathbb{Q}}$ is the pullback of ξ then $\tau_U(\xi_U) = 0$. Now U is a quasi-projective scheme, so τ_U is an isomorphism whence $\xi_U = 0$. We conclude that

$$\alpha \cdot \xi = \pi_*(\xi_U) = 0.$$

4.8. If $f : X' \to X$ is a proper representable morphism and if U' denotes the fiber product $U \times_X X'$ so $X' = [U'/G]$, then proper pushforward defined a morphism of $K^0(BG)$-modules

$$Rf_* : K_0(X') \to K_0(X),$$

and therefore also a morphism $\overline{K}_0(X')_{\mathbb{Q}} \to \overline{K}_0(X)_{\mathbb{Q}}$, which we again denote by Rf_*.

Proposition 4.9. *There exists a projective mophism $f : X' \to X$ with X' smooth such that the map*

$$Rf_* : \overline{K}_0(X')_{\mathbb{Q}} \to \overline{K}_0(X)_{\mathbb{Q}}$$

is surjective.

Proof. Since the Riemann-Roch transformation is compatible with proper pushforward [24, 2.2], it suffices to prove the analogous result for $A_*(-)$ in place of $\overline{K}_0(-)$.

Furthermore we may assume that X is reduced.

By [4, 7.3] there exists a projective surjective generically finite morphism $p : Y \to X$ with Y smooth. Let $U \subset X$ be a dense open subset over which this morphism is finite and flat, and let Y_U be the preimage in Y. If $Z \subset X$ (resp. $Y_Z \subset Y$) denotes the complement of U (resp. Y_U) then we have a commutative diagram with exact rows (see [10, Proposition 1.8] in the case of schemes)

$$\begin{tikzcd}
A_*(Y_Z)_{\mathbb{Q}} \arrow[r] \arrow[d, "Rp_*"] & A_*(Y)_{\mathbb{Q}} \arrow[r] \arrow[d, "Rp_*"] & A_*(Y_U)_{\mathbb{Q}} \arrow[r] \arrow[d, "Rp_*"] & 0, \\
A_*(Z)_{\mathbb{Q}} \arrow[r] & A_*(X)_{\mathbb{Q}} \arrow[r] & A_*(U)_{\mathbb{Q}} \arrow[r] & 0,
\end{tikzcd}$$

where the right vertical morphism is surjective as $Rp_* \circ p^* = \deg(Y_U/U)$. By noetherian induction we may assume that there exists a projective morphism $h : Z' \to Z$ such that

$A_*(Z')_{\mathbb{Q}} \to A_*(Z)_{\mathbb{Q}}$ is surjective. Setting $X' := Z' \coprod Y$ we then get the desired projective morphism $X' \to X$. $\square$

Remark 4.10. In fact we can even find a representation V of G such that if $\mathscr{V}$ denotes the corresponding locally free sheaf on X then we have an imbedding $X' \hookrightarrow \mathbb{P}\mathscr{V}$ over X. To see this let X'_U denote the fiber product $X' \times_X U$, so we have an action of G on X'_U over the action on U and such that $X' = [X'_U/G]$. Fix an imbedding $X'_U \hookrightarrow X \times_{\mathrm{Spec}(k)} \mathbb{P}W$ for a k-vector space W, and let $q : W \otimes_k \mathscr{O}_{X'_U} \to L$ be the surjection onto a line bundle corresponding to this imbedding. Let V denote the G-representation $\bigotimes_{g \in G} W$, where $h \in G$ acts by sending a tensor $\otimes_{g \in G} w_g$ to $\otimes_{g \in G} w_{hg}$. Similarly let M denote the G-equivariant sheaf on X' given by $\bigotimes_{g \in G} g^* L$. The map q induces a surjection of G-equivariant sheaves

$$V \otimes_k \mathscr{O}_{X'_U} \twoheadrightarrow M$$

defining a G-equivariant imbedding $X'_U \hookrightarrow U \times \mathbb{P}(V)$. Passing to the quotient by the G-action gives the claim.

Lemma 4.11. *Let* $\mathrm{ch} : K^0(BG) \to H^*(BG, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ *be the Chern character. Then* $\mathrm{ch}(\alpha) = |G|$.

Proof. Let $\mathrm{Spec}(k) \to BG$ be the morphism defined by the trivial torsor. Then the pullback map $H^*(BG, \mathbb{Q}_\ell) \to H^*(\mathrm{Spec}(k), \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ is an isomorphism and is the identity map on H^0. By compatibility of Chern classes with pullback it therefore suffices to prove the result over $\mathrm{Spec}(k)$ where the result is immediate. $\square$

Lemma 4.12. *Let* $X = [U/G]$ *as above, and let* $\pi : U \to X$ *be the projection. Then the pullback map*

$$\pi^* : K_0(X)_{\mathbb{Q}} \to K_0(U)_{\mathbb{Q}}$$

induces an isomorphism

$$\overline{K}_0(X)_{\mathbb{Q}} \to K_0(U)^G_{\mathbb{Q}}.$$

Proof. The composite map

$$K_0(U)_{\mathbb{Q}} \xrightarrow{\pi_*} K_0(X)_{\mathbb{Q}} \xrightarrow{\pi^*} K_0(U)_{\mathbb{Q}}$$

sends the class of a coherent sheaf F to the class of the sum $\oplus_{g \in G} g^* F$. From this it follows that the map π^* surjects onto $K_0(U)^G_{\mathbb{Q}}$, and that π^* factors through $\overline{K}_0(X)_{\mathbb{Q}}$. Since the composite map

$$\overline{K}_0(X)_{\mathbb{Q}} \xrightarrow{\pi^*} K_0(U)_{\mathbb{Q}} \xrightarrow{\pi_*} \overline{K}_0(X)_{\mathbb{Q}}$$

is equal to multiplication by $|G|$ (as can be seen by looking at the corresponding diagram of Chow groups), it follows that π^* induces an injective map on $\overline{K}_0(X)_{\mathbb{Q}}$. $\square$

Proposition 4.13. *Let* E *be a locally free sheaf of finite rank on* X. *Then* $\overline{K}_0(\mathbb{P}(E))_{\mathbb{Q}}$ *is generated as an abelian group by the classes of sheaves* $p^* F(r)$, *where* $p : \mathbb{P}(E) \to X$ *is the projection,* F *is a coherent sheaf on* X, *and* r *is an integer.*

Proof. In the case when X is a quasi-projective scheme the result follows from [10, Theorem 3.3 and 18.3.2].

For the general case, write $X = [U/G]$ as above, and let E_U denote the pullback of E to U so that $\mathbb{P}(E) = [\mathbb{P}_U E_U/G]$. Let $\xi \in \overline{K}_0(\mathbb{P}(E))$ be a class with pullback ξ_U in $K_0(\mathbb{P}_U E_U)$. By the case of a quasi-projective scheme, we can find coherent sheaves F_i, integers a_i and n_i, such that

$$\xi_U = \sum_i a_i[p_U^* F_i(n_i)],$$

where $p_U : \mathbb{P}_U E_U \to U$ is the projection. Let F_i' denote the G-equivariant sheaf $\oplus_{g\in G} g^* F_i$. Since ξ_U is G-invariant we have

$$|G| \cdot \xi_U = \sum_i a_i[p_U^* F_i'(n_i)]$$

in $K_0(\mathbb{P}_U E_U)^G$. Therefore if $\overline{F}_i$ denotes the sheaf on X corresponding to F_i' then we have

$$|G| \cdot \xi = \sum_i a_i[p^* \overline{F}_i(n_i)].$$

□

Remark 4.14. As pointed out to us by an anonymous referee, proposition 4.13 holds more generally for $K_0(\mathbb{P}(E))$ before tensoring with $\mathbb{Q}$ and passing to $\overline{K}_0(\mathbb{P}(E))$. The referee's argument is the following. Let $\mathscr{O}_\Delta$ denote the structure sheaf of the diagonal imbedding

$$\Delta : \mathbb{P}(E) \to \mathbb{P}(E) \times_X \mathbb{P}(E),$$

and let F be a coherent sheaf on $\mathbb{P}(E)$. Then we have

$$F \simeq R\mathrm{pr}_{1*}(\mathscr{O}_\Delta \otimes^{\mathbb{L}} \mathrm{pr}_2^* F).$$

On the other hand, Beilinson defines a finite resolution of $\mathscr{O}_\Delta$ by sheaves of the form $\mathscr{O}_{\mathbb{P}(E)}(-i) \boxtimes \Omega^i_{\mathbb{P}(E)/X}(i)$ (see for example [3, §5.7]). Therefore F is contained in the subgroup of $K_0(\mathbb{P}(E))$ generated by elements of the form

$$R\mathrm{pr}_{1*}(\mathscr{O}_{\mathbb{P}(E)}(-i) \boxtimes (\Omega^i_{\mathbb{P}(E)/X}(i) \otimes F)) \simeq \mathscr{O}_{\mathbb{P}(E)}(-i) \otimes Lp^* Rp_*(\Omega^i_{\mathbb{P}(E)/X}(i) \otimes F),$$

where the isomorphism is by flat base change. The image of the right side of this isomorphism in $K_0(\mathbb{P}(E))$ is contained in the subgroup generated twists of images of $K_0(X)$, whence the result.

4.15. **Compatibility with Riemann-Roch transformation.**

We continue with the notation and setup of 4.2.

4.16. In the case when Y is smooth of dimension d, we have $f^!\Lambda \simeq \Omega_X(-d)[-2d]$, so get an isomorphism

$$\widehat{H}^\cdot(X \to Y) \simeq \widetilde{H}_{d-\cdot}(X),$$

where $\widetilde{H}_\cdot(X)$ is defined as in 2.9.

The Grothendieck group of f-perfect complexes is in this case the same as the Grothendieck group $K_0(X)$ of coherent sheaves on X. Let

$$\tau_X : K_0(X) \to (A_* X)_{\mathbb{Q}}$$

be the Riemann-Roch transformation.

Theorem 4.17. *The diagram*

$$\begin{tikzcd} K_0(X) \arrow[r, "\tau_X"] \arrow[d, "\mathrm{td}(f^*T_Y)\cdot\tau_Y^X"'] & (A_*X)_{\mathbb{Q}} \arrow[d, "\mathrm{cl}_X"] \\ \widehat{H}^*(X \to Y) \arrow[r, "\simeq"] & \widetilde{H}_{d-*}(X) \end{tikzcd}$$

commutes.

The proof occupies 4.18–4.21.

4.18. First we reduce to the case when $f : X \to Y$ is a closed imbedding. For this choose a factorization

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr] & M \arrow[d, "g"] \\ & Y, \end{tikzcd}$$

with i a closed imbedding and g smooth. We then get a diagram

$$\begin{tikzcd} & A_*(X)_{\mathbb{Q}} \arrow[dr] & \\ K_0(X) \arrow[ur] \arrow[r, "\mathrm{td}(T_Y)\tau_Y^X"] \arrow[dr, "\mathrm{td}(T_M)\tau_M^X"'] & \widehat{H}^{\cdot}(X \to Y) \arrow[r] \arrow[d, "\simeq"] & \widetilde{H}_*(X) \\ & \widehat{H}^{\cdot}(X \hookrightarrow M), \arrow[ur] & \end{tikzcd}$$

where the two bottom triangles commute (and where we omit the shifting in degree to ease the notation). From this it follows that the theorem holds for $X \to Y$ if and only if it holds for $X \hookrightarrow M$. In particular, it suffices to prove the result in the case when $Y = \mathrm{Spec}(k)$.

Fix a presentation $X = [U/G]$ with U a quasi-projective k-scheme and G a finite group acting on U.

Lemma 4.19. *The composite map*

$$K_0(X)_{\mathbb{Q}} \xrightarrow{\tau_k^X} \widehat{H}^{\cdot}(X \to \mathrm{Spec}(k)) \xrightarrow{\simeq} \widetilde{H}_*(X)$$

factors through $\overline{K}_0(X)_{\mathbb{Q}}$.

Proof. Choosing an ample G-linearized invertible sheaf on U we can find a representation V of G over k and a G-equivariant imbedding $U \hookrightarrow \mathbb{P}(V)$. Let $W \subset \mathbb{P}(V)$ be a G-equivariant open subset such that $U \hookrightarrow \mathbb{P}(V)$ factors through a closed imbedding $U \hookrightarrow W$ and let M be the quotient $[W/G]$ so we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "g"'] & M \arrow[dl, "h"'] \\ BG. & \end{tikzcd}$$

If R denotes the sheaf on BG corresponding to the regular representation of G, then α is the class of g^*R. We therefore have (using the multiplicative property of local chern classes [15, 1.8])

$$\mathrm{ch}_M^X(g^*R) = \mathrm{ch}(R)\cdot \mathrm{ch}_M^X(\mathscr{O}_X),$$

and by 4.11 we have $\mathrm{ch}(R) = |G|$. This implies the result. □

Proposition 4.20. *Let $p : X' \to X$ be a projective morphism. Then the diagram*

(4.20.1)
$$\begin{tikzcd} \overline{K}_0(X') \arrow[r, "\tau_k^{X'}"] \arrow[d, "Rp_*"] & \widehat{H}^{\cdot}(X' \to \mathrm{Spec}(k)) \arrow[r, "\simeq"] & \widetilde{H}_*(X') \arrow[d, "p_*^{\mathrm{hom}}"] \\ \overline{K}_0(X) \arrow[r, "\tau_k^{X}"] & \widehat{H}^{\cdot}(X \to \mathrm{Spec}(k)) \arrow[r, "\simeq"] & \widetilde{H}_*(X) \end{tikzcd}$$

commutes.

Proof. Write $X = [U/G]$ as before, and choose a representation V of G with associated locally free sheaf $\mathscr{V}$ on X, and a closed imbedding $i : X' \hookrightarrow \mathbb{P}\mathscr{V}$ for some n, and let $q : \mathbb{P}\mathscr{V} \to X$ be the projection. We then get a diagram

$$\begin{tikzcd} \overline{K}_0(X') \arrow[r, "\tau_k^{X'}"] \arrow[d, "i_*"] \arrow[dd, bend right, "Rp_*"'] & \widehat{H}^{\cdot}(X' \to \mathrm{Spec}(k)) \arrow[r, "\simeq"] & \widetilde{H}_*(X') \arrow[d, "i_*^{\mathrm{hom}}"] \arrow[dd, bend left, "p_*^{\mathrm{hom}}"] \\ \overline{K}_0(\mathbb{P}\mathscr{V}) \arrow[r, "\tau_k^{\mathbb{P}\mathscr{V}}"] \arrow[d, "Rq_*"] & \widehat{H}^{\cdot}(\mathbb{P}\mathscr{V} \to \mathrm{Spec}(k)) \arrow[r, "\simeq"] & \widetilde{H}_*(\mathbb{P}\mathscr{V}) \arrow[d, "q_*^{\mathrm{hom}}"] \\ \overline{K}_0(X) \arrow[r, "\tau_k^{X}"] & \widehat{H}^{\cdot}(X \to \mathrm{Spec}(k)) \arrow[r, "\simeq"] & \widetilde{H}_*(X), \end{tikzcd}$$

where the top inside rectangle commutes by the definition of $\tau_k^{X'}$, and the right outside diagram commutes by associativity of the proper pushforward map.

From this it follows that it suffices to consider the case when $X' = \mathbb{P}\mathscr{V}$.

Since the pullback map $\widetilde{H}_*(X) \to \widetilde{H}_*(U)$ is injective, the commutativity of (4.20.1) can be verified after pulling back to U, so we may even assume that $X' = \mathbb{P}^n_X$ for some n (and that X is a scheme, though this is not necessary).

Now to verify that

(4.20.2)
$$\begin{tikzcd} \overline{K}_0(\mathbb{P}^n_X) \arrow[r, "\tau_k^{\mathbb{P}^n_X}"] \arrow[d, "Rq_*"] & \widehat{H}^{\cdot}(\mathbb{P}^n_X \to \mathrm{Spec}(k)) \arrow[r, "\simeq"] & \widetilde{H}_*(\mathbb{P}^n_X) \arrow[d, "q_*^{\mathrm{hom}}"] \\ \overline{K}_0(X) \arrow[r, "\tau_k^{X}"] & \widehat{H}^{\cdot}(X \to \mathrm{Spec}(k)) \arrow[r, "\simeq"] & \widetilde{H}_*(X), \end{tikzcd}$$

commutes, it suffices to look at generators of $\overline{K}_0(\mathbb{P}^n_X)$ so by 4.13 it suffices to consider the class of a sheaf $q^*F(r)$ for F coherent on X and r an integer. By construction of the local

Chern classes the image of $[q^*F(r)]$ under the composite map

$$\overline{K}_0(\mathbb{P}^n_X) \xrightarrow{\tau_k^{\mathbb{P}^n_X}} \widehat{H}^{\cdot}(\mathbb{P}^n_X \to \mathrm{Spec}(k)) \xrightarrow{\simeq} \widetilde{H}_*(\mathbb{P}^n_X) \xrightarrow{\simeq} \widetilde{H}_*(X) \otimes \widetilde{H}_*(\mathbb{P}^n_k)$$

is the product of the image of $[F] \in \overline{K}_0(X)$ under the map

$$\overline{K}_0(X) \xrightarrow{\tau_k^X} \widehat{H}^{\cdot}(X \to \mathrm{Spec}(k)) \xrightarrow{\simeq} \widetilde{H}_*(X)$$

and $\mathrm{td}(T_{\mathbb{P}^n_k/k}) \cdot \mathrm{ch}(\mathscr{O}_{\mathbb{P}^n_k}(r))$ in $H^*(\mathbb{P}^n_k, \mathbb{Q}_\ell) \simeq \widetilde{H}_*(\mathbb{P}^n_k)$. To verify that (4.20.2) commutes it therefore suffices to show that

$$\sum_i (-1)^i \dim H^i(\mathbb{P}^n_k, \mathscr{O}_{\mathbb{P}^n_k}(r)) = \int \mathrm{td}(T_{\mathbb{P}^n_k/k}) \cdot \mathrm{ch}(\mathscr{O}_{\mathbb{P}^n_k}(r))$$

which follows from the usual Grothendieck-Riemann-Roch theorem. □

4.21. To prove 4.17 in the case when $Y = \mathrm{Spec}(k)$, choose using 4.9 a projective surjection $p : X' \to X$ such that the induced map $\overline{K}_0(X')_{\mathbb{Q}} \to \overline{K}_0(X)_{\mathbb{Q}}$ is surjective and X' is smooth. We claim that if 4.17 holds for X' then it also holds for X. Indeed if $\omega \in \overline{K}_0(X)_{\mathbb{Q}}$ is a class choose a lifting $\omega' \in \overline{K}_0(X')_{\mathbb{Q}}$. Let $\gamma \in \widetilde{H}_*(X)$ (resp. $\gamma' \in \widetilde{H}_*(X')$) be the image of ω (resp. ω') under the map τ_k^X (resp. $\tau_k^{X'}$) composed with the isomorphism $\widehat{H}^{\cdot}(X \to \mathrm{Spec}(k)) \simeq \widetilde{H}_*(X)$ (resp. $\widehat{H}^{\cdot}(X' \to \mathrm{Spec}(k)) \simeq \widetilde{H}_*(X')$). Then by 4.20 we have $\gamma = p_*^{\mathrm{hom}}(\gamma')$. Since the diagram

$$\begin{tikzcd}
\overline{K}_0(X')_{\mathbb{Q}} \arrow[r, "\tau_{X'}"] \arrow[d, "Rp_*"] & (A_*X')_{\mathbb{Q}} \arrow[r, "\mathrm{cl}_{X'}"] \arrow[d, "p_*"] & \widetilde{H}_*(X') \arrow[d, "p_*^{\mathrm{hom}}"] \\
\overline{K}_0(X)_{\mathbb{Q}} \arrow[r, "\tau_X"] & (A_*X)_{\mathbb{Q}} \arrow[r, "\mathrm{cl}_X"] & \widetilde{H}_*(X)
\end{tikzcd}$$

commutes, if $\gamma' = \mathrm{cl}_{X'}(\tau_{X'}(\omega'))$, then we find that $\gamma = \mathrm{cl}_X(\tau_X(\omega))$. This therefore further reduces the proof of 4.17 to the case when X is smooth. In this case the result follows from the compatibility of the cycle class map with Chern classes and 3.25. □

4.22. **Compatibility with bivariant Chern classes.**

4.23. Let $f : X \to Y$ be a morphism quasi-projective finite group quotients and let $\mathcal{G}^{\cdot}$ be an f-perfect complex on X. We then get for every s a bivariant class $\tau^{FM}(\mathcal{G}^{\cdot})^s \in A^s(X \to Y)_{\mathbb{Q}}$, defined as in [10, p. 366]. Recall that this bivariant class is a rule which to any $Y' \to Y$ associates a map

$$\tau^{FM}(\mathcal{G}^{\cdot})^s \cap (-) : A_j(Y')_{\mathbb{Q}} \to A_{j-s}(X \times_Y Y')_{\mathbb{Q}},$$

and these maps have to satisfy various properties and compatibilities.

We can also consider the class $\tau_Y^X(\mathcal{G}^{\cdot})^s \in \widehat{H}^s(f : X \to Y)$. From this class and the map

$$f^*\Omega_Y \otimes f^!\Lambda_Y \to \Omega_X$$

induced by evaluation we get a map

$$\tau_Y^X(\mathcal{G}^{\cdot})^s \cap_{\mathrm{hom}} (-) : \widetilde{H}_j(Y) \to \widetilde{H}_{j-s}(X).$$

Proposition 4.24. *For every $\alpha \in A_j(Y)$ we have*

$$\mathrm{cl}_X(\tau^{\mathrm{FM}}(\mathcal{G}^\cdot)^s \cap \alpha) = \tau_Y^X(\mathcal{G}^\cdot)^s \cap_{\mathrm{hom}} (\mathrm{cl}_Y(\alpha)).$$

The proof occupies 4.25–4.27.

4.25. Let $q : Y' \to Y$ be a proper morphism and consider the resulting cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "p"] & Y' \arrow[d, "q"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{4.25.1}$$

Let $\mathcal{G}'^\cdot$ denote f'-perfect complex obtained by pullback from $\mathcal{G}^\cdot$.

Lemma 4.26. *The diagram*

$$\begin{tikzcd} \widetilde{H}_j(Y') \arrow[r, "\tau_{Y'}^{X'}(\mathcal{G}'^\cdot)^s \cap_{\mathrm{hom}}(-)"] \arrow[d, "q_{*\mathrm{hom}}"] & \widetilde{H}_{j-s}(X') \arrow[d, "p_{*\mathrm{hom}}"] \\ \widetilde{H}_j(Y) \arrow[r, "\tau_Y^X(\mathcal{G}^\cdot)^s \cap_{\mathrm{hom}}(-)"] & \widetilde{H}_{j-s}(X) \end{tikzcd}$$

commutes.

Proof. Let $\mathrm{tr}_q : q_*\Omega_{Y'} \to \Omega_Y$ (resp. $\mathrm{tr}_p : p_*\Omega_{X'} \to \Omega_X$) be the map inducing the proper pushforward. It then suffices to show that the diagram

$$\begin{tikzcd} f^!\Lambda \otimes f^*q_*\Omega_{Y'} \arrow[r] \arrow[d, "\mathrm{id}\otimes\mathrm{tr}_q"] & f^!\Lambda \otimes p_*f'^*\Omega_{Y'} \arrow[r] & p_*(f'^!\Lambda \otimes f'^*\Omega_{Y'}) \arrow[d, "\mathrm{ev}"] \\ f^!\Lambda \otimes f^*\Omega_Y \arrow[r, "\mathrm{ev}"] & \Omega_X & p_*\Omega_X \arrow[l, "\mathrm{tr}_p"'] \end{tikzcd}$$

commutes, where the top horizontal arrows are the base change morphisms. The Verdier dual of this diagram is given by

$$\begin{tikzcd} \mathscr{R}Hom(f^!\Lambda, f^!q_*\Lambda) & \mathscr{R}Hom(f^!\Lambda, p_*f'^!\Lambda) \arrow[l, "p_*f'^!\simeq f^!q_*"'] & p_*\mathscr{R}Hom(f'^!\Lambda, f'^!\Lambda) \arrow[l, "p^*f^!\Lambda\to f'^!\Lambda"'] \\ \mathscr{R}Hom(f^!\Lambda, f^!\Lambda) \arrow[u, "\Lambda\to q_*\Lambda"] & \Lambda \arrow[l, "\mathrm{id}"'] \arrow[r, "a"] & p_*\Lambda, \arrow[u, "b"'] \end{tikzcd}$$

where the composite map $ba : \Lambda \to p_*\mathscr{R}Hom(f'^!\Lambda, f'^!\Lambda)$ corresponds to the identity map $f'^!\Lambda \to f'^!\Lambda$. To prove the lemma it therefore suffices to note that the base change morphism

$$p^*f^!\Lambda \to f'^!\Lambda$$

can be described as the map induced by adjunction from the composite map

$$f^!\Lambda \xrightarrow{\Lambda\to q_*\Lambda} f^!q_*\Lambda \xrightarrow{f^!q_*\simeq p_*f'^!} p_*f'^!\Lambda.$$

□

4.27. The group $A^*(Y)_{\mathbb{Q}}$ is generated by classes of the form $q_*[Y']$, where $q : Y' \to Y$ is a proper morphism with Y' smooth. It therefore suffices to verify 4.24 for α of this form. Fix a morphism $q : Y' \to Y$, form the square 4.25.1, and let $\mathcal{G}'^{\cdot}$ denote the pullback of $\mathcal{G}^{\cdot}$ to X'. By the above we have

$$\tau_Y^X(\mathcal{G}^{\cdot})^s \cap_{\mathrm{hom}} (\mathrm{cl}_Y(q_*[Y'])) = p_{*\mathrm{hom}}(\tau_{Y'}^{X'}(\mathcal{G}'^{\cdot}) \cap_{\mathrm{hom}} \mathrm{cl}_{Y'}([Y'])),$$

and also by the compatibility of the cycle class map with proper pushforward we have

$$\mathrm{cl}_X(\tau^{\mathrm{FM}}(\mathcal{G}^{\cdot})^s \cap q_*[Y']) = p_{*\mathrm{hom}}\mathrm{cl}_{X'}(\tau^{\mathrm{FM}}(\mathcal{G}'^{\cdot}) \cap [Y']).$$

This reduces the proof of 4.24 to showing that

$$\mathrm{cl}_{X'}(\tau^{\mathrm{FM}}(\mathcal{G}'^{\cdot}) \cap [Y']) = \tau_{Y'}^{X'}(\mathcal{G}'^{\cdot}) \cap_{\mathrm{hom}} \mathrm{cl}_{Y'}([Y']),$$

which follows from 4.17. $\square$

5. Application to local terms

5.1. Let

$$c : C \to X \times X$$

be a correspondence, with C and X separated Deligne-Mumford stacks of finite type over k. Let F denote the fixed point stack $\mathrm{Fix}(c)$ of this correspondence, so we have a cartesian diagram

$$\begin{tikzcd} F \arrow[r, "\delta"] \arrow[d, "c'"] & C \arrow[d, "c"] \\ X \arrow[r, "\Delta"] & X \times X. \end{tikzcd}$$

The same argument as in [18, 4.5.3] (in the case of schemes this is [1, XVIII.3.1.12.2]) shows that for any $K, L \in D^b_{ctf}(X, \Lambda)$ we have a canonical isomorphism

$$c^! \mathscr{R}Hom_{X\times X}(\mathrm{pr}_1^* K, \mathrm{pr}_2^! L) \simeq \mathscr{R}Hom_C(c_1^* K, c_2^! L)$$

and as in [12, III.3.1.1], we have

$$\mathscr{R}Hom_{X\times X}(\mathrm{pr}_1^* K, \mathrm{pr}_2^! L) \simeq D(K) \boxtimes L.$$

We therefore obtain a canonical isomorphism

$$c^!(D(K) \boxtimes L) \simeq \mathscr{R}Hom(c_1^* K, c_2^! L). \tag{5.1.1}$$

In particular, for $K = L$, we can view a c-structure $u : c_1^* K \to c_2^! K$ as an element of $H^0(C, c^!(D(K) \boxtimes K))$.

Now we have a map

$$D(K) \boxtimes K \longrightarrow \Delta_*(D(K) \otimes K) \xrightarrow{\mathrm{eval}} \Delta_* \Omega_X.$$

Applying $c^!$ to this composite and using the base change theorem we get a morphism

$$c^!(D(K) \boxtimes K) \to \delta_* \Omega_F.$$

Taking global sections we get a map

$$\mathrm{Tr}_c : \mathrm{Hom}(c_1^* K, c_2^! K) \simeq H^0(C, c^!(D(K) \boxtimes K)) \to H^0(F, \Omega_F). \tag{5.1.2}$$

For any open and closed substack $Z \subset F$ proper over k, the *local term of u at Z*, denoted $\mathrm{lt}_Z(K,u) \in \Lambda$, is defined to be the image of $\mathrm{Tr}_c(u)$ under the map

$$H^0(F, \Omega_F) \xrightarrow{\text{restriction}} H^0(Z, \Omega_Z) \xrightarrow{\int} \Lambda.$$

Here $\int : H^0(Z, \Omega_Z) \to \Lambda$ is the proper pushforward map.

Remark 5.2. This definition of local terms differs from the one given in [12, III.4.2.7]. However, the equivalence of the two definitions is shown in [23, A.2].

Lemma 5.3. *Let X be a smooth quasi-projective finite group quotient of dimension d over k, and let $\Delta : X \to X \times X$ be the diagonal morphism.*

(i) Δ is a local complete intersection morphism of codimension d.

(ii) Via the natural isomorphisms $\Omega_X \simeq \Lambda(d)[2d]$ and $\Omega_{X\times X} \simeq \Lambda(2d)[4d]$ the map $\mathrm{cl}^{\mathrm{loc}}_{X\times X}(X) : \Omega_{X\times X} \to \Delta_*\Omega_X(d)[2d]$ is identified with the map

$$\Lambda_{X\times X}(2d)[4d] \to \Delta_*\Lambda_X(2d)[4d]$$

induces by the restriction map $\Lambda_{X\times X} \to \Delta_*\Lambda_X$.

Proof. Write X as a quotient $[U/G]$ with U a smooth quasi-projective k-scheme of dimension d and G a finite group acting on U. We then have a cartesian diagram

$$\begin{tikzcd} U \times G \arrow[r, "\gamma"] \arrow[d] & U \times U \arrow[d] \\ X \arrow[r, "\Delta_X"] & X \times X, \end{tikzcd}$$

where the restriction of γ to $U \times \{g\} \subset U \times G$ is the graph of g. This immediately implies (i), and statement (ii) can be verified étale locally on X so (ii) also follows from this and 2.19. □

5.4. In this paper we are interested in correspondences $c : C \to X \times X$ with X smooth of dimension d. In this case we have $c_2^!\Lambda \simeq \Omega_C(-d)[-2d]$, and

$$H^0(C, c_2^!\Lambda) \simeq H^{-2d}(C, \Omega_C(-d)) = \widetilde{H}_d(C).$$

The homological Gysin homomorphism gives a map

$$H^0(C, c_2^!\Lambda) \simeq \widetilde{H}_d(C) \xrightarrow{\Delta^!_{\mathrm{hom}}} \widetilde{H}_0(\mathrm{Fix}(c)) = H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)}). \tag{5.4.1}$$

Proposition 5.5. *The map (5.4.1) agrees with the map $\mathrm{Tr}_c : H^0(C, c_2^!\Lambda) \to H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)})$ defined as in (5.1.2).*

Proof. This follows from the construction and 5.3 (ii). □

Theorem 5.6. *Let $c : C \to X \times X$ be a correspondence with X smooth and X and C quasi-projective finite group quotients, and let d be the dimension of X. Let $\alpha \in A_d(C)_{\mathbb{Q}}$ be a d-cycle and let $u_\alpha : c_1^*\mathbb{Q}_{\ell,X} \to c_2^!\mathbb{Q}_{\ell,X}$ be the resulting action. Then for any proper connected component $\Gamma \subset \mathrm{Fix}(c)$ the local term $\mathrm{lt}_\Gamma(\mathbb{Q}_{\ell,X}, u_\alpha)$ is equal to the degree of the restriction of the refined intersection product $\alpha.\Delta_X \in A_0(\mathrm{Fix}(c))_{\mathbb{Q}}$ to Γ. In particular, this local term is in $\mathbb{Q}$ and independent of ℓ, and in $\mathbb{Z}$ if X and C are schemes and $\alpha \in A_d(C)$.*

Proof. This follows from combining 5.5 with 2.34. □

Appendix A. Finite group quotients and factorizations

Throughout this appendix we work over a field k.

Definition A.1. (i) A Deligne-Mumford stack X/k is a *finite group quotient* if there exists an algebraic space U/k with an action of a finite group G such that $X \simeq [U/G]$.

(ii) If X is a finite group quotient a *presentation of* X is a triple (U, G, σ), where U is an algebraic space with action of a finite group G and $\sigma : X \simeq [U/G]$ is an isomorphism of stacks.

(iii) A finite group quotient is called *quasi-projective* if there exists a presentation (U, G, σ) with U a quasi-projective k-scheme.

Remark A.2. Note that if (U, G, σ) is a presentation of a finite group quotient X, then there is a finite étale surjective morphism $\pi_U : U \to X$, realizing U as a G-torsor over X.

Proposition A.3. *Let $f : X \to Y$ be a morphism of finite group quotients over k, and let (W, H, η) be a presentation of Y. Then there exists a presentation (U, G, σ) of X and morphisms $g : U \to W$ and $\phi : G \to H$ such that the diagrams*

$$\begin{tikzcd} U \times G \arrow[r, "\text{action}"] \arrow[d, "g\times\phi"] & U \arrow[d, "g"] \\ W \times H \arrow[r, "\text{action}"] & W, \end{tikzcd}$$

and

$$\begin{tikzcd} {[U/G]} \arrow[r, "\bar{g}"] \arrow[d, "\sigma"] & {[W/H]} \arrow[d, "\eta"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commute, where $\bar{g}$ denotes the map induced by (g, ϕ).

Proof. Start with any presentations (U', G', σ') of X and let U denote the fiber product of the diagram

$$\begin{tikzcd} & & W \arrow[d, "\pi_W"] \\ U' \arrow[r, "\pi_{U'}"] & X \arrow[r, "f"] & Y. \end{tikzcd}$$

Since there is an action of G' on U' over X and an action of H on W over Y, there is an induced action of $G := G' \times H$ on U. Let $g : U \to W$ and $\phi : G \to H$ be the projections. Then $X \simeq [U/G]$ and the morphisms (g, ϕ) satisfy the conclusions of the proposition. □

Definition A.4. A pair (g, ϕ) as in the proposition will be referred to as a *chart* for the morphism f.

Remark A.5. The proof shows that we can even arrange that ϕ is surjective, and if X and Y are quasi-projective and W is a quasi-projective scheme then we can further arrange that U is a quasi-projective scheme.

Proposition A.6. *Let $f : X \to Y$ be a morphism of quasi-projective finite group quotients. Then there exists a factorization*

$$X \overset{i}{\hookrightarrow} M \xrightarrow{p} Y,$$

where M is also a quasi-projective finite group quotient, p is smooth, and i is a closed imbedding.

Proof. Choose (U, G, σ), (W, H, η), and (g, ϕ) as in A.3 with U and W quasi-projective and ϕ surjective. Let L be an ample line bundle on U, and let L' denote $\otimes_{g\in G} g^*L$, so L' is a G-linearized ample line bundle on U. Taking a suitable power of L' and a sub-representation $R \subset \Gamma(U, L'^{\otimes r})$ we can then find a G-representation and a G-invariant imbedding $U \hookrightarrow \mathbb{P}R$. Let $V \subset \mathbb{P}R$ be the complement of the G-invariant closed set $\overline{U} - U$ (where $\overline{U}$ is the closure of U). We then get a G-invariant closed imbedding

$$U \hookrightarrow W \times V,$$

where G acts on W through ϕ and on V by the induced action from $\mathbb{P}R$. The projection $h : W \times V \to W$ is compatible with the map $\phi : G \to H$ so setting $M = [W \times V/G]$ we get the desired factorization of f. □

Proposition A.7. *Let X be a quasi-projective finite group quotient. Then for every coherent sheaf F on X there exists a surjective morphism $\pi : E \to F$ with E a locally free sheaf on X of finite rank.*

Proof. Write $X = [Y/G]$, where Y is a quasi-projective k-scheme with action of a finite group G. Let F_Y be the pullback of F to Y, so F_Y is a G-linearized coherent sheaf on Y. Since Y is quasi-projective over k, there exists a surjection $\pi'_Y : E'_Y \to F_Y$ with E'_Y a locally free sheaf of finite rank on Y. Let E_Y denote the sheaf

$$\oplus_{g\in G} g^* E'_Y.$$

There is a natural G-linearization of E_Y given for $g_0 \in G$ by the isomorphism

$$g_0^* E_Y \to E_Y$$

which sends $g_0^* g^* E'_Y$ to $(gg_0)^* E'_Y$ by the natural isomorphism. For $g \in G$ let

$$\pi_Y'^{(g)} : g^* E_Y \to F$$

be the composite of $g^*\pi'_Y : g^*E_Y \to g^*F_Y$ and the isomorphism $g^*F_Y \to F_Y$ given by the G-linearization on F_Y. Taking the sum of the maps $\pi_Y'^{(g)}$ we get a surjective morphism of G-linearized sheaves $\pi : E_Y \to F_Y$. This map corresponds to a surjective map $\pi : E \to F$ on the stack X as desired. □

Lemma A.8. *Let $f : X \to Y$ be a morphism of quasi-projective finite group quotients, and let*

$$X \overset{s_i}{\hookrightarrow} P_i \xrightarrow{p_i} Y$$

be two factorizations of f $(i = 1, 2)$ with s_i an imbedding and p_i smooth. Then there exists a commutative diagram

$$\begin{tikzcd}
 & & P_1 \arrow[dr, "p_1"] & \\
X \arrow[r, hook, "s"] \arrow[urr, "s_1"] \arrow[drr, "s_2"'] & P \arrow[ur, "q_1"] \arrow[dr, "q_2"'] & & Y, \\
 & & P_2 \arrow[ur, "p_2"'] &
\end{tikzcd}$$

where s is an imbedding and q_1 and q_2 are smooth.

Proof. Fix a presentation $Y \simeq [V/H]$ of Y with V quasi-projective, and for $i = 1, 2$ choose presentations $P_i \simeq [U_i/G_i]$ and maps $g_i : U_i \to V$ and $\phi_i : G_i \to H$ as in A.3 so that p_i is identified with the induced map $[U_i/G_i] \to [V/H]$. We further assume that the ϕ_i are surjective and U_i is quasi-projective (which we may by A.5). Let $E_i \to X$ be the G_i-torsor obtained by pulling back $U_i \to P_i$ along s_i, and let $W \to X$ be the H-torsor obtained by pulling back $V \to Y$, so we have maps $E_i \to W$. Then $E_1 \times_W E_2$ is a $G_1 \times_H G_2$-torsor over X so $X \simeq [E_1 \times_X E_2/G_1 \times_H G_2]$. Furthermore, the natural map $E_1 \times_W E_2 \to U_1 \times_V U_2$ is quasi-finite. Indeed the map $E_i \to U_i$ is an imbedding for $i = 1, 2$ so the map $E_1 \times_V E_2 \to U_1 \times_V U_2$ is an imbedding. Since the square

$$\begin{tikzcd}
E_1 \times_W E_2 \arrow[r] \arrow[d] & E_1 \times_V E_2 \arrow[d] \\
W \arrow[r, "\Delta_X"] & W \times W
\end{tikzcd}$$

is cartesian and Δ_W is finite, this implies that the $G_1 \times_H G_2$-equivariant morphism $E_1 \times_W E_2 \to U_1 \times_V U_2$ factors as a finite morphism followed by an imbedding. Since $U_1 \times_V U_2$ is a quasi-projective scheme, this implies that $E_1 \times_W E_2$ is a quasi-projective scheme and that there exists a factorization

$$E_1 \times_W E_2 \xhookrightarrow{\lambda'} A' \xrightarrow{z'} U_1 \times_V U_2,$$

where λ is an imbedding and z' is smooth.. Let A denote the fiber product of the schemes g^*A' $(g \in G_1 \times_H G_2)$ over $U_1 \times_V U_2$, and let $\lambda : E_1 \times_W E_2 \hookrightarrow A$ denote the imbedding obtained from the product of the imbeddings $g^*\lambda' : E_1 \times_W E_2 \hookrightarrow g^*A'$. We then have a $G_1 \times_H G_2$-equivariant factorization

$$E_1 \times_W E_2 \xhookrightarrow{\lambda} A \xrightarrow{z} U_1 \times_V U_2,$$

with λ an imbedding and z smooth. Passing to the quotients by the group actions we get a commutative diagram

$$\begin{tikzcd}
 & & {[U_1/G_1]} \arrow[dr, "p_1"] & \\
{[E_1 \times_W E_2/G_1 \times_H G_2]} \arrow[r, hook] \arrow[urr, "s_1"] \arrow[drr, "s_2"'] & {[A/G_1 \times_H G_2]} \arrow[ur, "q_1"'] \arrow[dr, "q_2"] & & {[V/H],} \\
 & & {[U_2/G_2]} \arrow[ur, "p_2"'] &
\end{tikzcd}$$

which proves the lemma. □

A couple related results which we will need are the following:

Proposition A.9. *Let $i : X \hookrightarrow Y$ be a closed imbedding of quasi-projective finite group quotients, and let E be a locally free sheaf of finite rank on X. Then there exists a smooth projective morphism $G \to Y$, a lifting $\tilde{i} : X \hookrightarrow G$ of i, and a locally free sheaf $\mathscr{E}$ on G such that $\tilde{i}^*\mathscr{E} \simeq E$.*

Proof. By A.7 we can find a locally free sheaf of finite rank F on Y and a surjection $\pi : F \to i_*E$ on Y. By adjunction this defines a surjection $i^*\pi : i^*F \to E$. Let r be the rank of E, and let $G \to Y$ be the Grassmanian of rank r quotients of F. The surjection $i^*\pi$ then defines a lifting $\tilde{i} : X \hookrightarrow G$ and the universal quotient over G is the desired extension of E. □

Proposition A.10. *Let X be a quasi-projective finite group quotient and let $K \in D^b_{\text{coh}}(X)$ be an object of the bounded derived category of $\mathscr{O}_X$-modules with coherent cohomology sheaves. Let $Z \subset X$ be a closed substack such that the restriction of K to the complement $X - Z$ is 0. Then there exists a nilpotent thickening $Z \subset Z^* \subset X$ of Z in X such that K is in the essential image of the pushforward functor $i_* : D^b_{\text{coh}}(Z^*) \to D^b_{\text{coh}}(X)$.*

Proof. Since X has the resolution property we can represent K by a bounded complex $F^\cdot$ of coherent sheaves on X. We may without loss of generality assume that $F^j = 0$ for $j \notin [0, b]$ for some $b \geq 0$. We inductively construct a sequence of complexes of coherent sheaves

$$F^\cdot = F^\cdot_0 \to F^\cdot_1 \to F^\cdot_2 \to \cdots ,$$

such that each map $F^\cdot_i \to F^\cdot_{i+1}$ is a quasi-isomorphism, $F^j_i = 0$ for $j \notin [0, b]$, and F^j_i scheme-theoretically supported on some nilpotent thickening Z^* of Z for $j < i$.

We define $F^\cdot_0$ to be $F^\cdot$.

Given $F^\cdot_i$ we construct $F^\cdot_{i+1}$ as follows. Let Z^i_i (resp. B^i_i) denote the kernel of the map $F^i_i \to F^{i+1}_i$ (resp. the image of $F^{i-1}_i \to F^i_i$). We then have an exact sequence

$$0 \to B^i_i \to Z^i_i \to \mathscr{H}^i(F^\cdot) \to 0.$$

Since B^i_i is the image of a coherent sheaf supported on Z, we may assume given a nilpotent thickening $Z \subset Z^* \subset X$ such that all the $\mathscr{H}^i(F^\cdot)$, F^j_i for $j < i$, and Z^i_i are scheme-theoretically supported on Z^*. Let $\mathscr{I} \subset \mathscr{O}_X$ be the ideal of Z^* in X. Then for some integer N the composite map

$$Z^i_i \to F^i_i \to F^i_i/\mathscr{I}^N F^i_i$$

is injective. In particular, after possibly enlarging Z^* we can find a coherent sheaf G on Z^* and a surjective morphism $\pi : F_i^i \to G$ on X such that the composite $Z_i^i \to F_i^i \to G$ is injective. Define F_{i+1}^j to be F_i^j for $j \leq i-1$ and $j > i+1$, G for $j = i$, and the pushout $F_i^{i+1} \oplus_{F_i^i} G$ for $j = i+1$. We then have a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & F_i^{i-1} \arrow[r] \arrow[d, equal] & F_i^i \arrow[r] \arrow[d, two heads] & F_i^{i+1} \arrow[r] \arrow[d] & F_i^{i+2} \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & F_{i+1}^{i-1} \arrow[r] & F_{i+1}^i \arrow[r] & F_{i+1}^{i+1} \arrow[r] & F_{i+1}^{i+2} \arrow[r] & \cdots . \end{tikzcd}$$

By construction the complex $F_{i+1}^{\cdot}$ then has the desired properties. □

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