FUJIWARA'S THEOREM FOR EQUIVARIANT CORRESPONDENCES

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1. Statements of results

The subject of this paper is a generalization to stacks of Fujiwara's theorem [10, 5.4.5] (formerly known as Deligne's conjecture) on the traces of a correspondence acting on the compactly supported cohomology of a variety over a finite field.

Before discussing the stack-theoretic version, let us begin by reviewing Fujiwara's theorem.

Let q be a power of a prime p, and let $k = \overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. For objects over $\mathbb{F}_q$ we use a subscript 0, and unadorned letters denote the base change to k. For example, X_0 denotes a scheme (or stack) over $\mathbb{F}_q$ and X denotes the fiber product $X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(k)$.

Let X_0 be a separated finite type $\mathbb{F}_q$-scheme. A *correspondence* on X_0 is a diagram of separated finite type $\mathbb{F}_q$-schemes

$$\begin{tikzcd} & C_0 \arrow[ld, "c_1"'] \arrow[rd, "c_2"] & \\ X_0 & & X_0, \end{tikzcd}$$

or equivalently a morphism $c = (c_1, c_2) : C_0 \to X_0 \times X_0$.

For $n \geq 0$ we write

$$c^{(n)} = (c_1^{(n)}, c_2) : C_0^{(n)} \to X_0 \times X_0$$

for the correspondence

$$\begin{tikzcd} & & C_0 \arrow[ld, "c_1"'] \arrow[rd, "c_2"] \arrow[lldd, bend right, "c_1^{(n)}"'] & \\ & X_0 \arrow[ld, "F_{X_0}^n"'] & & X_0, \\ X_0 & & & \end{tikzcd}$$

where $F_{X_0} : X_0 \to X_0$ denotes the q-th power Frobenius morphism.

We write $\mathrm{Fix}(C)$ (or sometimes $\mathrm{Fix}(c)$ if we want to emphasize the reference to the morphism c) for the fiber product of the diagram (over k)

$$\begin{tikzcd} & C \arrow[d, "c"] \\ X \arrow[r, "\Delta"] & X \times_{\mathrm{Spec}(k)} X. \end{tikzcd}$$

If $\mathcal{F} \in D^b_c(X, \mathbb{Q}_\ell)$ we define a C-*structure* on $\mathcal{F}$ to be a map $u : c_{2!}c_1^*\mathcal{F} \to \mathcal{F}$ in $D^b_c(X, \mathbb{Q}_\ell)$ (or equivalently a map $c_1^*\mathcal{F} \to c_2^!\mathcal{F}$ in $D^b_c(C, \mathbb{Q}_\ell)$).

A *Weil complex on* X is a pair $(\mathcal{F}, \varphi)$, where $\mathcal{F} \in D^b_c(X, \mathbb{Q}_\ell)$ and $\varphi : F_X^*\mathcal{F} \to \mathcal{F}$ is an isomorphism.

If $(\mathcal{F}, \varphi, u)$ is a Weil complex with C-structure and $n \geq 0$, then $(\mathcal{F}, \varphi)$ has a $C^{(n)}$-structure given by the map

$$u^{(n)} : c_{2!}c_1^{(n)*}\mathcal{F} = c_{2!}c_1^*F_X^{n*}\mathcal{F} \xrightarrow{\varphi^n} c_{2!}c_1^*\mathcal{F} \xrightarrow{u} \mathcal{F}.$$

Assume now that c_1 is proper and that c_2 is quasi-finite.

For a fixed point $x \in \mathrm{Fix}(C)(k)$ we get for any Weil complex with C-structure $(\mathcal{F}, \varphi, u)$ an endomorphism

$$u_x : \mathcal{F}_{c_2(x)} \to \mathcal{F}_{c_2(x)}$$

defined as follows.

Since $c_2 : C \to X$ is quasi-finite, we have

$$(c_{2!}c_1^*\mathcal{F})_{c_2(x)} = \oplus_y \mathcal{F}_{c_1(y)},$$

where the sum is taken over the set of points $y \in C(k)$ with $c_2(y) = c_2(x)$. The map u_x is defined to be the composite

$$\mathcal{F}_{c_2(x)} = \mathcal{F}_{c_1(x)} \overset{x}{\hookrightarrow} \oplus_y \mathcal{F}_{c_1(y)} = (c_{2!}c_1^*\mathcal{F})_{c_2(x)} \xrightarrow{u} \mathcal{F}_{c_2(x)}.$$

Deligne's conjecture, proven by Fujiwara, is then the following:

Theorem 1.1 (Fujiwara [10, 5.4.5]). *There exists an integer n_0, independent of $(\mathcal{F}, \varphi, u)$, such that for any integer $n \geq n_0$ all the fixed points of $c^{(n)}$ are isolated, and*

$$\mathrm{tr}(c^{(n)}|R\Gamma_c(X, \mathcal{F})) = \sum_{x \in \mathrm{Fix}(C^{(n)}(k))} \mathrm{tr}(u_x^{(n)}|\mathcal{F}_{c_2(x)}). \tag{1.1.1}$$

Remark 1.2. Note that the right side of 1.1.1 is a finite sum.

With the recent work on cohomology with compact supports for Artin stacks [13, 14], it is natural to ask for a generalization of 1.1 to Artin stacks. In this paper we propose a conjectural generalization for arbitrary stacks, and we prove this conjecture in a number of cases (in particular for equivariant correspondences).

Fujiwara's theorem is most naturally viewed in two parts. The first part is a geometric statement that the fixed points of $c^{(n)}$ are isolated and that the sum of the "naive local terms" $\sum_{x \in \mathrm{Fix}(C^{(n)}(k))} \mathrm{tr}(u_x^{(n)}|\mathcal{F}_{c_2(x)})$ is equal to the sum of the "true local terms" as defined in [3, III §4]. The second part is a reduction to the Lefschetz trace formula [3, III.4.7], which holds when X is proper.

Similarly our work on stacks breaks up naturally as a study of the geometry of the stack of fixed points, and then a study of the global trace formula. Following this breakdown of the problem we now discuss our results.

1.3. Statement of local theorem.

1.4. As above, let q denote a power of a prime p, and let $k = \overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. Let $\mathscr{X}_0/\mathbb{F}_q$ be an algebraic stack of finite type (but not necessarily separated). As in the case of schemes, a *correspondence on* $\mathscr{X}_0$ is a diagram of finite type algebraic stacks over $\mathbb{F}_q$

$$\begin{tikzcd} & \mathscr{C}_0 \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ \mathscr{X}_0 & & \mathscr{X}_0, \end{tikzcd}$$

or equivalently a morphism $c = (c_1, c_2) : \mathscr{C}_0 \to \mathscr{X}_0 \times \mathscr{X}_0$, and for an integer $n \geq 0$ we write

$$c^{(n)} = (c_1^{(n)}, c_2) : \mathscr{C}_0^{(n)} \to \mathscr{X}_0 \times \mathscr{X}_0$$

for the correspondence

$$\begin{tikzcd} & & \mathscr{C}_0 \arrow[dl, "c_1"'] \arrow[dr, "c_2"] \arrow[ddll, bend right, "c_1^{(n)}"'] & \\ & \mathscr{X}_0 \arrow[dl, "F_{\mathscr{X}_0}^n"'] & & \mathscr{X}_0, \\ \mathscr{X}_0 & & & \end{tikzcd}$$

where $F_{\mathscr{X}_0} : \mathscr{X}_0 \to \mathscr{X}_0$ denotes the q-th power Frobenius morphism. We denote by $\mathrm{Fix}(\mathscr{C})$ (or sometimes $\mathrm{Fix}(c)$) the fiber product of the diagram (over k)

$$\begin{tikzcd} & \mathscr{C} \arrow[d, "c"] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_{\mathrm{Spec}(k)} \mathscr{X}. \end{tikzcd}$$

Note that often the diagonal Δ is not quasi-finite, and therefore $\mathrm{Fix}(\mathscr{C})$ is usually not quasi-finite over $\mathscr{C}$.

1.5. If $c : \mathscr{C}_0 \to \mathscr{X}_0 \times \mathscr{X}_0$ is a correspondence and $f : \mathscr{X}_0' \to \mathscr{X}_0$ is a morphism of algebraic stacks over $\mathbb{F}_q$, we define the pullback $c' : \mathscr{C}_0' \to \mathscr{X}_0' \times \mathscr{X}_0'$ of c along f to be the correspondence obtained from the top line of the following fiber product diagram

$$\begin{tikzcd} \mathscr{C}_0' \arrow[r, "c'"] \arrow[d] & \mathscr{X}_0' \times \mathscr{X}_0' \arrow[d, "f \times f"] \\ \mathscr{C}_0 \arrow[r, "c"] & \mathscr{X}_0 \times \mathscr{X}_0. \end{tikzcd}$$

For later use it will be convenient to introduce the following non-standard terminology.

Definition 1.6. A morphism $f : \mathscr{Z} \to Y$ from an algebraic stack $\mathscr{Z}$ to a scheme Y is *pseudo-finite* if for every algebraically closed field Ω the map

$$|\mathscr{Z}(\Omega)| \to Y(\Omega)$$

is finite-to-one (where $|\mathscr{Z}(\Omega)|$ denotes the set of isomorphism classes in $\mathscr{Z}(\Omega)$), and for every $x \in \mathscr{Z}(\Omega)$ the Ω-group scheme G_x of automorphisms of x is finite of k.

The main local result is then the following:

Theorem 1.7. *Let $\mathscr{X}_0/\mathbb{F}_q$ be an algebraic stack of finite type, and let $c : \mathscr{C}_0 \to \mathscr{X}_0 \times \mathscr{X}_0$ be a correspondence with c_2 representable and quasi-finite. Then there exists an integer n_0 such that for every $n \geq n_0$ the stack* $\mathrm{Fix}(\mathscr{C}^{(n)})$ *is pseudo-finite over* $\mathrm{Spec}(k)$*, and for every $x \in \mathrm{Fix}(\mathscr{C}^{(n)})(k)$ the automorphism group scheme $\underline{\mathrm{Aut}}_{\mathrm{Fix}(\mathscr{C}^{(n)})}(x)$ is étale over k.*

Remark 1.8. The statements that $\mathrm{Fix}(\mathscr{C}^{(n)})$ is pseudo-finite over $\mathrm{Spec}(k)$ and that the automorphism group schemes $\underline{\mathrm{Aut}}_{\mathrm{Fix}(\mathscr{C}^{(n)})}(x)$ are étale over k, imply that the maximal reduced closed substack of $\mathrm{Fix}(\mathscr{C}^{(n)})$ is isomorphic to a disjoint union of classifying stacks BH of finite groups H.

1.9. **Global results and conjectures.**

Before stating the stack-theoretic version of Deligne's conjecture, we need to introduce some technical results and definitions developed in the body of the paper.

Action of correspondences.

1.10. Let $c = (c_1, c_2) : \mathscr{C}_0 \to \mathscr{X}_0 \times \mathscr{X}_0$ be a correspondence of algebraic stacks of finite type over $\mathbb{F}_q$, with c_1 proper with finite diagonal and c_2 representable and quasi-finite.

As in the case of schemes (except we now consider possibly unbounded complexes), a *$\mathscr{C}$-structure* on $\mathcal{F} \in D_c^-(\mathscr{X}, \mathbb{Q}_\ell)$ is a map $u : c_{2!}c_1^*\mathcal{F} \to \mathcal{F}$ in $D_c^-(\mathscr{X}, \mathbb{Q}_\ell)$ (or equivalently $c_1^*\mathcal{F} \to c_2^!\mathcal{F}$ in $D_c^-(\mathscr{C}, \mathbb{Q}_\ell)$).

A *Weil complex on $\mathscr{X}$* is a pair $(\mathcal{F}, \varphi)$, where $\mathcal{F} \in D_c^-(\mathscr{X}, \mathbb{Q}_\ell)$ and $\varphi : F_{\mathscr{X}}^*\mathcal{F} \to \mathcal{F}$ is an isomorphism. We say that a Weil complex $(\mathcal{F}, \varphi)$ is *bounded* if $\mathcal{F} \in D_c^b(\mathscr{X}, \mathbb{Q}_\ell)$.

1.11. Let $\mathcal{F} \in D_c^-(\mathscr{X}, \mathbb{Q}_\ell)$ be a complex and let $u : c_{2!}c_1^*\mathcal{F} \to \mathcal{F}$ be a $\mathscr{C}$-structure. In order to obtain an action of u on $R\Gamma_c(\mathscr{X}, \mathcal{F})$ we need an isomorphism

$$c_{1*}c_1^*\mathcal{F} \simeq c_{1!}c_1^*\mathcal{F}.$$

In the case of schemes this follows from the fact that c_1 is proper, and the very definition of $c_{1!}$. In the case of stacks, this is far from obvious, and in fact false for torsion coefficients. However, in section 5 we prove the following:

Theorem 1.12 (Corollary 5.17)**.** *Let $f : \mathscr{X} \to \mathscr{Y}$ be a proper morphism with finite diagonal between algebraic stacks. Then for any $F \in D_c^-(\mathscr{X}, \mathbb{Q}_\ell)$ there is a canonical isomorphism $f_!F \to f_*F$.*

Convergent complexes.

1.13. Let $D_c^-(\mathbb{Q}_\ell)$ denote the bounded above derived category of complexes of $\mathbb{Q}_\ell$-vector spaces with finite dimensional cohomology groups. Let $K \in D_c^-(\mathbb{Q}_\ell)$ be an object and $\varphi : K \to K$ an endomorphism.

Now fix an embedding $\iota : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, and for $i \in \mathbb{Z}$ let $\mathrm{Eg}^i(\varphi)$ (or $\mathrm{Eg}^i_\iota(\varphi)$) denote the set of eigenvalues of $H^i(\varphi \otimes_\iota \mathbb{C})$ acting on $H^i(K \otimes_\iota \mathbb{C})$.

Definition 1.14. The pair (K, φ) is *ι-convergent* (or simply *convergent* if the reference to ι is clear) if the sum

$$\sum_{p \in \mathbb{Z}} \sum_{\lambda \in \mathrm{Eg}^p(\varphi)} |\lambda|$$

converges.

1.15. If (K, φ) is ι-convergent, then the sum

$$\sum_p (-1)^p \iota(\mathrm{tr}(\varphi | H^p(K)))$$

converges absolutely. We denote the limit by $\mathrm{tr}_\iota(\varphi|K)$.

Definition of local terms.

1.16. Now consider an algebraic stack $\mathscr{X}_0$ of finite type over $\mathbb{F}_q$, and a correspondence $c = (c_1, c_2) : \mathscr{C}_0 \to \mathscr{X}_0$, with c_1 proper and c_2 quasi-finite and representable.

If $(\mathcal{F}, \varphi, u)$ is a Weil complex with $\mathscr{C}$-structure and $n \geq 0$, then $(\mathcal{F}, \varphi)$ has a $\mathscr{C}^{(n)}$-structure given by the map

$$u^{(n)} : c_{2!}c_1^{(n)*}\mathcal{F} = c_{2!}c_1^* F_{\mathscr{X}}^{n*}\mathcal{F} \xrightarrow{\varphi^n} c_{2!}c_1^*\mathcal{F} \xrightarrow{u} \mathcal{F}.$$

1.17. For a fixed point $(x, \lambda) \in \mathrm{Fix}(\mathscr{C})(k)$, with $x \in \mathscr{C}(k)$ and $\lambda : c_2(x) \to c_1(x)$ an isomorphism in $\mathscr{X}(k)$, we get for any $\mathcal{F} \in D_c^-(\mathscr{X}, \mathbb{Q}_\ell)$ with a $\mathscr{C}$-structure $u : c_{2!}c_1^*\mathcal{F} \to \mathcal{F}$ an endomorphism

$$u_{(x,\lambda)} : \mathcal{F}_{c_2(x)} \to \mathcal{F}_{c_2(x)}$$

defined as follows.

Since $c_2 : \mathscr{C} \to \mathscr{X}$ is representable and quasi-finite, we have

$$(c_{2!}c_1^*\mathcal{F})_{c_2(x)} = \oplus_{(y,\tau)}\mathcal{F}_{c_1(y)},$$

where the sum is taken over isomorphism classes of pairs (y, τ) with $y \in \mathscr{C}(k)$ and $\tau : c_2(y) \simeq c_2(x)$ an isomorphism in $\mathscr{X}(k)$. The map $u_{(x,\lambda)}$ is defined to be the composite

$$\begin{tikzcd}
\mathcal{F}_{c_2(x)} \arrow[r, "\lambda"] & \mathcal{F}_{c_1(x)} \arrow[r, hook, "x"] & \oplus_{(y,\tau)}\mathcal{F}_{c_1(y)} \arrow[r, equal] & (c_{2!}c_1^*\mathcal{F})_{c_2(x)} \arrow[r, "u"] & \mathcal{F}_{c_2(x)}.
\end{tikzcd}$$

1.18. Let $(\mathcal{F}, \varphi, u)$ be a bounded Weil complex with $\mathscr{C}$-structure, and choose (using 1.7) an integer n_0 such that for every $n \geq n_0$ the stack $\mathrm{Fix}(\mathscr{C}^{(n)})$ is pseudo-finite over $\mathrm{Spec}(k)$ with étale stabilizers. In particular, any connected component β of the maximal reduced closed substack $\mathrm{Fix}(\mathscr{C}^{(n)})_{\mathrm{red}} \subset \mathrm{Fix}(\mathscr{C}^{(n)})$ is isomorphic to the classifying stack BH of some finite group H. For such a component β, define the *naive local term at* β (or just the *local term at* β if no confusion seems likely to arise) to be

$$\mathrm{LT}_\iota(\beta, (\mathcal{F}, \varphi, u)) := \frac{1}{|H_{(x,\lambda)}|}\mathrm{tr}_\iota(u_{(x,\lambda)}|\mathcal{F}_{c_2(x)}),$$

where $(x, \lambda) : \mathrm{Spec}(k) \to \beta$ is any k-valued point and $H_{(x,\lambda)}$ is the automorphism group of (x, λ). Note that any two k-valued points (x, λ) of β are isomorphic, so this definition is independent of the choice of (x, λ).

Statement of conjecture and global results.

Conjecture 1.19. *Let $(\mathcal{F}, \varphi, u)$ be a bounded Weil complex with $\mathscr{C}$-structure on $\mathscr{X}$. Then there exists an integer n_0 (independent of $(\mathcal{F}, \varphi, u)$) such that for every $n \geq n_0$ we have:*

(i) The complex of $\mathbb{Q}_\ell$*-vector spaces* $R\Gamma_c(\mathcal{F}) \in D_c^-(\mathbb{Q}_\ell)$ *with the endomorphism* $R\Gamma_c(u^{(n)})$ *is* ι*-convergent.*

(ii) $\mathrm{Fix}(\mathscr{C}^{(n)})$ *is pseudo-finite over* $\mathrm{Spec}(k)$ *with étale stabilizers, and*

$$\mathrm{tr}_\iota(R\Gamma_c(u^{(n)})|R\Gamma_c\mathcal{F}) = \sum_{\beta\subset \mathrm{Fix}(\mathscr{C}^{(n)})} \mathrm{LT}_\iota(\beta, (\mathcal{F}, \varphi, u^{(n)})). \tag{1.19.1}$$

Remark 1.20. If $r \geq 1$ is an integer, then to verify 1.19 for $(\mathcal{F}, \varphi, u)$ it suffices to verify 1.19 after making the extension $\mathbb{F}_q \to \mathbb{F}_{q^r}$ for the Weil sheaves with $\mathscr{C}_j$-structure $(\mathcal{F}, \varphi^r, u \circ \varphi^j)$ defined over $\mathbb{F}_{q^r}$ (for $j = 0, \ldots, r-1$), where $\mathscr{C}_j$ is the correspondence $(F^j_{\mathscr{X}} \circ c_1, c_2) : \mathscr{C} \to \mathscr{X} \times \mathscr{X}$.

Remark 1.21. In the case of Frobenius acting on an algebraic stack, the trace formula in full generality follows from the work of Behrend [4], combined with the definition of cohomology with compact support in [13, 14].

One might hope for a notion of a convergent complex Weil complex $(\mathcal{F}, \varphi)$ with $\mathscr{C}$-structure on an algebraic stack $\mathscr{X}$ generalizing Behrend's notion for Frobenius in [4], and then a generalization of 1.19 to a relative statement saying that the pushforward of a convergent Weil complex is again a convergent Weil complex (see section 6 for how to push forward Weil complexes with action of a correspondence). However, we have been unable to find a suitable notion of convergent complex for actions of correspondences.

1.22. **Equivariant correspondences.** In this paper we will prove 1.19 in the special case of equivariant correspondences (as well as a few other cases, see sections 11 and 12).

1.23. Let $X_0/\mathbb{F}_q$ be a separated scheme of finite type, and let G_0 be a finite type group scheme over $\mathbb{F}_q$ which acts on X_0. Let $\alpha : G_0 \to G_0$ be a finite homomorphism, and let $c = (c_1, c_2) : C_0 \to X_0 \times X_0$ be a correspondence such that c_1 is proper, c_2 is quasi-finite. Assume that G_0 also acts on C_0 such that for every scheme-valued point $x \in C_0$ and $g \in G_0$ we have $\alpha(g) * c_1(x) = c_1(g * x)$ (resp. $g * c_2(x) = c_2(g * x)$). Let $\mathscr{X}_0$ (resp. $\mathscr{C}_0$) denote $[X_0/G_0]$ (resp. $[C_0/G_0]$) so we have a correspondence $\mathscr{C}_0 \to \mathscr{X}_0 \times \mathscr{X}_0$ which we again denote by c.

As we explain in 10.1, the map $c_1 : \mathscr{C}_0 \to \mathscr{X}_0$ is proper with finite diagonal and $c_2 : \mathscr{C}_0 \to \mathscr{X}_0$ is representable and quasi-finite.

Theorem 1.24. *Conjecture 1.19 holds for any bounded Weil complex with* $\mathscr{C}$*-structure on* $\mathscr{X}$*.*

1.25. **Organization of the paper.** In section 2 we prove a group theory result which plays a key technical role in a number of the results that follow.

In section 3 we prove theorem 1.7.

In sections 4 and 5 we extend some results on trace morphisms from [2, XVII and XVIII] to stacks, as well as prove the comparison result 1.12 between $f_!$ and f_* for proper morphisms with finite diagonal. These results may be of some use elsewhere, so we discuss both torsion and ℓ-adic coefficients. The main results are 4.1 and 5.1.

In section 6, we make some observations about pushforwards of Weil sheaves with action of correspondences.

In sections 7–9 we prove 1.24 in the special case of a classifying stack of a finite type group scheme and for a correspondence induced by an endomorphism of the group, as well as for correspondences arising from a diagram of finite groups.

Then in section 10 we prove 1.24 by reducing to the case of the classifying stack of a group.

In sections 11 and 12 we discuss 1.19 for algebraic spaces and Deligne-Mumford stacks. It is an open question whether there is a version of Nagata's compactification theorem for algebraic spaces. Nonetheless, in the case of an automorphism of an algebraic space 1.19 still holds as we explain in section 11. Furthermore, as we explain in section 12 the validity of 1.19 for a correspondence of Deligne-Mumford stacks is equivalent to the validity for the coarse moduli spaces (in particular, 1.19 holds when the coarse moduli spaces are schemes, or if the correspondence arises from an automorphism).

Remark 1.26. Fujiwara and F. Kato have recently announced a proof of Nagata's theorem for algebraic spaces. Granting this, one then also has Fujiwara's theorem for correspondences on algebraic spaces, and by the discussion in section 12 therefore also for Deligne-Mumford stacks.

In sections 13 and 14, we illustrate the general theory with examples. The first example comes from the theory of toric varieties, where in the smooth case the equivariant cohomology is the so-called Stanley-Reisner ring. The second example is a higher dimensional version of the formula [4, 6.4.11] of Behrend and Deligne, concerning traces of Hecke operators on modular forms.

There is also an appendix concerning a technical point about extending Rf_* to the unbounded below derived category.

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1.28. **Notation.** By an algebraic stack $\mathscr{X}$ over a scheme S, we mean a stack over the category of S-schemes with the étale topology such that the following hold:

(i) The diagonal $\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable, quasi-compact, and quasi-separated.

(ii) There exists a smooth surjection $X \to \mathscr{X}$ with X a scheme.

An algebraic stack $\mathscr{X}$ is called *Deligne-Mumford* if Δ is finite, and there exists an étale surjection $X \to \mathscr{X}$ with X a scheme.

Consider a pair (S, Λ), where S is a scheme and Λ is a ring. We say that (S, Λ) is *admissible* if the following hold:

(i) S is an affine excellent scheme of finite Krull dimension.

(ii) Λ is either $\mathbb{Q}_\ell$ or a Gorenstein local ring of dimension 0 with residue characteristic ℓ, where ℓ is a prime invertible in S.

(iii) There exists a function F on non-negative integers, such that for any finite type S-scheme X of dimension d we have $\mathrm{cd}_\ell(X) < F(d)$ (see [13, 1.0.1] for more discussion of this condition).

We remark that $(S, \mathbb{Z}/(\ell^r))$ or $(S, \mathbb{Q}_\ell)$ is an admissible pair if S is the spectrum of a finite or separably closed field, or if S is the spectrum of a complete discrete valuation ring with residue field either finite or separably closed, and ℓ invertible in S.

If (S, Λ) is an admissible pair, then for any algebraic stack $\mathscr{X}/S$ of finite type the theory developed in [13, 14] applies. In particular, there exists a dualizing complex $\Omega_{\mathscr{X}} \in D^b_c(\mathscr{X}, \Lambda)$ on $\mathscr{X}$. We write

$$D_{\mathscr{X}} : D_c(\mathscr{X}, \Lambda) \to D_c(\mathscr{X}, \Lambda)$$

for the resulting dualizing functor.

2. Some group theory in positive characteristic

2.1. Let $\mathbb{F}_q$ be a finite field with $q = p^r$ elements, and let k be an algebraic closure of $\mathbb{F}_q$. Let $F : \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ denote the q-th power Frobenius morphism, and fix the following data:

(1) An integer $n \geq 1$.
(2) Two group schemes G and G' of finite type over k.
(3) An open and closed subgroup scheme $j : H \hookrightarrow G$.
(4) A homomorphism $\alpha : H \to G'$.
(5) An isomorphism $\lambda : F^{n*}G' \to G$.

Let $F^n : G' \to F^{n*}G'$ denote the map induced by the n-th power of the q-power Frobenius on G'. The morphism F^n sends a T-valued point $\gamma : T \to G'$ (for some k-scheme T) to the unique dotted arrow filling in the diagram

$$\begin{tikzcd}
T \arrow[rrd, bend left, "\gamma\circ F_T^n"] \arrow[rd, dashed, "F^n(\gamma)"] \arrow[rdd] & & \\
& F^{n*}G' \arrow[r] \arrow[d] & G' \arrow[d] \\
& \mathrm{Spec}(k) \arrow[r, "F^n"] & \mathrm{Spec}(k),
\end{tikzcd}$$

where F_T denotes the q-power Frobenius on T. Let

$$\alpha^{(n)} : H \to G$$

denote the composite map $\lambda \circ F^n \circ \alpha$.

Proposition 2.2. *For every element $g \in G(k)$ the map of schemes (not necessarily respecting the group structure)*

$$\rho_g : H \to G, \quad h \mapsto \alpha^{(n)}(h) g j(h)^{-1}$$

is étale.

Proof. For a local ring R with maximal ideal $\mathfrak{m} \subset R$ and $k \geq 0$, let $\mathrm{gr}^k(R)$ denote $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ and let $\mathrm{gr}^*(R)$ denote $\oplus_{k \geq 0} \mathrm{gr}^k(R)$.

To prove the proposition, it suffices to show that for every $\gamma \in H(k)$ the map

$$\rho_g^* : \mathrm{gr}^*(\mathscr{O}_{G,\rho_g(h)}) \to \mathrm{gr}^*(\mathscr{O}_{H,h})$$

is an isomorphism.

Lemma 2.3. *For any $\gamma \in G'(k)$ the map*

$$F^{n*} : \mathrm{gr}^k(\mathcal{O}_{F^{n*}G',F^{n*}(\gamma)}) \to \mathrm{gr}^k(\mathcal{O}_{G',\gamma})$$

is zero for $k \geq 1$.

Proof. It suffices to prove the result in the case $k = 1$, or equivalently that the map on tangent spaces is zero. This in turn is equivalent to showing that if $\gamma \in G'(k[\epsilon])$ is a point over the ring of dual numbers reducing to the identity in $G'(k)$, then $F^n(\gamma) \in F^{n*}G'(k[\epsilon])$ is the identity. This is clear because

$$F^n_{k[\epsilon]} : \mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k[\epsilon])$$

factors through the closed immersion $\mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(k[\epsilon])$ since $n \geq 1$. ☐

Lemma 2.4. *For any $\gamma \in H(k)$ the map*

$$\alpha^{(n)*} : \mathrm{gr}^k(\mathcal{O}_{G,\alpha^{(n)}(\gamma)}) \to \mathrm{gr}^k(\mathcal{O}_{H,\gamma})$$

is zero for $k \geq 1$.

Proof. Note that $\alpha^{(n)*}$ factors as

$$\mathrm{gr}^k(\mathcal{O}_{G,\alpha^{(n)}(\gamma)}) \xrightarrow{\lambda^*} \mathrm{gr}^k(\mathcal{O}_{F^{n*}G',F^n(\alpha(\gamma))}) \xrightarrow{F^{n*}} \mathrm{gr}^k(\mathcal{O}_{G',\alpha(\gamma)}) \xrightarrow{\alpha^*} \mathrm{gr}^k(\mathcal{O}_{H,\gamma}),$$

where the middle arrow is the zero map. ☐

If X and Y are k-schemes of finite type and $x \in X(k)$ and $y \in Y(k)$ are points, then the pullback map

$$\mathrm{pr}_2^* : \mathrm{gr}^*(\mathcal{O}_{Y,y}) \to \mathrm{gr}^*(\mathcal{O}_{X\times Y,x\times y})$$

admits a retraction induced by the morphism

$$Y \to X \times Y, \quad z \mapsto (x, z).$$

It follows that there is a canonical decomposition of graded vector spaces

$$\mathrm{gr}^*(\mathcal{O}_{X\times Y,x\times y}) = \mathrm{gr}^*(\mathcal{O}_{X\times Y,x\times y})^\dagger \oplus \mathrm{gr}^*(\mathcal{O}_{Y,y}).$$

If $f : X \to X'$ and $g : Y \to Y'$ are two morphisms then the pullback

$$(f \times g)^* : \mathrm{gr}^*(\mathcal{O}_{X'\times Y',f(x)\times g(y)}) \to \mathrm{gr}^*(\mathcal{O}_{X\times Y,x\times y})$$

respects the decompositions. Furthermore there is a commutative diagram

$$\begin{tikzcd}
\mathrm{gr}^*(\mathcal{O}_{X',f(x)}) \otimes_k \mathrm{gr}^*(\mathcal{O}_{Y',g(y)}) \arrow[r, "f^*\otimes g^*"] \arrow[d] & \mathrm{gr}^*(\mathcal{O}_{X,x}) \otimes_k \mathrm{gr}^*(\mathcal{O}_{Y,y}) \arrow[d] \\
\mathrm{gr}^*(\mathcal{O}_{X'\times Y',f(x)\times g(y)}) \arrow[r, "(f\times g)^*"] & \mathrm{gr}^*(\mathcal{O}_{X\times Y,x\times y}),
\end{tikzcd}$$

where the vertical arrows are surjective. It follows that if $f^*:\mathrm{gr}^k(\mathcal{O}_{X',f(x)})\rightarrow \mathrm{gr}^k(\mathcal{O}_{X,x})$ is the zero map for $k\geq 1$ then the map $(f\times g)^*$ is of the form

$$\begin{pmatrix}0 & 0\\ 0 & g^*\end{pmatrix}:\mathrm{gr}^*(\mathcal{O}_{X'\times Y',f(x)\times g(y)})^\dagger\oplus \mathrm{gr}^*(\mathcal{O}_{Y',f(y)})\rightarrow \mathrm{gr}^*(\mathcal{O}_{X\times Y,x\times y})^\dagger\oplus \mathrm{gr}^*(\mathcal{O}_{Y,y}).$$

We apply this discussion with $X'=Y'=G$, $X=Y=H$, and f (resp. g) equal to the map $\alpha^{(n)}$ (resp. $g\cdot j(-)^{-1}$), and the point $x=y=\gamma$. We conclude that the map

$$\rho_g^*:\mathrm{gr}^*(\mathcal{O}_{G,\alpha^{(n)}(\gamma)gj(\gamma)^{-1}})\rightarrow \mathrm{gr}^*(\mathcal{O}_{H,\gamma})$$

factors as

$$\begin{tikzcd}
\mathrm{gr}^*(\mathcal{O}_{G,\alpha^{(n)}(\gamma)gj(\gamma)^{-1}}) \arrow[d, "m^*"] \\
\mathrm{gr}^*(\mathcal{O}_{G\times G,\alpha^{(n)}(\gamma)\times gj(\gamma)^{-1}})^\dagger\oplus \mathrm{gr}^*(\mathcal{O}_{G,gj(\gamma)^{-1}}) \arrow[d, "\begin{pmatrix}0 & 0\\ 0 & \simeq\end{pmatrix}"] \\
\mathrm{gr}^*(\mathcal{O}_{H\times H,\gamma\times\gamma})^\dagger\oplus \mathrm{gr}^*(\mathcal{O}_{H,\gamma}) \arrow[d, "\Delta^*"] \\
\mathrm{gr}^*(\mathcal{O}_{H,\gamma}).
\end{tikzcd}$$

To prove that ρ_g is étale at γ it therefore suffices to show that the composites

$$\mathrm{gr}^*(\mathcal{O}_{G,\alpha^{(n)}(\gamma)gj(\gamma)^{-1}})\xrightarrow{m^*}\mathrm{gr}^*(\mathcal{O}_{G\times G,\alpha^{(n)}(\gamma)\times gj(\gamma)^{-1}})^\dagger\oplus \mathrm{gr}^*(\mathcal{O}_{G,gj(\gamma)^{-1}})\xrightarrow{(x,y)\mapsto y}\mathrm{gr}^*(\mathcal{O}_{G,gj(\gamma)^{-1}})$$

and

$$\mathrm{gr}^*(\mathcal{O}_{H,\gamma})\xrightarrow{z\mapsto(0,z)}\mathrm{gr}^*(\mathcal{O}_{H\times H,\gamma\times\gamma})^\dagger\oplus \mathrm{gr}^*(\mathcal{O}_{H,\gamma})\xrightarrow{\Delta^*}\mathrm{gr}^*(\mathcal{O}_{H,\gamma})$$

are isomorphisms. This follows from consideration of the commutative diagrams

$$\begin{tikzcd}
G \arrow[d, "{(\alpha^{(n)}(\gamma),\mathrm{id})}"'] \arrow[dr, "\alpha^{(n)}(\gamma)\cdot(-)"] & \\
G\times G \arrow[r, "m"] & G
\end{tikzcd}$$

and

$$\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[rr, bend left, "\mathrm{id}"] & H\times H \arrow[r, "\mathrm{pr}_2"] & H.
\end{tikzcd}$$

∎

Remark 2.5. Proposition 2.2 also holds in the case when $n=0$ if the homomorphism α is nowhere étale (which implies that 2.4 holds also for $n=0$).

3. Proof of Theorem 1.7

Let $c : \mathscr{C}_0 \to \mathscr{X}_0 \times \mathscr{X}_0$ be as in 1.7.

Lemma 3.1. *Let $f : \mathscr{X}'_0 \to \mathscr{X}_0$ be a representable and quasi-finite morphism of algebraic stacks over $\mathbb{F}_q$. Then if $c' : \mathscr{C}'_0 \to \mathscr{X}'_0 \times \mathscr{X}'_0$ denotes the pullback of c along f (see 1.5), then the projection $c'_2 : \mathscr{C}'_0 \to \mathscr{X}'_0$ is representable and quasi-finite.*

Proof. Consider the commutative diagram of algebraic stacks where all squares are cartesian

$$\begin{tikzcd}
\mathscr{C}'_0 \arrow[r, "c'"] \arrow[d, "a"'] \arrow[rrd, bend left, "c'_2"] & \mathscr{X}'_0 \times \mathscr{X}'_0 \arrow[d, "f\times 1"] & \\
\mathscr{D}_0 \arrow[r, "c_{\mathscr{D}}"] \arrow[d] & \mathscr{X}_0 \times \mathscr{X}'_0 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "1\times f"] & \mathscr{X}'_0 \arrow[d, "f"] \\
\mathscr{C} \arrow[r, "c"] \arrow[rr, bend right, "c_2"'] & \mathscr{X}_0 \times \mathscr{X}_0 \arrow[r, "\mathrm{pr}_2"] & \mathscr{X}_0.
\end{tikzcd}$$

Since c_2 is representable and quasi-finite, the morphism $\mathrm{pr}_2 \circ c_{\mathscr{D}} : \mathscr{D}_0 \to \mathscr{X}'_0$ is representable and quasi-finite, and since f is representable and quasi-finite the morphism a is representable and quasi-finite. It follows that the composite

$$c'_2 = \mathrm{pr}_2 \circ c_{\mathscr{D}} \circ a$$

is also representable and quasi-finite. □

3.2. We can in particular apply 3.1 to the morphism $f : \mathscr{X}_{0,\mathrm{red}} \to \mathscr{X}_0$ from the maximal reduced closed substack of $\mathscr{X}_0$. If $\mathscr{C}'_0 \to \mathscr{X}_{0,\mathrm{red}} \times \mathscr{X}_{0,\mathrm{red}}$ denotes the pullback of $\mathscr{C}_0$, then the natural map

$$\mathrm{Fix}(\mathscr{C}_0'^{(n)}) \to \mathrm{Fix}(\mathscr{C}_0^{(n)})$$

is a closed immersion defined by a nilpotent ideal. Since the notion of a pseudo-finite morphism in 1.6 is insensitive to the infinitesimal structure on $\mathscr{X}$, it therefore suffices to prove 1.7 under the additional assumption that $\mathscr{X}_0$ is reduced.

3.3. If $\mathscr{U}_0 \subset \mathscr{X}_0$ is an open substack with complement $\mathscr{Z}_0$, and $\mathscr{C}_{\mathscr{U},0} \to \mathscr{U}_0 \times \mathscr{U}_0$ (resp. $\mathscr{C}_{\mathscr{Z},0} \to \mathscr{Z}_0 \times \mathscr{Z}_0$) is the pullback of $\mathscr{C}_0$ to $\mathscr{U}_0$ (resp. $\mathscr{Z}_0$), then the two correspondences

$$\mathscr{C}_{\mathscr{U},0} \to \mathscr{U}_0 \times \mathscr{U}_0, \;\; \mathscr{C}_{\mathscr{Z},0} \to \mathscr{Z}_0 \times \mathscr{Z}_0 \tag{3.3.1}$$

also satisfy the assumptions of 1.7 by 3.1. Moreover, $\mathrm{Fix}(\mathscr{C}_{\mathscr{U}}^{(n)})$ is an open substack of $\mathrm{Fix}(\mathscr{C}^{(n)})$ with complement $\mathrm{Fix}(\mathscr{C}_{\mathscr{Z}}^{(n)})$. To prove 1.7 for $\mathscr{C}_0 \to \mathscr{X}_0 \times \mathscr{X}_0$ it therefore suffices to prove 1.7 for the two correspondences 3.3.1.

3.4. By noetherian induction it therefore suffices to show that 1.7 holds for a nonempty open substack of our reduced stack $\mathscr{X}_0$.

Let $\mathscr{I}_0 \to \mathscr{X}_0$ be the inertia stack. Since $\mathscr{X}_0$ is reduced there exists a dense open substack $\mathscr{U}_0 \subset \mathscr{X}_0$ such that the restriction $\mathscr{I}_{\mathscr{U},0} \to \mathscr{U}_0$ is flat. We can then form the rigidification of $\mathscr{U}_0$ with respect to $\mathscr{I}_{\mathscr{U},0}$ as for example in [17, §1.5] to get an algebraic space U_0 with a morphism $\pi : \mathscr{U}_0 \to U_0$ which is universal for morphisms to algebraic spaces. Replacing $\mathscr{X}_0$ by $\mathscr{U}_0$, we may assume that the inertia stack $\mathscr{I}_0 \to \mathscr{X}_0$ is flat over $\mathscr{X}_0$. Let $\pi : \mathscr{X}_0 \to X_0$ be the rigidification. Then X_0 is an algebraic space with quasi-compact diagonal, and therefore by [12, II.6.7] X_0 contains a dense open affine subscheme. Replacing $\mathscr{X}_0$ by the inverse image of this open subscheme, we may assume that X_0 is an affine scheme.

Next observe that we may also assume that the inertia stack $\mathscr{I}_{\mathscr{C},0} \to \mathscr{C}_0$ is flat over $\mathscr{C}_0$. Indeed note that if $\mathscr{C}_{0,\mathrm{red}} \subset \mathscr{C}_0$ is the maximal reduced closed substack then the correspondence

$$\mathscr{C}_{0,\mathrm{red}} \to \mathscr{X}_0$$

also satisfies the assumptions of 1.7 and for every $n \geq 0$ the map $\mathrm{Fix}(\mathscr{C}^{(n)}_{\mathrm{red}}) \to \mathrm{Fix}(\mathscr{C}^{(n)})$ is a closed immersion defined by a nilpotent ideal. We may therefore assume that $\mathscr{C}_0$ is reduced.

Furthermore, if $\mathscr{V}_0 \subset \mathscr{C}_0$ is an open substack with complement $\mathscr{T}_0 \subset \mathscr{C}_0$, then the correspondences

$$\mathscr{V}_0 \to \mathscr{X}_0 \times \mathscr{X}_0, \quad \mathscr{T}_0 \to \mathscr{X}_0 \times \mathscr{X}_0$$

also satisfy the assumptions of 1.7 and $\mathrm{Fix}(\mathscr{V}^{(n)})$ is an open substack of $\mathrm{Fix}(\mathscr{C}^{(n)})$ with complement $\mathrm{Fix}(\mathscr{T}^{(n)})$. Stratifying $\mathscr{C}_0$ by substacks over which the inertia stack is flat we reduce to the case when $\mathscr{I}_{\mathscr{C},0} \to \mathscr{C}_0$ is flat. Let $\mathscr{C}_0 \to C_0$ denote the rigidification. After further shrinking on $\mathscr{C}_0$ we may assume that C_0 is also an affine scheme. We then have a commutative diagram

$$\begin{tikzcd}
 & \mathscr{C}_0 \arrow[dl, "c_1"'] \arrow[dr, "c_2"] \arrow[dd, "\pi_{\mathscr{C}}"] & \\
\mathscr{X}_0 \arrow[dd, "\pi_{\mathscr{X}}"'] & & \mathscr{X}_0 \arrow[dd, "\pi_{\mathscr{X}}"] \\
 & C_0 \arrow[dl, "\bar{c}_1"'] \arrow[dr, "\bar{c}_2"] & \\
X_0 & & X_0.
\end{tikzcd}$$

This diagram induces for every $n \geq 0$ a morphism

$$\rho : \mathrm{Fix}(\mathscr{C}^{(n)}) \to \mathrm{Fix}(C^{(n)}).$$

3.5. Fix a point $x \in \mathrm{Fix}(C^{(n)})(k)$. The groupoid $\rho^{-1}(x) \subset \mathrm{Fix}(\mathscr{C}^{(n)})(k)$ can be described as follows.

Fix an object $\tilde{x} \in \mathscr{C}(k)$ mapping to the image of x in $C(k)$. Since $\tilde{x}$ maps to the same element of $X(k)$ under either c_2 or $F^n_{\mathscr{X}} c_1$, it follows that $c_2(\tilde{x})$ is isomorphic to $F^n_{\mathscr{X}} c_1(\tilde{x})$. Let $G_{c_2(\tilde{x})}$ be the automorphism group scheme of $c_2(\tilde{x}) \in \mathscr{X}(k)$, and let P denote the $G_{c_2(\tilde{x})}$-torsor of isomorphisms in $\mathscr{X}(k)$

$$\lambda : c_2(\tilde{x}) \to F^n_{\mathscr{X}} c_1(\tilde{x}).$$

Let $G_{\tilde{x}}$ be the automorphism group scheme of $\tilde{x} \in \mathscr{C}(k)$. There is an action of $G_{\tilde{x}}$ on P for which $\gamma \in G_{\tilde{x}}$ sends λ to the composite

$$c_2(\tilde{x}) \xrightarrow{c_2(\gamma)} c_2(\tilde{x}) \xrightarrow{\lambda} F^n_{\mathscr{X}} c_1(\tilde{x}) \xrightarrow{F^n_{\mathscr{X}} c_1(\gamma)^{-1}} F^n_{\mathscr{X}} c_1(\tilde{x}) .$$

The groupoid $\rho^{-1}(x)$ is then isomorphic to the quotient groupoid $[P(k)/G_{\tilde{x}}(k)]$. In particular, the set of isomorphism classes of objects in $\rho^{-1}(x)$ is equal to the set $P(k)/G_{\tilde{x}}(k)$.

Proposition 3.6. *If $n \geq 1$ then the set $P(k)/G_{\tilde{x}}(k)$ is finite.*

Proof. For any $\lambda \in P(k)$, let

$$\sigma_\lambda : G_{\tilde{x}} \to P, \quad \gamma \mapsto \gamma * \lambda$$

be the map of schemes defined by the action. To prove the proposition it suffices to show that σ_λ is étale. For then the image is open, and since P is quasi-compact there exists finitely many $\lambda_1, \ldots, \lambda_r \in P(k)$ such that the images $\sigma_{\lambda_i}(G_{\tilde{x}})$ cover P.

Lemma 3.7. *The map $p_2 : G_{\tilde{x}} \to G_{c_2(\tilde{x})}$ induced by c_2 is an open and closed immersion.*

Proof. Since c_2 is representable and quasi-finite, the fiber product of the diagram

$$\begin{tikzcd} & \mathscr{C} \arrow[d, "c_2"] \\ \operatorname{Spec}(k) \arrow[r, "c_2(\tilde{x})"] & \mathscr{X} \end{tikzcd}$$

is an affine scheme $\operatorname{Spec}(A)$ finite over $\operatorname{Spec}(k)$. The maximal reduced closed subscheme $\operatorname{Spec}(A_{\text{red}}) \subset \operatorname{Spec}(A)$ is therefore equal to a disjoint union

$$\operatorname{Spec}(A_{\text{red}}) = \coprod_{s \in S} \operatorname{Spec}(k)$$

of copies of $\operatorname{Spec}(k)$. The group scheme $G_{c_2(\tilde{x})}$ acts on $\operatorname{Spec}(A_{\text{red}})$ and therefore also on the set S. This defines a homomorphism

$$u : G_{c_2(\tilde{x})} \to \operatorname{Aut}(S).$$

The point $\tilde{x}$ corresponds to a distinguished element $s_0 \in S$, and $G_{\tilde{x}}$ is the stabilizer in $G_{c_2(\tilde{x})}$ of this element. From this the lemma follows. □

We now apply 2.2 with $G = G_{c_2(\tilde{x})}$, $H = G_{\tilde{x}}$, $G' = G_{c_1(\tilde{x})}$, $j : H \hookrightarrow G$ the inclusion defined by c_2, and α the map given by c_1. An element $\lambda \in P(k)$ defines an isomorphism $F^{n*}G' \simeq G$ which we again denote by λ. This isomorphism has the property that the composite map

$$\begin{tikzcd} H \arrow[r, "\sigma_\lambda"] & P \arrow[r, "\lambda"] & G \end{tikzcd}$$

is equal to the map

$$H \to G, \quad h \mapsto j(h)\alpha^{(n)}(h)^{-1}.$$

That σ_λ is étale therefore follows from 2.2. This completes the proof of 3.6. □

Note also that if we fix an isomorphism $\lambda : c_2(\tilde{x}) \to F^n_{\mathscr{X}} c_1(\tilde{x})$ so that $(\tilde{x}, \lambda)$ is an object of $\operatorname{Fix}(\mathscr{C}^{(n)})$, then the group scheme $\underline{\operatorname{Aut}}_{\operatorname{Fix}(\mathscr{C}^{(n)})}(\tilde{x}, \lambda)$ is equal to the inverse image of the identity under $\rho_e : G_{\tilde{x}} \to G_{c_2(\tilde{x})}$. Since ρ_e is étale it follows that the automorphism group scheme of any object in $\operatorname{Fix}(\mathscr{C}^{(n)})$ is finite étale over k.

To complete the proof of 1.7, it therefore suffices to choose an integer n_0 such that for any $n \geq n_0$ the set $\operatorname{Fix}(C^{(n)})$ is finite. This is possible by [19, 1.2.2]. □

4. Trace map for quasi-finite morphisms of stacks

Fix an admissible pair (S, Λ) as in 1.28 .

The main result of this section is the following.

Theorem 4.1. *There exists a unique way to associate to any quasi-finite flat morphism $f : \mathscr{X} \to \mathscr{Y}$ of algebraic stacks and constructible sheaf of Λ-modules F on $\mathscr{Y}$ a morphism*

$$\mathrm{tr}_f : f_! f^* F \to F \tag{4.1.1}$$

such that the following hold:

(i) (Functoriality) The morphism tr_f is functorial in F.

(ii) (Compatibility with base change) For every cartesian diagram

$$\begin{tikzcd} \mathscr{X}' \arrow[r, "g'"] \arrow[d, "f'"] & \mathscr{X} \arrow[d, "f"] \\ \mathscr{Y}' \arrow[r, "g"] & \mathscr{Y} \end{tikzcd}$$

with f flat and quasi-finite, and F a constructible sheaf on $\mathscr{Y}$ the diagram

$$\begin{tikzcd} R^0 f'_! f'^* g^* F \arrow[r, equal] \arrow[dr, "\mathrm{tr}_{f'}"'] & R^0 f'_! g'^* f^* F \arrow[r, "a"] & g^* R^0 f_! f^* F \arrow[dl, "\mathrm{tr}_f"] \\ & g^* F & \end{tikzcd} \tag{4.1.2}$$

commutes, where the morphism a is the base change isomorphism.

(iii) (Compatibility with composition) Let

$$\begin{tikzcd} \mathscr{X} \arrow[r, "g"] & \mathscr{Y} \arrow[r, "f"] & \mathscr{Z} \end{tikzcd}$$

be a diagram of quasi-finite flat morphisms of algebraic stacks and let F be a constructible sheaf on $\mathscr{Z}$. Then the diagram

$$\begin{tikzcd} f_! g_! g^* f^* F \arrow[r, "\mathrm{tr}_g"] \arrow[d, "\simeq"] & f_! f^* F \arrow[d, "\mathrm{tr}_f"] \\ (fg)_!(fg)^* F \arrow[r, "\mathrm{tr}_{fg}"] & F \end{tikzcd} \tag{4.1.3}$$

commutes.

(iv) (Normalization) If $\mathscr{Y} = Y$ is a scheme, and $p : W \to \mathscr{X}$ is a smooth surjection with constant fiber dimension d, then the diagram

$$\begin{tikzcd} f_! p_! p^* f^* F \arrow[r, "t"] \arrow[d, "\simeq"] & f_! p_! p^! f^* F(-d)[-2d] \arrow[r, "p_! p^! \to \mathrm{id}"] & f_! f^* F(-d)[-2d] \arrow[r, "\mathrm{tr}_f"] & F(-d)[-2d] \\ (pf)_!(pf)^* F \arrow[urrr, "\mathrm{tr}_{pf}"'] & & & \end{tikzcd} \tag{4.1.4}$$

commutes, where tr_{pf} is the trace map defined in [2, XVIII.2.9] *and the map t is the map defined by the isomorphism $p^* \simeq p^!(-d)[-2d]$.*

In the case of a quasi-finite morphism of schemes $f : X \to Y$ *and* F *on* Y *the trace morphism* tr_f *agrees with the morphism defined in* [2, XVII, 6.2.3].

The proof will be broken into several steps 4.2-4.12.

Note that by adjunction giving the morphism 4.1.1 is equivalent to giving a morphism $\widetilde{\mathrm{tr}}_f : f^*F \to f^!F$.

4.2. *Some cohomological observations.*

Lemma 4.3. *(i) For any* $G \in D_c^{[0,\infty)}(\mathscr{Y}, \Lambda)$ *we have* $f^!G \in D_c^{[0,\infty)}(\mathscr{X}, \Lambda)$.

(ii) For any constructible sheaf of Λ*-modules* F *on* $\mathscr{Y}$*, we have*

$$\mathscr{E}xt^i(f^*F, f^!F) = 0 \quad \textit{for} \quad i < 0.$$

Proof. Statement (ii) follows immediately from (i).

To prove (i), note that the assertion is local in the smooth topology on $\mathscr{Y}$, and the Zariski topology on $\mathscr{X}$. It follows that it suffices to consider the case when $\mathscr{Y}$ is a scheme and there exists a smooth surjection $p : W \to \mathscr{X}$ with W a scheme such that the fibers of p are all of dimension d, for some integer d. Since $p^!f^!G \simeq p^*f^!G(d)[2d]$ it therefore suffices to show that $(fp)^!G \in D_c^{\geq -2d}(\mathscr{X})$ which is [2, XVIII, 3.1.7]. ☐

Lemma 4.4. *Let* $s \in \mathbb{Z}$ *be an integer. Then for any* $F \in D_c^{(-\infty,s]}(\mathscr{X}, \Lambda)$ *we have* $f_!F \in D_c^{(-\infty,s]}(\mathscr{Y}, \Lambda)$.

Proof. It clearly suffices to consider the case $s = 0$.

Let d be the maximal integer such that $R^df_!F \neq 0$. Then for any constructible sheaf of Λ-modules G on $\mathscr{Y}$ we have

$$\mathrm{Ext}^{-d}(f_!F, G) = \mathrm{Hom}(\mathscr{H}^d(f_!F), G).$$

On the other hand, we have by adjunction

$$\mathrm{Ext}^{-d}(f_!F, G) = \mathrm{Ext}^{-d}(F, f^!G),$$

which since $f^!G \in D^{[0,\infty)}(\mathscr{X}, \Lambda)$ is zero if $d > 0$. Thus if $d > 0$ we find that

$$\mathrm{Hom}(\mathscr{H}^d(f_!F), G) = 0$$

for every constructible sheaf G. This is a contradiction (take $G = \mathscr{H}^d(f_!F)$). ☐

Remark 4.5. It follows that if F is a constructible sheaf of Λ-modules on $\mathscr{Y}$, then giving a morphism $f_!f^*F \to F$ is equivalent to giving a morphism $R^0f_!f^*F \to F$.

Remark 4.6. Both 4.3 and 4.4 hold without the assumption that f is flat, with the same proofs.

4.7. *A special case.*

Consider first the case when $\mathscr{Y}$ is a scheme and there exists a scheme W and a smooth surjection $p : W \to \mathscr{X}$ with constant fiber dimension an integer d. Let Z denote $W \times_{\mathscr{X}} W$ so we have morphisms

$$\mathrm{pr}_1, \mathrm{pr}_2 : Z \to W, \quad q : Z \to \mathscr{X}.$$

The morphisms pr_i are smooth of constant relative dimension d and the morphism q is smooth of relative dimension $2d$. To define the map $\widetilde{\mathrm{tr}}_f : f^*F \to f^!F$ it suffices to construct a morphism

$$\epsilon : p^*f^*F \to p^*f^!F$$

such that the diagram

$$\begin{tikzcd}
\mathrm{pr}_1^*p^*f^*F \arrow[r, "\mathrm{pr}_1^*\epsilon"] \arrow[d, "\simeq"] & \mathrm{pr}_1^*p^*f^!F \arrow[d, "\simeq"] \\
\mathrm{pr}_2^*p^*f^*F \arrow[r, "\mathrm{pr}_2^*\epsilon"] & \mathrm{pr}_2^*p^*f^!F
\end{tikzcd} \tag{4.7.1}$$

commutes.

To define the map ϵ, note that since p is smooth of relative dimension d we have $p^*f^!F \simeq p^!f^!F(-d)[-2d] \simeq (fp)^!F(-d)[-2d]$. Therefore giving ϵ is equivalent to giving a morphism

$$(fp)^*F \to (fp)^!F(-d)[-2d].$$

For this we take the map $\widetilde{\mathrm{tr}}_{fp}$ defined by the trace map in [2, XVIII, 2.9].

Remark 4.8. The commutativity of the diagram 4.1.4 is equivalent by adjunction to the commutativity of the diagram

$$\begin{tikzcd}
p^*f^*F \arrow[r, "\simeq"] \arrow[d, "\simeq"] & p^!f^*F(-d)[-2d] \arrow[r, "p^!\widetilde{\mathrm{tr}}_f"] & p^!f^!F(-d)[-2d] \\
(pf)^*F. \arrow[urr, "\widetilde{\mathrm{tr}}_{pf}"'] & &
\end{tikzcd}$$

Thus the normalization condition (iv) is equivalent to the condition that $p^*\widetilde{\mathrm{tr}}_f$ is equal to the map ϵ defined above.

To verify that 4.7.1 commutes, note that for $i = 1, 2$ the diagram

$$\begin{tikzcd}
\mathrm{pr}_i^*p^*f^*F \arrow[r, "\mathrm{pr}_i^*\epsilon"] \arrow[rr, bend left, "\widetilde{\mathrm{tr}}_{fp}"] \arrow[d, "\simeq"] & \mathrm{pr}_i^*p^*f^!F \arrow[r, "\simeq"] & \mathrm{pr}_i^*(fp)^!F(-d)[-2d] \arrow[d, "\widetilde{\mathrm{tr}}_{\mathrm{pr}_i}"] \\
(fq)^*F \arrow[rr, "\widetilde{\mathrm{tr}}_{fq}"] & & (fq)^!F(-2d)[-4d]
\end{tikzcd}$$

commutes by [2, XVIII, 2.9 (Var 3)] and the map $\widetilde{\mathrm{tr}}_{\mathrm{pr}_i}$ is an isomorphism. This implies that $\mathrm{pr}_1^*\epsilon = \mathrm{pr}_2^*\epsilon$.

Lemma 4.9. *The induced map $\widetilde{\mathrm{tr}}_f : f^*F \to f^!F$ is independent of the choice of the covering $p : W \to \mathscr{X}$.*

Proof. Let $\widetilde{\mathrm{tr}}_f$ be the map defined using $p : W \to \mathscr{X}$, and let $p' : W' \to \mathscr{X}$ be a second smooth covering of relative dimension d', and let $\widetilde{\mathrm{tr}}'_f$ be the map defined using p'. By considering the product $W \times_{\mathscr{X}} W'$, to show that $\widetilde{\mathrm{tr}}_f = \widetilde{\mathrm{tr}}'_f$ we may assume that there exists a smooth morphism $h : W' \to W$ over $\mathscr{X}$.

Then using [13, 2.3.4] to prove that $\widetilde{\mathrm{tr}}_f = \widetilde{\mathrm{tr}}'_f$ it suffices to show that $p'^*\widetilde{\mathrm{tr}}_f = p'^*\widetilde{\mathrm{tr}}'_f$. This follows from [2, XVIII, 2.9 (Var 3)] and consideration of the diagram

$$\begin{tikzcd}
p'^*f^*F \arrow[r, "\widetilde{\mathrm{tr}}'_f"] \arrow[rr, bend left, "\widetilde{\mathrm{tr}}_{fp'}"] \arrow[d, "\simeq"] & p'^*f^!F \arrow[r, "\simeq"] & (fp')^!F(-d')[-2d'] \\
h^*p^*f^*F \arrow[r, "\widetilde{\mathrm{tr}}_f"] \arrow[rr, bend right, "\widetilde{\mathrm{tr}}_{fp}"'] & h^*p^*f^!F \arrow[r, "\simeq"] & h^*(fp)^!F(-d)[-2d]. \arrow[u, "\widetilde{\mathrm{tr}}_h"']
\end{tikzcd}$$

□

Lemma 4.10. *Property (ii) holds for a morphism of schemes $g : \mathscr{Y}' \to \mathscr{Y}$.*

Proof. Let $p : W \to \mathscr{X}$ be a smooth morphism of constant fiber dimension d as above, and let W' denote $\mathscr{X}' \times_{\mathscr{X}} W$ so we have a commutative diagram

$$\begin{tikzcd}
W' \arrow[r, "g''"] \arrow[d, "p'"] & W \arrow[d, "p"] \\
\mathscr{X}' \arrow[r, "g'"] \arrow[d, "f'"] & \mathscr{X} \arrow[d, "f"] \\
\mathscr{Y}' \arrow[r, "g"] & \mathscr{Y}.
\end{tikzcd}$$

To verify the commutativity of 4.1.2 it suffices to verify the commutativity of the diagram

$$\begin{tikzcd}
f'^*g^*F \arrow[r, "\simeq"] \arrow[dr, "\widetilde{\mathrm{tr}}_{f'}"'] & g'^*f^*F \arrow[r, "bc"] & f'^!g^*f_!f^*F \arrow[dl, "\mathrm{tr}_f"] \\
& f'^!g^*F, &
\end{tikzcd}$$

where "bc" denotes the map induced by adjunction from the base change isomorphism. For this in turn it suffices to verify that it commutes after applying p'^*. Consider the following diagram

$$\begin{tikzcd}
 & & (f'p')^!g^*(fp)_!(fp)^*F \arrow[dddl, "\mathrm{tr}_p"'] \\
p'^*f'^*g^*F \arrow[r, "\simeq"] \arrow[urr, "bc"] \arrow[d, "\widetilde{\mathrm{tr}}_{f'}"] \arrow[dd, bend right=60, "\widetilde{\mathrm{tr}}_{f'p'}"'] & p'^*g'^*f^*F \arrow[d, "bc"] & \\
p'^*f'^!g^*F \arrow[d, "\widetilde{\mathrm{tr}}_{p'}"] & p'^*f'^!g^*f_!f^*F \arrow[l, "\mathrm{tr}_f"] \arrow[d, "\widetilde{\mathrm{tr}}_{p'}"] & \\
(f'p')^!g^*F(-d)[-2d] & (f'p')^!g^*f_!f^*F(-d)[-2d]. \arrow[l, "\mathrm{tr}_f"] &
\end{tikzcd}$$

By [2, XVIII, 2.9 (Var 2)] the big outside diagram commutes, and all the small inside diagrams commute by the construction of the trace map, except possibly the top square. It follows that the top square also commutes. □

4.11. *The general case.*

Now consider the case of a general quasi-finite flat morphism $f : \mathscr{X} \to \mathscr{Y}$.

Let $Y \to \mathscr{Y}$ be a smooth surjection with Y a scheme, such that there exists a smooth covering $W \to \mathscr{X}_Y$ by a scheme W with constant relative dimension d. We then obtain a canonical morphism

$$f_!f^*F|_Y \to F|_Y$$

such that the two pullbacks to $Y \times_{\mathscr{Y}} Y$ agree (by 4.10). Using 4.5 this map over Y descends uniquely to a morphism $\mathrm{tr}_f : f_!f^*F \to F$. As before we let $\widetilde{\mathrm{tr}}_f : f^*F \to f^!F$ denote the map obtained by adjunction.

We now verify (i)-(iv). Property (i) is immediate, and property (iv) follows from 4.9.

Let us verify property (iii). To show that 4.1.3 commutes, it suffices to show that the diagram

$$\begin{tikzcd}
g^*f^*F \arrow[r, "\widetilde{\mathrm{tr}}_g"] \arrow[dd, "\simeq"] & g^!f^*F \arrow[d, "\widetilde{\mathrm{tr}}_f"] \\
 & g^!f^!F \arrow[d, "\simeq"] \\
h^*F \arrow[r, "\widetilde{\mathrm{tr}}_h"] & h^!F
\end{tikzcd} \tag{4.11.1}$$

commutes, where we write $h := gf$. Furthermore, it suffices to verify the commutativity of this diagram after pulling back along any smooth surjective morphism $X \to \mathscr{X}$.

Let $p : Z \to \mathscr{Z}$ be a smooth morphism with Z a scheme, and set $\mathscr{Y}_Z := \mathscr{Y} \times_{\mathscr{Z}} Z$ and $\mathscr{X}_Z := \mathscr{X} \times_{\mathscr{Z}} Z$ so we have a commutative diagram

$$\begin{tikzcd}
\mathscr{X}_Z \arrow[r, "g'"] \arrow[d, "p''"] \arrow[rr, bend left, "h'"] & \mathscr{Y}_Z \arrow[r, "f'"] \arrow[d, "p'"] & Z \arrow[d, "p"] \\
\mathscr{X} \arrow[r, "g"] \arrow[rr, bend right, "h"'] & \mathscr{Y} \arrow[r, "f"] & \mathscr{Z}.
\end{tikzcd}$$

We then get a diagram

$$\begin{tikzcd}
g'^*p'^*f^*F \arrow[r, "\widetilde{\mathrm{tr}}_{g'}"] \arrow[d, "\simeq"] & g'^!p'^*f^*F \arrow[r, "\simeq"] \arrow[d, "\simeq"] \arrow[ddr, "\widetilde{\mathrm{tr}}_f"] & g'^*f'^*p^*F \arrow[dddl, bend left, "\widetilde{\mathrm{tr}}_{f'}"] \\
p''^*g^*f^*F \arrow[r, "\widetilde{\mathrm{tr}}_g"] \arrow[d, "\simeq"] & p''^*g^!f^*F \arrow[d, "\widetilde{\mathrm{tr}}_f"] & \\
p''^*h^*F \arrow[r, "\widetilde{\mathrm{tr}}_h"] \arrow[d, "\simeq"] & p''^*g^!f^!F \arrow[r, "\simeq"] \arrow[d, "\simeq"] & g'^!p'^*f^!F \\
g'^*f'^*p^*F \arrow[r, "\widetilde{\mathrm{tr}}_{h'}"] & g'^!f'^!p^*F. &
\end{tikzcd}$$

It follows from the construction of the trace map that all the small inside diagrams commute, except possibly the middle square in the left column. To verify that this last square commutes it therefore suffices to show that the big outside diagram commutes. This reduces the proof to the case when $\mathscr{Z}$ is a scheme.

By a similar argument, one reduces to the case when $\mathscr{X}$ is also a scheme.

So now consider the case when $\mathscr{X}$ and $\mathscr{Z}$ are schemes, which we denote by roman letters X and Z respectively. Let $p : Y \to \mathscr{Y}$ be a smooth morphism of constant relative dimension d and with Y a scheme. Let X_Y denote $X \times_{\mathscr{Y}} Y$. Then X_Y is an algebraic space. Consider the commutative diagram

$$\begin{tikzcd}
X_Y \arrow[r, "g'"] \arrow[d, "p'"] \arrow[drr, bend left, "h'"] & Y \arrow[d, "p"] \arrow[dr, "f'"] & \\
X \arrow[r, "g"] \arrow[rr, bend right, "h"'] & \mathscr{Y} \arrow[r, "f"] & Z.
\end{tikzcd}$$

To verify that 4.11.1 commutes it suffices as mentioned above to verify that it commutes after pulling back along p', for if this holds for all smooth (not necessarily surjective) morphisms $Y \to \mathscr{Y}$ of constant relative dimension, then there exists a smooth surjection $X \to \mathscr{X}$ such that it holds after pulling back to X. Now consider the diagram

$$\begin{tikzcd}
g'^*p^*f^*F \arrow[r, "\widetilde{\mathrm{tr}}_{g'}"] \arrow[d, "\simeq"] & g'^!p^*f^*F \arrow[d, "\simeq"] \arrow[ddr, "\widetilde{\mathrm{tr}}_{f'}"] & \\
p'^*g^*f^*F \arrow[r, "\widetilde{\mathrm{tr}}_g"] \arrow[dd, "\simeq"] & p'^*g^!f^*F \arrow[d, "\widetilde{\mathrm{tr}}_f"] & \\
 & p'^*g^!f^!F \arrow[r, "\simeq"] \arrow[d, "\simeq"] & g'^!p^!f^!F(-d)[-2d] \\
p'^*h^*F \arrow[r, "\widetilde{\mathrm{tr}}_h"] \arrow[urr, bend right, "\widetilde{\mathrm{tr}}_{h'}"'] & p'^*h^!F. \arrow[ur, "\simeq"] &
\end{tikzcd}$$

Again it follows from the construction of the trace map that all the small inside diagrams commute except possibly for the bottom-left pentagon and the big outside diagram commutes. This reduces the proof to the case when X is an algebraic space and Y and Z are schemes. To verify that 4.11.1 commutes in this case, we may work étale locally on X which finally reduces to the case of X, Y, and Z all schemes which follows from [2, XVIII, 2.9 (Var 3)]. This completes the verification of property (iii).

As we now explain, the verification of property (ii) proceeds using a similar reduction to the case of schemes, and the observation that (ii) holds by the construction of the trace map in the case when $\mathscr{Y}' \to \mathscr{Y}$ is a smooth morphism.

Reduction to the case when $\mathscr{Y}$ is a scheme. Let $p : Y \to \mathscr{Y}$ be a smooth surjection with Y a scheme, and set $Y' := \mathscr{Y}' \times_{\mathscr{Y}} Y$, $\mathscr{X}_Y := \mathscr{X} \times_{\mathscr{Y}} Y$, and $\mathscr{X}'_{Y'} := \mathscr{X}' \times_{\mathscr{Y}'} Y'$ so we have a commutative diagram

$$\begin{tikzcd}
 & \mathscr{X}'_{Y'} \arrow[rr, "h'"] \arrow[dl, "q'"'] \arrow[dd, "w'"] & & \mathscr{X}_Y \arrow[dl, "q"] \arrow[dd, "w"] \\
\mathscr{X}' \arrow[rr, "g'"] \arrow[dd, "f'"] & & \mathscr{X} \arrow[dd, "f"'] & \\
 & Y' \arrow[rr, "h"] \arrow[dl, "p'"'] & & Y \arrow[dl, "p"] \\
\mathscr{Y}' \arrow[rr, "g"] & & \mathscr{Y}. &
\end{tikzcd}$$

We then obtain a diagram

$$\begin{tikzcd}
 & p'^*f'_!f'^*g^*F \arrow[r, "\simeq"] \arrow[dl, "\simeq"'] \arrow[d, "\mathrm{tr}_{f'}"] & p'^*f'_!g'^*f^*F \arrow[d, "bc"] \\
w'_!w'^*p'^*g^*F \arrow[r, "\mathrm{tr}_{w'}"] \arrow[d, "\simeq"] & p'^*g^*F \arrow[d, "\simeq"] & p'^*g^*f_!f^*F \arrow[l, "\mathrm{tr}_f"'] \arrow[d, "\simeq"] \\
w'_!w'^*h^*p^*F \arrow[r, "\mathrm{tr}_{w'}"] \arrow[dr, leftrightarrow, "\simeq", "bc"'] & h^*p^*F & h^*p^*f_!f^*F \arrow[l, "\mathrm{tr}_f"] \arrow[dl, "bc"] \\
 & h^*w_!w^*p^*F, \arrow[u, "\mathrm{tr}_w"] &
\end{tikzcd}$$

where the morphisms labelled "*bc*" are base change isomorphisms.

As usual to verify the commutativity of 4.1.2 it suffices to verify that it commutes after applying p'^*, and therefore it suffices to show that the top right square in the preceding diagram commutes.

It is clear that all the small inside diagrams commute except possibly the top right square (whose commutativity we are trying to verify), and the bottom left triangle. Since the big outside diagram commutes by associativity of the base change isomorphism, it therefore suffices to verify that the bottom left triangle commutes, which reduces the proof to the case when $\mathscr{Y} = Y$ is a scheme. We assume this henceforth.

Reduction to the case when $\mathscr{Y}'$ is a scheme. Let $p : Y' \to \mathscr{Y}'$ be a smooth surjection with Y' a scheme, and set $\mathscr{X}_{Y'} := \mathscr{X}' \times_{\mathscr{Y}'} Y'$ so we have a commutative diagram

$$\begin{tikzcd}
\mathscr{X}_{Y'} \arrow[r, "p'"] \arrow[d, "f''"] & \mathscr{X}' \arrow[r, "g'"] \arrow[d, "f'"] & \mathscr{X} \arrow[d, "f"] \\
Y' \arrow[r, "p"] & \mathscr{Y}' \arrow[r, "g"] & Y.
\end{tikzcd}$$

We then obtain a diagram

$$\begin{tikzcd}
p^*f'_!f'^*g^*F \arrow[r, "\simeq"] \arrow[d, "\mathrm{tr}_{f'}"] \arrow[dd, bend right=60, "\simeq"'] & p^*f'_!g'^*f^*F \arrow[d, "bc"] \\
p^*g^*F & p^*g^*f_!f^*F \arrow[l, "\mathrm{tr}_f"'] \\
f''_!f''^*p^*g^*F, \arrow[u, "\mathrm{tr}_{f''}"] \arrow[ur, "bc"] &
\end{tikzcd}$$

where as before we denote a base change isomorphism by "*bc*". The big outside pentagon commutes by the associativity of the base change isomorphisms, and therefore to verify that the top square commutes it suffices to verify that the bottom triangle commutes. This then reduces the proof to the case when $\mathscr{Y}' = Y'$ is also a scheme.

Reduction to the case of schemes. Finally one reduces to the case when $\mathscr{X}$ (and hence also $\mathscr{X}'$) is a scheme, by the same argument used in the proof of 4.10. This completes the proof that the maps tr_f we have constructed satisfy properties (i)-(iv).

The uniqueness of the maps tr_f follows from the vanishing of the negative $\mathscr{E}xt$-groups as in the beginning of the proof of 4.1 which reduces the proof of the uniqueness to the case of schemes.

4.12. *Proof of uniqueness.* By 4.5, the trace map $\mathrm{tr}_f : f_!f^*F \to F$ is determined by the map $R^0f_!f^*F \to F$. Therefore the trace map is determined by its restriction to a smooth cover of $\mathscr{Y}$. We may therefore assume that $\mathscr{Y}$ is a scheme and there exists a smooth surjection $p : W \to \mathscr{X}$ of pure relative dimension d. In this case the uniqueness follows from 4.8.

This completes the proof of 4.1. ☐

4.13. Theorem 4.1 can be generalized to complexes $F \in D_c^-(\mathscr{Y}, \Lambda)$ as follows. The map

$$\mathrm{tr}_f : f_!\Lambda \to \Lambda$$

defines by duality a morphism

$$\Omega_{\mathscr{Y}} \to f_*\Omega_{\mathscr{X}},$$

which by adjunction corresponds to a morphism

$$\mathrm{tr}_f^t : f^*\Omega_{\mathscr{Y}} \to \Omega_{\mathscr{X}}.$$

For $F \in D_c^-(\mathscr{Y}, \Lambda)$, we therefore get a map

$$f^*D_{\mathscr{Y}}(F) \xrightarrow{\alpha} R\mathscr{H}om(f^*F, f^*\Omega_{\mathscr{Y}}) \xrightarrow{\mathrm{tr}_f^t} R\mathscr{H}om(f^*F, \Omega_{\mathscr{X}}) = D_{\mathscr{X}}(f^*F), \tag{4.13.1}$$

where the map α is the canonical map

$$f^*R\mathscr{H}om(F, \Omega_{\mathscr{Y}}) \to R\mathscr{H}om(f^*F, f^*\Omega_{\mathscr{Y}}).$$

By adjunction this defines a morphism

$$D_{\mathscr{Y}}(F) \to f_*D_{\mathscr{X}}(f^*F),$$

and we define

$$\mathrm{tr}_f : f_!f^*F \to F$$

to be the map obtained by applying $D_{\mathscr{Y}}$.

4.14. In the case of a constructible sheaf F on $\mathscr{Y}$, we then have two possible definitions of a trace map $f_!f^*F \to F$. Let tr_f (resp. tr_f^*) denote the map in 4.1 (resp. 4.13).

Proposition 4.15. *The maps* tr_f *and* tr_f^* *are equal.*

Proof. By 4.5 the assertion is local on $\mathscr{Y}$ in the smooth topology, so we may assume that $\mathscr{Y} = Y$ is a scheme, and there exists a smooth surjection $p : W \to \mathscr{X}$ of constant fiber dimension d. Let $\psi_f : f^*D_Y(-) \to D_{\mathscr{X}}f^*(-)$ be the transformation 4.13.1. By construction, the adjoint map $\widetilde{\mathrm{tr}}_f^* : f^*F \to f^!F$ is the composite

$$f^*F \xrightarrow{\simeq} f^*D_Y^2(F) \xrightarrow{\psi_f} D_{\mathscr{X}}f^*D_Y(F) = f^!F. \tag{4.15.1}$$

Let $g : W \to Y$ denote the composite fp. By the same argument as in 4.13, the trace map $\mathrm{tr} : g_!g^*\Lambda(d)[2d] \to \Lambda$ defined in [2, XVIII.2.9] induces a natural transformation

$$\psi_g : g^*D_Y(-) \to D_Wg^*(-)(-d)[-2d].$$

By 4.1 (iv), the diagram

$$\begin{tikzcd}
p^*f^*\Lambda \arrow[r, "\widetilde{\mathrm{tr}}_f"] \arrow[d, "\simeq"'] & p^*f^!\Lambda \arrow[r, "\gamma"] & p^!f^!\Lambda(-d)[-2d] \arrow[d, "\simeq"] \\
g^*\Lambda \arrow[rr, "\widetilde{\mathrm{tr}}_g"] & & g^!\Lambda(-d)[-2d]
\end{tikzcd}$$

commutes, where the map γ is induced by the isomorphism $p^! = p^*(d)[2d]$. From this it follows that $p^*\widetilde{\mathrm{tr}}_f^*$, which we view as a map $g^*F \to g^!F(-d)[-2d]$ is equal to the composite

$$g^*F \xrightarrow{\simeq} g^*D_Y^2F \xrightarrow{\psi_g} D_Wg^*D_Y(W)(-d)[-2d] = g^!F(-d)[-2d].$$

On the other hand, the map $g^*F \to g^!F(-d)[-2d]$ obtained by pullback back $\widetilde{\mathrm{tr}}_f$ is by the construction of $\widetilde{\mathrm{tr}}_f$ equal to $\widetilde{\mathrm{tr}}_g$. Proposition 4.15 therefore follows from the following result. ☐

Lemma 4.16. *Let $g : W \to Y$ be a flat morphism of schemes of finite type over S and with constant fiber dimension d. Then for any constructible sheaf F on Y the map $\widetilde{\mathrm{tr}}_g : g^*F \to g^!F(-d)[-2d]$ is equal to the composite*

$$g^*F \xrightarrow{\simeq} g^*D_Y^2F \xrightarrow{\psi_g} D_Wg^*D_YF(-d)[-2d] \xrightarrow{\simeq} g^!F(-d)[-2d].$$

Proof. Using the argument of [2, XVIII, proof of 2.9 (c) and (d)], we may assume there exists a factorization of g

$$W \xrightarrow{a} \mathbb{A}^d_Y \xrightarrow{b} Y,$$

where a is quasi-finite and flat and b is the projection.

By [2, XVIII, 2.9 (Var 3)], the diagram

$$\begin{tikzcd}
g^*\Lambda \arrow[rr, "\widetilde{\mathrm{tr}}_g"] \arrow[d, "\simeq"] & & g^!\Lambda(-d)[-2d] \arrow[d, "\simeq"] \\
a^*b^*\Lambda \arrow[r, "\widetilde{\mathrm{tr}}_b"] & a^*b^!\Lambda(-d)[-2d] \arrow[r, "\widetilde{\mathrm{tr}}_a"] & a^!b^!\Lambda(-d)[-2d]
\end{tikzcd}$$

commutes. This implies that the diagram

$$\begin{tikzcd}
g^*D_Y \arrow[rr, "\psi_g"] \arrow[d, "\simeq"'] & & D_Wg^*(-d)[-2d] \arrow[d, "\simeq"] \\
a^*b^*D_Y \arrow[r, "\psi_b"] & a^*D_{\mathbb{A}^d_Y}b^*(-d)[-2d] \arrow[r, "\psi_a"] & D_Wa^*b^*(-d)[-2d]
\end{tikzcd}$$

commutes. From this it follows that it suffices to prove the lemma for the morphisms a and b.

The statement for b is immediate.

Thus we are reduced to proving the lemma in the case when g is also quasi-finite. In this case, let k be a separably closed field and $\bar{y} : \mathrm{Spec}(k) \to Y$ a geometric point. Then it follows from the constructions that both tr_g and tr^*_g induce the summation map

$$\oplus_{\bar{w}} F_{\bar{y}} \simeq g_! g^*(F)_{\bar{y}} \to F_{\bar{y}},$$

where the sum is taken over liftings $\bar{w} : \mathrm{Spec}(k) \to W$ of $\bar{y}$. □

5. Comparison of $f_!$ and f_*

Fix an admissible pair (S, Λ). Throughout this section we work with stacks over S and Λ-coefficients.

The main result of this section is the following:

Theorem 5.1. *(i) Let $f : \mathscr{X} \to \mathscr{Y}$ be a quasi-finite morphism of algebraic stacks with $\Delta : \mathscr{X} \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ finite. Then for any constructible sheaf of Λ-modules F on $\mathscr{X}$, there is a canonical morphism*

$$\epsilon_F : f_! F \to f_* F. \tag{5.1.1}$$

(ii) Assume further that f is proper and that one of the following conditions hold:

(a) $\Lambda = \mathbb{Q}_\ell$,

(b) *For every algebraically closed field Ω and point $x : \mathrm{Spec}(\Omega) \to \mathscr{X}$ with image $y \in \mathscr{Y}(\Omega)$ the étale part of the group scheme (finite over Ω since the diagonal of f is finite)*

$$G := \mathrm{Ker}(\underline{\mathrm{Aut}}_{\mathscr{X}}(x) \to \underline{\mathrm{Aut}}_{\mathscr{Y}}(y))$$

has order invertible in k.

Then the map $\epsilon_F : f_! F \to f_* F$ is an isomorphism.

Remark 5.2. If we wish to emphasize the morphism f we write ϵ^f_F for ϵ_F.

Proof of 5.1. Giving the map 5.1.1 is equivalent to giving a map $F \to f^!(f_* F)$. Since $\mathscr{H}^i(f^! f_* F) = 0$ for $i < 0$ by 4.6, this is in turn equivalent to a morphism $F \to \mathscr{H}^0(f^! f_* F)$.

Consider first the case when $\mathscr{Y}$ is a quasi-compact scheme. By Chow's lemma [16, 1.1] there exists a proper surjection $p : Z \to \mathscr{X}$ with Z a scheme. Let W denote $Z \times_{\mathscr{X}} Z$ and let $q : W \to \mathscr{X}$ be the projection. Define sheaves

$$H := R^0 p_* p^* F, \quad G := R^0 q_* q^* F.$$

We then have an exact sequence on $\mathscr{X}$

$$0 \to F \to H \to G.$$

This sequence induces an exact sequence

$$0 \to R^0 f_* F \to R^0 f_* H \to R^0 f_* G$$

and then an exact sequence (using 4.3)

$$0 \to \mathscr{H}^0(f^! R^0 f_* F) \to \mathscr{H}^0(f^! R^0 f_* H) \to \mathscr{H}^0(f^! R^0 f_* G).$$

Since

$$\mathscr{H}^0(f^!R^0f_*H) \simeq \mathscr{H}^0(f^!f_*p_*p^*F), \quad \mathscr{H}^0(f^!R^0f_*G) \simeq \mathscr{H}^0(f^!f_*q_*q^*F),$$

we obtain an identification

$$\mathscr{H}^0(f^!f_*F) \simeq \mathrm{Ker}(\mathscr{H}^0(f^!f_*p_*p^*F) \to \mathscr{H}^0(f^!f_*q_*q^*F)).$$

Therefore to define a map $s : F \to f^!f_*F$ it suffices to define a map $s_p : F \to f^!f_*p_*p^*F$ such that the two maps

$$F \to f^!f_*q_*q^*F$$

obtained by composing s_p with the two pullbacks

$$\mathrm{pr}_i^* : f^!f_*p_*p^*F \to f^!f_*q_*q^*F$$

are equal. We claim that the map $s_p : F \to f^!f_*p_*p^*F$ defined as the composite

$$F \longrightarrow p_*p^*F \xrightarrow{\simeq} p_!p^*F \xrightarrow{a} f^!f_*p_*p^*F, \tag{5.2.1}$$

has this property, where the canonical ismorphism $p_*p^*F \simeq p_!p^*F$ is by [13, 5.2.1] using the fact that p is proper and representable, and the map a is the map obtained by adjunction from the map

$$f_!p_!p^*F \simeq (f \circ p)_!p^*F \to (f \circ p)_*p^*F \simeq f_*p_*p^*F,$$

where the map $(fp)_! \to (fp)_*$ is the natural map defined for morphisms of schemes.

To see this note that by a similar construction there is a canonical map $s_q : F \to f^!f_*q_*q^*F$ defined as the composite

$$F \longrightarrow q_*q^*F \xrightarrow{\simeq} q_!q^*F \xrightarrow{b} f^!f_*q_*q^*F.$$

Lemma 5.3. *For $i = 1, 2$ the composite map*

$$F \xrightarrow{s_p} f^!f_*p_*p^*F \xrightarrow{\mathrm{pr}_i^*} f^!f_*q_*q^*F$$

is equal to s_q.

Proof. Let $\rho : p^*F \to \mathrm{pr}_{i*}q^*F$ be the morphism induced by adjunction. To prove the lemma it suffices to show that the following diagram commutes

$$\begin{tikzcd}
F \arrow[r] \arrow[rdd] \arrow[rd, phantom, "①"] & p_*p^*F \arrow[r, "\simeq"] \arrow[dd, "\rho"] \arrow[rd, phantom, "②"] & p_!p^*F \arrow[r, "a"'] \arrow[d, "\rho"] \arrow[rd, phantom, "③"] & f^!f_*p_*p^*F \arrow[d, "\rho"] \\
 & & p_!\mathrm{pr}_{i*}q^*F \arrow[r, "a"'] \arrow[d, "\simeq"] \arrow[rd, phantom, "④"] & f^!f_*p_*\mathrm{pr}_{i*}q^*F \arrow[d, "\simeq"] \\
 & q_*q^*F \arrow[r, "\simeq"] & q_!q^*F \arrow[r, "b"'] & f^!f_*q_*q^*F.
\end{tikzcd}$$

The small inside diagrams 1 and 3 clearly commute. The commutativity of 2 follows by noting that we have commutative diagrams of functors

$$\begin{tikzcd}
p_*p^* \arrow[r, "p_!\simeq p_*"] \arrow[d, "\rho"] & p_!p^* \arrow[d, "\rho"] \\
p_*\mathrm{pr}_{i*}\mathrm{pr}_i^*p^* \arrow[r, "p_!\simeq p_*"] & p_!\mathrm{pr}_{i*}\mathrm{pr}_i^*p^*,
\end{tikzcd}$$

and

$$\begin{tikzcd}
p_*\mathrm{pr}_{i*} \arrow[r, "p_!\simeq p_*"] \arrow[d, "\simeq"] & p_!\mathrm{pr}_{i*} \arrow[r, "\mathrm{pr}_{i*}\simeq\mathrm{pr}_{i!}"] & p_!\mathrm{pr}_{i!} \arrow[d, "\simeq"] \\
q_* \arrow[rr, "q_*\simeq q_!"] & & q_!.
\end{tikzcd}$$

The commutativity of 4 follows from noting that the diagram of functors

$$\begin{tikzcd}
f_!p_!\mathrm{pr}_{i*} \arrow[r, "f_!p_!\to f_*p_*"] \arrow[d, "\mathrm{pr}_{i*}\simeq\mathrm{pr}_{i!}"] & f_*p_*\mathrm{pr}_{i*} \arrow[d, "\simeq"] \\
f_!q_! \arrow[r, "f_!q_!\to f_*q_*"] & f_*q_*
\end{tikzcd}$$

commutes. □

Let $s_F : F \to f^!f_*F$ be the resulting morphism, and let $\epsilon_F : f_!F \to f_*F$ be the map obtained by adjunction.

In order to define ϵ_F in the case when $\mathscr{Y}$ is a stack, we need the following lemma which we will generalize in 5.5 below.

Lemma 5.4. *Let $g : \mathscr{Y}' \to \mathscr{Y}$ be a smooth morphism of schemes, let $\mathscr{X}'$ denote $\mathscr{Y}' \times_{\mathscr{Y}} \mathscr{X}$ so we have a cartesian square*

$$\begin{tikzcd}
\mathscr{X}' \arrow[r, "g'"] \arrow[d, "f'"] & \mathscr{X} \arrow[d, "f"] \\
\mathscr{Y}' \arrow[r, "g"] & \mathscr{Y}.
\end{tikzcd}$$

Then for any constructible sheaf of Λ-modules F on $\mathscr{X}$ the diagram

$$\begin{tikzcd}
f'_!g'^*F \arrow[d, "\epsilon_{g'^*F}"'] & g^*f_!F \arrow[l, "\alpha"'] \arrow[d, "g^*\epsilon_F"] \\
f'_*g'^*F & g^*f_*F \arrow[l, "\beta"']
\end{tikzcd}$$

commutes, where the morphisms α and β are the base change isomorphisms.

Proof. Consider the diagram

$$\begin{tikzcd}
 & g'^* f^! f_! F \arrow[dr, "\simeq"] \arrow[ddddr, bend left=60, "c"] & \\
g'^* F \arrow[ur, "\mathrm{can}"] \arrow[d, "s_{g'^* F}"] \arrow[dd, bend right=60, "c'"'] & & f'^! g^* f_! F \arrow[d, "\epsilon_F"] \\
f'^! f'_* g'^* F \arrow[d, "\mathrm{id} \to p'_* p'^*"] & & f'^! g^* f_* F \arrow[ll, "\mathrm{can}"'] \arrow[d, "\mathrm{id} \to p_* p^*"'] \\
f'^! f'_* p'_* p'^* g'^* F & & g'^* f^! f_* p_* p^* F, \arrow[ll, "\mathrm{can}"]
\end{tikzcd}$$

where the map c is the composite

$$f^! f_! F \xrightarrow{5.2.1} f^! f_! f^! f_* p_* p^* F \xrightarrow{f_! f^! \to \mathrm{id}} f^! f_* p_* p^* F,$$

and c' is the map defined as in 5.2.1.

The small diagrams on the sides commute by definition of $s_{g'^* F}$ and ϵ_F, and the bottom square clearly commutes. To prove the lemma it suffices by adjunction to verify that the top inside pentagon commutes, and since

$$\mathscr{H}^0(f'^! f'_* g'^* F) \to \mathscr{H}^0(f'^! f'_* p'_* p'^* g'^* F)$$

is injective, it therefore suffices to verify that the big outside diagram commutes. This reduces the proof to the case when $\mathscr{X}$ is a scheme, where the result is classical. □

This enables us to define the morphism $f_! F \to f_* F$ for general f and F a constructible sheaf. Indeed as noted earlier such a morphism is specified by a morphism of sheaves $F \to \mathscr{H}^0(f^! f_* F)$, and so it suffices to construct a morphism $f_! F \to f_* F$ locally in the smooth topology which is compatible with base change. We therefore have maps $\epsilon_F : f_! F \to f_* F$ and $s_F : F \to f^! f_* F$ also for general morphisms f and constructible sheaves F.

Lemma 5.5. *Let $g : \mathscr{Y}' \to \mathscr{Y}$ be a morphism of algebraic stacks, and let $\mathscr{X}'$ denote $\mathscr{Y}' \times_{\mathscr{Y}} \mathscr{X}$ so we have a cartesian square*

$$\begin{tikzcd}
\mathscr{X}' \arrow[r, "g'"] \arrow[d, "f'"] & \mathscr{X} \arrow[d, "f"] \\
\mathscr{Y}' \arrow[r, "g"] & \mathscr{Y}.
\end{tikzcd}$$

Then for any constructible sheaf of Λ-modules F on $\mathscr{X}$ the diagram

(5.5.1)
$$\begin{tikzcd}
R^0 f'_! g'^* F \arrow[d, "\epsilon_{g'^* F}"'] & g^* R^0 f_! F \arrow[l, "\alpha"'] \arrow[d, "g^* \epsilon_F"] \\
R^0 f'_* g'^* F & g^* R^0 f_* F \arrow[l, "\beta"']
\end{tikzcd}$$

commutes, where the morphisms α and β are the base change morphisms.

Proof. As usual, we may work locally in the lisse-étale topology on both $\mathscr{Y}$ and $\mathscr{Y}'$. We may therefore assume that both $\mathscr{Y}$ and $\mathscr{Y}'$ are schemes, and that there exists a proper surjection $p : Z \to \mathscr{X}$. Let $p' : Z' \to \mathscr{X}'$ be the pullback of p so there is a commutative diagram

$$\begin{tikzcd}
Z' \arrow[r, "g''"] \arrow[d, "p'"] & Z \arrow[d, "p"] \\
\mathscr{X}' \arrow[r, "g'"] \arrow[d, "f'"] & \mathscr{X} \arrow[d, "f"] \\
\mathscr{Y}' \arrow[r, "g"] & \mathscr{Y}.
\end{tikzcd}$$

Consider the diagram

(5.5.2)
$$\begin{tikzcd}
 & f'_! g'^* F \arrow[dl, "\mathrm{id}\to p'_*p'^*"'] \arrow[d, "\epsilon_{g'^*F}"] & g^* f_! F \arrow[l, "\alpha"] \arrow[d, "\epsilon_F"] \arrow[dr, "\mathrm{id}\to p_*p^*"] & \\
f'_! p'_* p'^* g'^* F \arrow[d, "\simeq"] & f'_* g'^* F \arrow[dd, "\mathrm{id}\to p'_*p'^*"] & g^* f_* F \arrow[l, "\beta"'] \arrow[dd, "\mathrm{id}\to p_*p^*"'] & g^* f_! p_* p^* F \arrow[d, "\simeq"] \\
f'_! p'_! p'^* g'^* F \arrow[dr, "\epsilon_{(p'g')^*F}"'] & & & g^* f_! p_! p^* F \arrow[lll, "\gamma"] \arrow[dl, "\epsilon_{p^*F}"] \\
 & f'_* p'_* p'^* g'^* F & g^* f_* p_* p^* F, \arrow[l, "\delta"] &
\end{tikzcd}$$

where the square

$$\begin{tikzcd}
f'_! p'_! p'^* g'^* F \arrow[d, "\epsilon_{(p'g')^*F}"'] & g^* f_! p_! p^* F \arrow[l, "\gamma"'] \arrow[d, "\epsilon_{p^*F}"] \\
f'_* p'_* p'^* g'^* F & g^* f_* p_* p^* F, \arrow[l, "\delta"]
\end{tikzcd}$$

is the analog of 5.5.1 for the cartesian diagram

$$\begin{tikzcd}
Z' \arrow[r, "g''"] \arrow[d, "f'p'"'] & Z \arrow[d, "fp"] \\
\mathscr{Y}' \arrow[r, "g"] & \mathscr{Y}.
\end{tikzcd}$$

By the case of schemes and F concentrated in degree 0 this diagram commutes. Furthermore, all the small inside diagrams in 5.5.2 clearly commute except the top center square whose commutativity we are trying to verify. Since the map

$$\mathscr{H}^0(f'_* g'^* F) \to \mathscr{H}^0(f'_* p'_* p'^* g'^*)$$

is injective, this implies that this top center square also commutes which proves the lemma. □

Lemma 5.6. *Let*

$$\mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z}$$

be a diagram of algebraic stacks and let F be a constructible sheaf on $\mathscr{X}$. Assume that f and g are quasi-finite with finite diagonals. Then the diagram

(5.6.1)
$$\begin{tikzcd}
g_!f_!F \arrow[r, "\epsilon^f_F"] \arrow[d, "\simeq"] & g_!f_*F \arrow[r, "\epsilon^g_{f_*F}"] & g_*f_*F \arrow[d, "\simeq"] \\
(gf)_!F \arrow[rr, "\epsilon^{gf}_F"] & & (gf)_*F
\end{tikzcd}$$

commutes.

Proof. As usual it suffices to show the result locally on $\mathscr{Z}$. We may therefore assume that $\mathscr{Z}$ is a quasi-compact scheme and that there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, "q"] & Y \arrow[d, "p"] \arrow[dr, "g'"] & \\
\mathscr{X} \arrow[r, "f"] & \mathscr{Y} \arrow[r, "g"] & Z,
\end{tikzcd}$$

where p and q are proper surjections and X and Y are schemes. By the construction of the maps ϵ^f_F and the case of schemes the following diagram commutes

$$\begin{tikzcd}
g_!f_!F \arrow[r, "\epsilon^f_F"] \arrow[d, "\mathrm{id}\to q_*q^*"] & g_!f_*F \arrow[r, "\epsilon^g_{f_*F}"] \arrow[d, "\mathrm{id}\to q_*q^*"] & g_*f_*F \arrow[d, "\mathrm{id}\to q_*q^*"] \\
g_!f_!q_*q^*F \arrow[r, "\epsilon^f"] \arrow[d, "q_*\simeq q_!"] & g_!f_*q_*q^*F \arrow[r, "\epsilon^g_{f_*q_*q^*F}"] \arrow[d, "\mathrm{id}\to p_*p^*"] & g_*f_*q_*q^*F \arrow[d, "\mathrm{id}\to p_*p^*"'] \arrow[dd, bend left=60, "f_*q_*\simeq p_*f'_*"] \\
g_!p_*f'_!q^*F \arrow[dr, "\epsilon^{f'}_{q^*F}"'] \arrow[drr, bend right=50, "\epsilon^{g'f'}_{q^*F}"'] & g_!p_*p^*f_*q_*q^*F \arrow[r, "\epsilon^g_{p_*p^*f_*q_*q^*F}"] \arrow[d, "\rho"] & g_*p_*p^*f_*q_*q^*F \arrow[d, "\rho"] \\
 & g_!p_*f'_*q^*F \arrow[r, "\epsilon^g_{p_*f'_*q^*F}"] & g_*p_*f'_*q^*F,
\end{tikzcd}$$

where $\rho : p^*f_*q_* \to f'_*$ is the morphism of functors induced by adjunction from the natural isomorphism $f_*q_* \simeq p_*f'_*$. Now to verify that the diagram 5.6.1 commutes it suffices to show that the composite map

$$g_!f_!F \xrightarrow{\epsilon^f_F} g_!f_*F \xrightarrow{\epsilon^g_{f_*F}} g_*f_*F \xrightarrow{\mathrm{id}\to q_*q^*} g_*f_*q_*q^*F \xrightarrow{f_*q_*\simeq p_*f'_*} g_*p_*f'_*q^*F$$

is equal to the composite

$$g_!f_!F \xrightarrow{\epsilon^{gf}_F} g_*f_*F \xrightarrow{\mathrm{id}\to q_*q^*} g_*f_*q_*q^*F \xrightarrow{f_*q_*\simeq p_*f'_*} g_*p_*f'_*q^*F.$$

This follows from noting that by the construction of ϵ^{gf}_F the following diagram commutes

$$\begin{tikzcd}
g_!f_!F \arrow[r, "\epsilon^{gf}_F"] \arrow[d, "\mathrm{id}\to q_*q^*"'] & g_*f_*F \arrow[r, "\mathrm{id}\to q_*q^*"] & g_*f_*q_*q^*F \arrow[r, "f_*q_*\simeq p_*f'_*"] & g_*p_*f'_*q^*F \\
g_!f_!q_*q^*F \arrow[r, "f_!q_*\simeq p_*f'_!"] & g_!p_*f'_!q^*F. \arrow[urr, "\epsilon^{g'f'}_{q^*F}"'] & &
\end{tikzcd}$$

$\square$

Corollary 5.7. *Let $g : \mathscr{X}' \to \mathscr{X}$ be a universal homeomorphism, and let $f' : \mathscr{X}' \to \mathscr{Y}$ denote the composite fg. Then for a constructible sheaf of Λ-modules F on $\mathscr{X}$ the map $\epsilon^f_F : f_!F \to f_*F$ is an isomorphism if and only if the map $\epsilon^{f'}_{g^*F} : f'_!g^*F \to f'_*g^*F$ is an isomorphism.*

Proof. Indeed by 5.6 there is a commutative diagram

$$\begin{tikzcd}
 & f_!F \arrow[r, "\epsilon^f_F"] \arrow[d, "\mathrm{id}\to g_*g^*"] & f_*F \arrow[d, "\mathrm{id}\to g_*g^*"] \\
f_!g_!g^*F \arrow[r, "\epsilon^g_{g^*F}"] \arrow[d, "\simeq"] & f_!g_*g^*F \arrow[r, "\epsilon^f_{g_*g^*F}"] & f_*g_*g^*F \arrow[d, "\simeq"] \\
f'_!g^*F \arrow[rr, "\epsilon^{f'}_{g^*F}"] & & f'_*g^*F,
\end{tikzcd}$$

where the vertical arrows as well as the map $\epsilon^g_{g^*F}$ are isomorphisms since g is a universal homeomorphism. $\square$

We use this to prove that in the case when f is also proper and one of the conditions in 5.1 (ii) hold the map $f_!F \to f_*F$ is an isomorphism. Using 5.5 and the base change theorems [13, 5.5.6] and [16, 1.3] to prove that the map $f_!F \to f_*F$ is an isomorphism it suffices to consider the case when $\mathscr{Y} = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field. We may also assume that $\mathscr{X}$ is connected.

Pick a section $s \in \mathscr{X}(k)$ and let G denote the finite automorphism group scheme of s. We then have a closed immersion $j : BG \hookrightarrow \mathscr{X}$ defined by a nilpotent ideal. By 5.7 it therefore further suffices to consider the case when $\mathscr{X} = BG$ for some finite k-group scheme G, which in the case of torsion coefficients has étale part of order invertible in k.

Let $G_{\mathrm{red}} \subset G$ denote the maximal reduced closed subscheme. Then G_{red} is an étale subgroup scheme, and the map $BG_{\mathrm{red}} \to BG$ is a universal homeomorphism. We can therefore further assume that G is equal to its maximal étale quotient. In this case if F is a sheaf on BG corresponding to a G-representation V, then we have canonical isomorphisms $f_!F \simeq V_H$ (coinvariants) and $f_*F \simeq V^H$ (invariants). We claim that with these identifications the map ϵ^f_F becomes identified with the map

$$\sum_{g\in G} : V_H \to V^H$$

induced by the map

$$V \to V^H, \quad v \mapsto \sum_{g \in G} g \cdot v.$$

This will certainly prove 5.1 (ii).

To see this let $p : \mathrm{Spec}(k) \to BG$ be the proper étale surjection defined by the trivial torsor. Then by definition of ϵ^f the following diagram commutes

$$\begin{tikzcd}
V_H \simeq f_!F \arrow[r, "\mathrm{id}\to p_*p^*"] \arrow[dd, "\epsilon_F"] & f_!p_*p^*F \arrow[r, "\simeq"] & f_!p_!p^*F \arrow[dr, "\simeq"] & \\
& & & V \\
V^H \simeq f_*F \arrow[r, "\mathrm{id}\to p_*p^*"] & f_*p_*p^*F. \arrow[urr, "\simeq"'] & &
\end{tikzcd}$$

Now the representation of G corresponding to the sheaf p_*p^*F is the product $\prod_{g\in G} V$ with action of $g_0 \in G$ given by $g_0 * (v_g) = (v_{g_0 g})$. The map $V \to \prod_{g\in G} V$ corresponding to the adjunction map is the map $v \mapsto (g \cdot v)$. It follows that the following diagram commutes

$$\begin{tikzcd}
V_H \arrow[r, "\sum_{g\in G}"] \arrow[d, "\epsilon_F"] & V \arrow[d, "\mathrm{id}"] \\
V^H \arrow[r, hookrightarrow] & V
\end{tikzcd}$$

which completes the proof of 5.1. □

Let us also note the following consequence of the proof:

Corollary 5.8. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a quasi-finite proper morphism of algebraic stacks with finite diagonal, and assume one of the conditions (a) or (b) in 5.1 (ii) hold. Then for any constructible sheaf F on $\mathscr{X}$ we have $R^q f_! F = 0$ for $q \neq 0$.*

Proof. This follows from observing that $f_*F \in D^{[0,\infty)}(\mathscr{X}, \Lambda)$ and $f_!F \in D^{(-\infty,0]}(\mathscr{X}, \Lambda)$. □

5.9. Let $f : \mathscr{X} \to \mathscr{Y}$ be a quasi-finite proper morphism of algebraic stacks with finite diagonal, and assume one of the conditions (a) or (b) in 5.1 (ii) hold. Taking $F = \Lambda$ in 5.1 we obtain an isomorphism

$$\epsilon_\Lambda^{-1} : f_*\Lambda \to f_!\Lambda,$$

which upon composition with the natural map $\Lambda \to f_*\Lambda$ defines a morphism

$$\tau : \Lambda \to f_!\Lambda.$$

Applying $D_{\mathscr{Y}}$ we obtain a morphism

$$\gamma_f : f_*\Omega_{\mathscr{X}} \to \Omega_{\mathscr{Y}}.$$

This defines a morphism of functors

$$\Gamma_f : f_*D_{\mathscr{X}}(-) \to D_{\mathscr{Y}}(f_*(-))$$

by taking the composite

$$f_*D_{\mathscr{X}}(-) \longrightarrow R\mathscr{H}om(f_*(-), f_*\Omega_{\mathscr{X}}) \xrightarrow{\gamma} R\mathscr{H}om(f_*(-), \Omega_{\mathscr{Y}}) = D_{\mathscr{Y}}(f_*(-)),$$

where the first map is the canonical morphism defined in A.6

$$f_*R\mathscr{H}om(-, \Omega_{\mathscr{X}}) \to R\mathscr{H}om(f_*(-), f_*\Omega_{\mathscr{X}}). \tag{5.9.1}$$

Applying $D_{\mathscr{Y}}$ to Γ_f we obtain for every $F \in D_c^-(\mathscr{X})$ a canonical morphism (if we want to emphasize the morphism f we sometimes also write λ_F^f)

$$\lambda_F : f_*F \to f_!F.$$

By the same argument, using [13, 5.2.2], if $g : \mathscr{Z} \to \mathscr{X}$ is a proper representable morphism then we have a morphism $\gamma_g : g_*\Omega_{\mathscr{Z}} \to \Omega_{\mathscr{X}}$ which induces a morphism of functors $\Gamma_g : g_*D_{\mathscr{Z}}(-) \to D_{\mathscr{X}}g_*(-)$.

Remark 5.10. In the preceding paragraph and in the discussion that follows, we use the results of appendix A extending f_* to the full unbounded derived category $D_c(\mathscr{X}, \Lambda)$.

Lemma 5.11. *Consider a composite*

$$\mathscr{Z} \xrightarrow{g} \mathscr{X} \xrightarrow{f} \mathscr{Y},$$

with g and gf proper and representable, and f proper and quasi-finite satisfying either (a) or (b) in 5.1 (ii). Then the diagram of functors $D_c^+(\mathscr{Z}) \to D_c^-(\mathscr{Y})$

(5.11.1)
$$\begin{tikzcd}
f_*g_*D_{\mathscr{Z}}(-) \arrow[r, "\Gamma_g"] \arrow[d, "\simeq"] & f_*D_{\mathscr{X}}(g_*(-)) \arrow[r, "\Gamma_f"] & D_{\mathscr{Y}}(f_*g_*(-)) \arrow[d, "\simeq"] \\
(fg)_*D_{\mathscr{Z}}(-) \arrow[rr, "\Gamma_{fg}"] & & D_{\mathscr{Y}}((fg)_*(-))
\end{tikzcd}$$

commutes.

Proof. Consider the diagram

$$\begin{tikzcd}
f_*g_*R\mathscr{H}om(-, \Omega_{\mathscr{Z}}) \arrow[r, "\text{can}"] \arrow[dr, "\text{can}"] \arrow[rr, bend left, "\Gamma_g", "\textcircled{1}"'] & f_*R\mathscr{H}om(g_*(-), g_*\Omega_{\mathscr{Z}}) \arrow[r, "\gamma_g", "\textcircled{2}"'] \arrow[d, "\text{can}", "\textcircled{5}"'] & f_*R\mathscr{H}om(g_*(-), \Omega_{\mathscr{Z}}) \arrow[d, "\text{can}"] \arrow[dd, bend left=60, "\Gamma_f", "\textcircled{3}"'] \\
& R\mathscr{H}om(f_*g_*(-), f_*g_*\Omega_{\mathscr{Z}}) \arrow[r, "\gamma_g"] \arrow[dr, "\gamma_{fg}"] & R\mathscr{H}om(f_*g_*(-), f_*\Omega_{\mathscr{X}}) \arrow[d, "\gamma_f", "\textcircled{4}"'] \\
& & R\mathscr{H}om(f_*g_*(-), \Omega_{\mathscr{Z}}),
\end{tikzcd}$$

where the arrows labelled "can" denote the canonical morphisms of functors as in 5.9.1. By construction the inside diagrams 1 and 3 commute, and the inside diagrams 2 and 5 clearly commute. Therefore it suffices to show that the inside diagram 4 commutes which is equivalent to the statement that the diagram

(5.11.2)
$$\begin{tikzcd}
f_*g_*\Omega_{\mathscr{X}} \arrow[r, "\gamma_g"] \arrow[d, "\simeq"] & f_*\Omega_{\mathscr{X}} \arrow[r, "\gamma_f"] & \Omega_{\mathscr{Y}} \\
(fg)_*\Omega_{\mathscr{X}} \arrow[urr, "\gamma_{fg}"'] & &
\end{tikzcd}$$

commutes. Let

$$\tilde{\gamma}_g : \Lambda \to g_!\Lambda \ \ (\text{resp. } \ \tilde{\gamma}_f : \Lambda \to f_!\Lambda)$$

denote the composite

$$\Lambda \longrightarrow g_*\Lambda \xrightarrow{\epsilon^{g-1}} g_!\Lambda \qquad (\text{resp.} \quad \Lambda \longrightarrow f_*\Lambda \xrightarrow{\epsilon^{f-1}} f_!\Lambda).$$

Then by duality to show that the diagram 5.11.2 commutes it suffices to show that the diagram

$$\begin{tikzcd}
\Lambda \arrow[r, "\tilde{\gamma}_f"] \arrow[drr, "\tilde{\gamma}_{fg}"'] & f_!\Lambda \arrow[r, "\tilde{\gamma}_g"] & f_!g_!\Lambda \arrow[d, "\simeq"] \\
& & (fg)_!\Lambda
\end{tikzcd}$$

commutes. This diagram fits into the larger diagram

$$\begin{tikzcd}
\Lambda \arrow[r] \arrow[dr] \arrow[rr, bend left, "\tilde{\gamma}_f"] & f_*\Lambda \arrow[r, "\epsilon_{\Lambda}^{f-1}"] \arrow[d] & f_!\Lambda \arrow[d] \arrow[dr, "f_!(\tilde{\gamma}_g)"] & \\
& f_*g_*\Lambda \arrow[r, "\epsilon_{g_*\Lambda}^{f-1}"] \arrow[rr, bend right, "\epsilon_{\Lambda}^{fg-1}"'] & f_!g_*\Lambda \arrow[r, "\epsilon_{\Lambda}^{g-1}"] & f_!g_!\Lambda.
\end{tikzcd}$$

From this it follows that it suffices to show that the diagram

$$\begin{tikzcd}
f_!g_!\Lambda \arrow[r, "\epsilon_{\Lambda}^{g}"] \arrow[d, "\simeq"] & f_!g_*\Lambda \arrow[r, "\epsilon_{g_*\Lambda}^{f}"] & f_*g_*\Lambda \arrow[d, "\simeq"] \\
(fg)_!\Lambda \arrow[rr, "\epsilon_{\Lambda}^{fg}"] & & (fg)_*\Lambda
\end{tikzcd}$$

commutes, and for this in turn it suffices to show that the diagram

$$\begin{tikzcd}
\Lambda \arrow[r, "s^g"] \arrow[drr, "s^{fg}"] & g^!g_*\Lambda \arrow[r, "s^f"] & g^!f^!f_*g_*\Lambda \arrow[d, "\simeq"] \\
& & (fg)^!(fg)_*\Lambda
\end{tikzcd}$$

commutes. This can be verified after making a smooth base change on $\mathscr{Y}$, so we may assume that $\mathscr{Y}$ is a scheme. The result in this case follows from the construction of the map s at the beginning of the proof of 5.1. □

Corollary 5.12. *With notation and assumptions as in 5.11, for any constructible sheaf of Λ-modules F on $\mathscr{Z}$ the diagram*

$$\begin{tikzcd}
f_*g_*F \arrow[r, "\lambda^g_F"] \arrow[d, "\simeq"] & f_*g_!F \arrow[r, "\lambda^f_{g_!F}"] & f_!g_!F \arrow[d, "\simeq"] \\
(fg)_*F \arrow[rr, "\lambda^{fg}_F"] & & (fg)_!F
\end{tikzcd}$$

commutes.

Proof. This follows from the commutativity of 5.11.1 applied to $D_{\mathscr{Z}}(F)$. ☐

Proposition 5.13. *If F is a constructible sheaf of Λ-modules on $\mathscr{X}$, then the map*

$$\lambda_F : f_*F \to f_!F$$

is equal to the inverse of ϵ_F.

Proof. Since both f_*F and $f_!F$ are concentrated in degree 0 (by 5.8), the assertion is local in the smooth topology on $\mathscr{Y}$, so we may assume that $\mathscr{Y} = Y$ is a scheme, and that there exists a proper surjection $p : Z \to \mathscr{X}$ with Z a scheme.

Consider the diagrams

$$\begin{tikzcd}
R^0f_*F \arrow[r, "\lambda^f_F"] \arrow[d, hook] & R^0f_!F \arrow[d, hook] \\
R^0f_*p_*p^*F \arrow[r, "\lambda^f_{p_*p^*F}"] \arrow[dr, "\lambda^{fp}_{p^*F}"'] & R^0f_!p_*p^*F \arrow[d, "\lambda^p_{p^*F}"] \\
& R^0f_!p_!p^*F
\end{tikzcd}
\qquad
\begin{tikzcd}
R^0f_*F \arrow[r, "\epsilon^{f-1}_F"] \arrow[d, hook] & R^0f_!F \arrow[d, hook] \\
R^0f_*p_*p^*F \arrow[r, "\epsilon^{f-1}_{p_*p^*F}"] \arrow[dr, "\epsilon^{fp-1}_{p^*F}"'] & R^0f_!p_*p^*F \arrow[d, "\epsilon^{p-1}_{p^*F}"] \\
& R^0f_!p_!p^*F.
\end{tikzcd}$$

The bottom triangles commute by 5.12 and 5.6 respectively, and the top squares clearly commute. Since all the vertical arrows are injections, it therefore suffices to show that the map

$$\lambda^{fp}_{p^*F} : R^0f_*p_*p^*F \to R^0f_!p_!p^*F$$

is equal to $\epsilon^{fp-1}_{p^*F}$. This follows from the following lemma. ☐

Lemma 5.14. *Let $f : X \to Y$ be a proper morphism of schemes, and let F be a constructible sheaf of Λ-modules on X. Then $\lambda_F : f_*F \to f_!F$ is equal to ϵ_F^{-1} (which by the construction is the classically defined isomorphism $f_!F \to f_*F$).*

Proof. The isomorphism ϵ_F is characterized by the fact that the diagram

$$\begin{tikzcd}
f_!(F) \otimes f_*D_X(F) \arrow[r, "P"] \arrow[d, "\epsilon_F\otimes 1"] & \Omega_Y \\
f_*F \otimes f_*D_X(F) \arrow[r, "Q"] & f_*\Omega_X \arrow[u, "\gamma_f"']
\end{tikzcd}$$

commutes, where P is the Poincaré duality pairing and Q is the canonical pairing. It therefore suffices to show that the diagram

$$\begin{tikzcd}
f_!(F) \otimes f_*D_X(F) \arrow[r, "P"] & \Omega_Y \\
f_*F \otimes f_*D_X(F) \arrow[r, "Q"] \arrow[u, "\lambda_F \otimes 1"] & f_*\Omega_X \arrow[u, "\gamma_f"']
\end{tikzcd}$$

commutes. This is equivalent to the statement that the diagram

$$\begin{tikzcd}
f_*F \arrow[rr, "\Gamma_f"] \arrow[d, "\mathrm{can}"] & & D_Y f_* D_X(F) \\
R\mathscr{H}om(f_*D_X(F), f_*\Omega_X) \arrow[urr, "\gamma_f"'] & &
\end{tikzcd}$$

commutes, which is immediate from the definition of Γ_f. ☐

Corollary 5.15. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a quasi-finite proper morphism of algebraic stacks with finite diagonal, and assume one of the conditions (a) or (b) in 5.1 (ii) hold. Then for any $F \in D_c^-(\mathscr{X}, \Lambda)$ the map*

$$\lambda_F : f_*F \to f_!F$$

is an isomorphism.

Proof. By a standard reduction using the distinguished triangles

$$\tau_{\leq n}F \to F \to \tau_{>n}F \to \tau_{\leq n}F[1]$$

it suffices to consider the case when F is a constructible sheaf, where the result follows from 5.13. ☐

Remark 5.16. In what follows, we write $\epsilon_F : f_!F \to f_*F$ for the inverse of λ_F. Proposition 5.13 ensures that this is consistent with our earlier notation.

Corollary 5.17. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a proper morphism with finite diagonal between algebraic stacks and assume one of the conditions (a) or (b) in 5.1 (ii) hold. Then for any $F \in D_c^-(\mathscr{X}, \Lambda)$ there is a canonical isomorphism $f_!F \to f_*F$.*

Proof. Let

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "\pi"] \arrow[rr, bend left, "f"] & \overline{\mathscr{X}} \arrow[r, "g"] & \mathscr{Y}
\end{tikzcd}$$

be the relative coarse moduli space of f, as defined for example in [1, §3]. Then π is proper and quasi-finite and satisfies one of the conditions in 5.1 (ii) and g is proper and representable. By 5.1 we have

$$\pi_!F \simeq \pi_*F,$$

and by the proper representable case [13, 5.2.1] we have $g_!\pi_*F \simeq g_*\pi_*F$. ☐

Finally let us discuss the connection with the trace map of the previous section.

Lemma 5.18. *Let*

$$\begin{tikzcd} X_0 \arrow[r, hook, "j"] \arrow[rd, "f_0"'] & X \arrow[d, "f"] \\ & S \end{tikzcd}$$

be a commutative diagram of schemes with f and f_0 flat with equidimensional fibers of dimension d, and j a closed immersion defined by a nilpotent ideal I such that for every $k \geq 1$ the $\mathscr{O}_{X_0}$-module I^k/I^{k+1} is locally free of constant rank. Let $\mathrm{gr}_I^*(\mathscr{O}_X)$ denote the locally free $\mathscr{O}_{X_0}$-module $\oplus_{k\geq 0}(I^k/I^{k+1})$. Then for any constructible sheaf of Λ-modules F on S the diagram

$$\begin{tikzcd} f_!f^*F(d)[2d] \arrow[r, "\mathrm{tr}_f"] \arrow[d, "\simeq"] & F \\ f_{0!}f_0^*F(d)[2d] \arrow[r, "\mathrm{tr}_{f_0}"] & F \arrow[u, "\cdot\mathrm{rank}(\mathrm{gr}_I^*(\mathscr{O}_X))"'] \end{tikzcd}$$

commutes, where $\mathrm{rank}(\mathrm{gr}_I^*(\mathscr{O}_X))$ *denotes the rank of* $\mathrm{gr}_I^*(\mathscr{O}_X)$ *as an* $\mathscr{O}_{X_0}$*-module.*

Proof. Let

$$\gamma : f^*D_S \rightarrow (D_Xf^*)(-d)[-2d], \quad \gamma_0 : f_0^*D_S \rightarrow (D_{X_0}f^*)(-d)[-2d]$$

be the maps defined as in 4.13.1 from the trace map for Λ. If we prove the lemma for $F = \Lambda$, then it follows that for general F the diagram

$$\begin{tikzcd} f_*D_Xf^*F(-d)[-2d] \arrow[d, "\simeq"] & f_*f^*D_SF \arrow[l, "\gamma"'] \\ f_{0*}D_{X_0}f_0^*F(-d)[-2d] & f_{0*}f_0^*D_SF \arrow[l, "\gamma_0"'] \arrow[u, "\cdot\mathrm{rank}(\mathrm{gr}_I^*(\mathscr{O}_X))"'] \end{tikzcd}$$

commutes. This in turn implies that the diagram

$$\begin{tikzcd} f_!f^*F(d)[2d] \arrow[r, equal] \arrow[d, "\simeq"] & D_Sf_*D_Xf^*F(d)[2d] \arrow[r, "\gamma"] \arrow[d, "\simeq"] & D_Sf_*f^*D_SF \arrow[r, "\mathrm{id}\rightarrow f_*f^*"] & D_S^2F \arrow[r, "\simeq"] & F \\ f_{0!}f_0^*F(d)[2d] \arrow[r, equal] & D_Sf_{0*}D_{X_0}f_0^*F(d)[2d] \arrow[r, "\gamma_0"] & D_Sf_{0*}f_0^*D_SF \arrow[r, "\mathrm{id}\rightarrow f_{0*}f_0^*"] \arrow[u, "\cdot\mathrm{rank}(\mathrm{gr}_I^*(\mathscr{O}_X))"'] & D_S^2F \arrow[r, "\simeq"] & F \arrow[u, "\cdot\mathrm{rank}(\mathrm{gr}_I^*(\mathscr{O}_X))"'] \end{tikzcd}$$

commutes, and by 4.15 the composite along the top row (resp. bottom row) is equal to tr_f (resp. tr_{f_0}).

It therefore suffices to consider the case when $F = \Lambda$.

Consider first the case when $S = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field and f (and hence also f_0) is finite. In this case X can be written as

$$X = \coprod_{x \in X_0(k)} \mathrm{Spec}(A^x),$$

where A^x is an artinian local k-algebra. Let $I^x \subset A^x$ denote the ideal defining $X_0 \times_X \mathrm{Spec}(A^x)$ so that

$$X_0 = \coprod_{x \in X_0(k)} \mathrm{Spec}(A^x/I^x).$$

Using [2, XVII, 6.2.3.1] to prove the result in this case it suffices to consider the case when $X = \mathrm{Spec}(A)$ has just one component. In this case by [2, XVII, 6.2.3 (Var 4)] the map

$$\Lambda = f_*\Lambda = f_!\Lambda \xrightarrow{\mathrm{tr}_f} \Lambda$$

is equal to multiplication by the length of A, and similarly for tr_{f_0}. Since

$$\mathrm{length}(A) = \mathrm{length}(\oplus_{k\geq 0} I^k/I^{k+1}) = \mathrm{length}(A_0)\cdot \mathrm{rank}(\mathrm{gr}_I^*(A)).$$

this implies the result in this case.

Since the verification of 5.18 can be made after making a base change $\bar{s} \to S$ with $\bar{s}$ the spectrum of an algebraically closed field, this also implies the lemma in the case when S is arbitrary but f is quasi-finite.

We now reduce the general case to this special case. By similar considerations to prove the general case it suffices to consider the case when $S = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field. If $U \subset X$ is a dense open subset and U_0 denotes $U \times_X X_0$ then we have a commutative diagram

$$\begin{tikzcd}
f_{U!}\Lambda_U \arrow[r] \arrow[d, "\simeq"] & f_!\Lambda \arrow[d, "\simeq"] \\
f_{U_0!}\Lambda_{U_0} \arrow[r] & f_{0!}\Lambda
\end{tikzcd}$$

where $f_U : U \to \mathrm{Spec}(k)$ (resp. $f_{U_0} : U_0 \to \mathrm{Spec}(k)$) denotes the restriction of f. By the same argument used in [2, XVIII, proof of 2.9 part (d)], to prove the lemma we may assume that there exists a commutative diagram over k

$$\begin{tikzcd}
X_0 \arrow[r, hook, "j"] \arrow[dr, "\sigma_0"] & X \arrow[d, "\sigma"] \\
 & \mathbb{A}^d_k,
\end{tikzcd}$$

where σ and σ_0 are quasi-finite and flat. By the construction of the trace map in [2, XVIII, proof of 2.9 (b)] it follows that it suffices to show that the diagram

$$\begin{tikzcd}
\sigma_!\Lambda \arrow[r, "\mathrm{tr}_\sigma"] \arrow[d, "\simeq"] & \Lambda \\
\sigma_{0!}\Lambda \arrow[r, "\mathrm{tr}_{\sigma_0}"] & \Lambda \arrow[u, "\cdot\mathrm{rank}(\mathrm{gr}_I^*(\mathscr{O}_X))"']
\end{tikzcd}$$

commutes, which follows from the quasi-finite case already considered. $\square$

Lemma 5.19. *Consider a commutative diagram of stacks*

$$\begin{tikzcd}
\mathscr{X}_0 \arrow[r, hook, "j"] \arrow[dr, "f_0"'] & \mathscr{X} \arrow[d, "f"] \\
 & \mathscr{Y},
\end{tikzcd}$$

where f and f_0 are quasi-finite and flat, and j is a closed defined by a nilpotent ideal $I \subset \mathscr{O}_{\mathscr{X}}$ such that $\mathrm{gr}^*_I(\mathscr{O}_{\mathscr{X}}) := \oplus_{k \geq 0} I^k/I^{k+1}$ is a locally free sheaf of finite rank on $\mathscr{X}_0$. Then for any constructible sheaf of Λ-modules F on $\mathscr{Y}$ the diagram

$$\begin{tikzcd}
f_!f^*F \arrow[r, "\mathrm{tr}_f"] \arrow[d, "\simeq"] & F \\
f_{0!}f_0^*F \arrow[r, "\mathrm{tr}_{f_0}"] & F \arrow[u, "\cdot\mathrm{rank}(\mathrm{gr}^*_I(\mathscr{O}_{\mathscr{X}}))"']
\end{tikzcd}$$

commutes.

Proof. As in the proof of 5.18 it suffices to consider the case $F = \Lambda$. Also, making a base change on $\mathscr{Y}$ as in the proof of 5.18, it suffices to consider the case when $\mathscr{Y} = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field. Furthermore by adjunction it suffices to show that the diagram on $\mathscr{X}$

(5.19.1)
$$\begin{tikzcd}
 & \Lambda \arrow[dl, "\widetilde{\mathrm{tr}}_{f_0}"'] \arrow[dr, "\widetilde{\mathrm{tr}}_f"] & \\
f_0^!\Lambda \arrow[rr, "\cdot\mathrm{rank}(\mathrm{gr}^*_I(\mathscr{O}_{\mathscr{X}}))"] & & f^!\Lambda
\end{tikzcd}$$

commutes (where in the bottom line we have identified $D_c(\mathscr{X})$ with $D_c(\mathscr{X}_0)$). Let $p : X \to \mathscr{X}$ be a smooth surjection of constant fiber dimension d, and let $p_0 : X_0 \to \mathscr{X}_0$ be the pullback of p to $\mathscr{X}_0$. To verify that 5.19.1 commutes it suffices to do so after applying p^*. The result therefore follows from consideration of the diagram, where all but the top triangle are known to commute,

$$\begin{tikzcd}
 & \Lambda \arrow[dl, "\widetilde{\mathrm{tr}}_{f_0}"'] \arrow[dr, "\widetilde{\mathrm{tr}}_f"] \arrow[ddl, bend right=60, "\widetilde{\mathrm{tr}}_{f_0p_0}"'] \arrow[ddr, bend left=60, "\widetilde{\mathrm{tr}}_{fp}"] & \\
p_0^*f_0^!\Lambda \arrow[rr, "\cdot\mathrm{rank}(\mathrm{gr}^*_I(\mathscr{O}_{\mathscr{X}}))"] \arrow[d, "\simeq"'] & & p^*f^!\Lambda \arrow[d, "\simeq"] \\
(f_0p_0)^!\Lambda(-d)[-2d] \arrow[rr, "\cdot\mathrm{rank}(\mathrm{gr}^*_I(\mathscr{O}_{\mathscr{X}}))"] & & (fp)^!\Lambda(-d)[-2d]
\end{tikzcd}$$

and 5.18. □

Let k be an algebraically closed field, and let $\mathscr{X}/k$ be a quasi-finite connected stack of finite type with finite diagonal. Choose a point $s : \mathrm{Spec}(k) \to \mathscr{X}$, and let G_s denote the finite automorphism group scheme of s. We then have a closed immersion $j : BG_s \hookrightarrow \mathscr{X}$. Let $J \subset \mathscr{O}_{\mathscr{X}}$ denote the ideal sheaf defining j, and let $\mathscr{A}$ denote the graded ring $\mathrm{gr}^*_J(\mathscr{O}_{\mathscr{X}})$. Then $\mathscr{A}$ is a locally free sheaf of finite rank on BG_s. We define the *length* of $\mathscr{X}$ to be the rational number

$$\mathrm{ln}(\mathscr{X}) := \mathrm{rank}(\mathscr{A})/\mathrm{rank}(G_s).$$

Since any two points of $\mathscr{X}(k)$ are isomorphic this is independent of the choice of s.

If $\mathscr{X}/k$ is quasi-finite with finite diagonal, but not necessarily connected, we define the length of $\mathscr{X}$ to be the sum of the lengths of the connected components of $\mathscr{X}$.

Now let $f : \mathscr{X} \to \mathscr{Y}$ be a quasi-finite morphism with finite diagonal of algebraic stacks, and let $y : \mathrm{Spec}(k) \to \mathscr{Y}$ be a morphism with k an algebraically closed field. Let $\mathscr{X}_y$ denote the fiber $\mathscr{X} \times_{\mathscr{Y}} \mathrm{Spec}(k)$.

Proposition 5.20. *If f is flat and proper and $\mathscr{Y}$ is connected then the number $\mathrm{ln}(\mathscr{X}_y)$ is independent of y, and for any constructible sheaf of $\mathbb{Q}_\ell$-modules F on $\mathscr{Y}$ the composite map*

$$F \xrightarrow{\mathrm{id}\to f_*f^*} f_*f^*F \xrightarrow{\epsilon^{-1}_{f^*F}} f_!f^*F \xrightarrow{\mathrm{tr}_f} F$$

is equal to multiplication by $\mathrm{ln}(\mathscr{X}_y)$.

Proof. Note that it suffices to prove the last statement. This statement holds in the case of a representable morphism by the classical theory. Furthermore, to prove it in general it suffices to consider the case when $\mathscr{Y} = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field and $\mathscr{X}$ is connected. Let $s \in \mathscr{X}(k)$ be a point and let G be the automorphism group scheme. Let $j : BG \hookrightarrow \mathscr{X}$ be the resulting closed immersion, and let $f_0 : BG \to \mathrm{Spec}(k)$ be the structural morphism. By 5.19 we then have a commutative diagram

$$\begin{tikzcd}
F \arrow[r, "\mathrm{id}\to f_*f^*"] \arrow[dr, "\mathrm{id}\to f_{0*}f_0^*"'] & f_*f^*F \arrow[r, "\epsilon^{-1}_{f^*F}"] \arrow[d, "\simeq"] & f_!f^*F \arrow[r, "\mathrm{tr}_f"] \arrow[d, "\simeq"] & F \\
 & f_{0*}f_0^*F \arrow[r, "\epsilon^{-1}_{f_0^*F}"] & f_{0!}f_0^*F \arrow[r, "\mathrm{tr}_{f_0}"] & F. \arrow[u, "\cdot\mathrm{rank}(\mathrm{gr}^*_J(\mathscr{O}_{\mathscr{X}}))"']
\end{tikzcd}$$

It therefore suffices to consider the case when $\mathscr{X} = BG$. Let $p : \mathrm{Spec}(k) \to BG$ denote the projection defined by the trivial torsor. The result then follows from the representable case and consideration of the diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[dr] & f_*f^*F \arrow[r, "\epsilon^{-1}_{f^*F}"] \arrow[d] & f_!f^*F \arrow[r, "\mathrm{rank}(G)\cdot\mathrm{tr}_f"] \arrow[d] \arrow[ddr, "k"] & F \\
 & f_*p_*p^*f^*F \arrow[dr, "\epsilon^{fp,-1}_{(fp)^*F}"] & f_!p_*p^*f^*F \arrow[d, "\epsilon^{p,-1}_{p^*f^*F}"] & \\
 & & f_!p_!p^*f^*F \arrow[r, "\mathrm{tr}^p_{f^*F}"] & f_!f^*F, \arrow[uu, "\mathrm{tr}_f"']
\end{tikzcd}$$

where the map labelled k is multiplication by $\mathrm{rank}(G)$, and all the small inside diagrams commute by our earlier results. □

Remark 5.21. If f is flat, proper, and quasi-finite then we call the rational number occurring in 5.20 the *degree* of f, and denote it by $\mathrm{deg}(f)$.

Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of k-stacks and let F (resp. G) be a constructible sheaf of Λ-modules on $\mathscr{Y}$ (resp. $\mathscr{X}$). Let $u : f^*F \to G$ be a morphism of sheaves on $\mathscr{X}$. Applying $D_{\mathscr{X}}$ we obtain a morphism

$$D_{\mathscr{X}}(G) \to D_{\mathscr{X}}(f^*F) = f^!D_{\mathscr{Y}}(F).$$

We denote by

$$u_* : f_!D_{\mathscr{X}}(G) \to D_{\mathscr{Y}}(F)$$

the map obtained by adjunction.

Let

$$\mathrm{ev}_{\mathscr{Y}} : R\Gamma_c(\mathscr{Y}, D_{\mathscr{Y}}(F)) \otimes R\Gamma(\mathscr{Y}, F) \rightarrow \Lambda$$

and

$$\mathrm{ev}_{\mathscr{X}} : R\Gamma_c(\mathscr{X}, D_{\mathscr{X}}(G)) \otimes R\Gamma(\mathscr{X}, G) \rightarrow \Lambda$$

denote the canonical pairings obtained from the identifications $R\Gamma_c(\mathscr{Y}, -) = D_{\mathrm{Spec}(k)} \circ R\Gamma \circ D_{\mathscr{Y}}$ and $R\Gamma_c(\mathscr{X}, -) = D_{\mathrm{Spec}(k)} \circ R\Gamma \circ D_{\mathscr{X}}$.

Lemma 5.22. *The diagram*

$$\begin{tikzcd}
R\Gamma_c(\mathscr{X}, D_{\mathscr{X}}(G)) \otimes R\Gamma(\mathscr{Y}, F) \arrow[r, "u_*\otimes 1"] \arrow[d, "1\otimes u^*"] & R\Gamma_c(\mathscr{Y}, D_{\mathscr{Y}}(F)) \otimes R\Gamma(\mathscr{Y}, F) \arrow[d, "\mathrm{ev}_{\mathscr{Y}}"] \\
R\Gamma_c(\mathscr{X}, D_{\mathscr{X}}(G)) \otimes R\Gamma(\mathscr{X}, G) \arrow[r, "\mathrm{ev}_{\mathscr{X}}"] & \Lambda
\end{tikzcd}$$

commutes.

Proof. Write also u for the map $F \rightarrow f_*G$ obtained by adjunction from the map $f^*F \rightarrow G$, and to ease notation write D_k for $D_{\mathrm{Spec}(k)}$. Also for any $F \in D_c^-(\mathrm{Spec}(k), \Lambda)$ write

$$\mathrm{ev} : D_k(F) \otimes F \rightarrow \Lambda$$

for the canonical evaluation map.

The map $u_* : f_!D_{\mathscr{X}}(G) \rightarrow D_{\mathscr{Y}}(F)$ is equal to the composite

$$f_!D_{\mathscr{X}}(G) =\!=\!= D_{\mathscr{Y}}f_*D^2_{\mathscr{X}}(G) \xrightarrow{D^2_{\mathscr{X}}\simeq \mathrm{id}} D_{\mathscr{Y}}f_*G \xrightarrow{D_{\mathscr{Y}}(u)} D_{\mathscr{Y}}(F).$$

Combining this with consideration of the diagram

$$\begin{tikzcd}
D_k\Gamma_{\mathscr{X}}D^2_{\mathscr{X}}(G) \otimes \Gamma_{\mathscr{Y}}(F) \arrow[r, "D^2_{\mathscr{Y}}\simeq \mathrm{id}"] \arrow[d, "1\otimes u"] & D_k\Gamma_{\mathscr{Y}}D^2_{\mathscr{Y}}f_*D^2_{\mathscr{X}}(G) \otimes \Gamma_{\mathscr{Y}}(F) \arrow[r, "D^2_{\mathscr{X}}\simeq \mathrm{id}"] \arrow[d, "1\otimes u"] & D_k\Gamma_{\mathscr{Y}}D^2_{\mathscr{Y}}f_*G \otimes \Gamma_{\mathscr{Y}}(F) \arrow[d, "D_{\mathscr{Y}}(u)"] \\
D_k\Gamma_{\mathscr{X}}D^2_{\mathscr{X}}(G) \otimes \Gamma_{\mathscr{Y}}f_*G \arrow[r, "D^2_{\mathscr{Y}}\simeq \mathrm{id}"] \arrow[d, "\Gamma_{\mathscr{Y}}f_*=\Gamma_{\mathscr{X}}"] & D_k\Gamma_{\mathscr{Y}}D^2_{\mathscr{Y}}f_*D^2_{\mathscr{X}}(G) \otimes \Gamma_{\mathscr{Y}}f_*G \arrow[d, "D^2_{\mathscr{X}}=\mathrm{id}"] & D_k\Gamma_{\mathscr{Y}}D^2_{\mathscr{Y}}(F) \otimes \Gamma_{\mathscr{Y}}(F) \arrow[d, "D^2_{\mathscr{Y}}=\mathrm{id}"] \\
D_k\Gamma_{\mathscr{X}}D^2_{\mathscr{X}}(G) \otimes \Gamma_{\mathscr{X}}(G) \arrow[d, "D^2_{\mathscr{X}}=\mathrm{id}"] & D_k\Gamma_{\mathscr{Y}}D^2_{\mathscr{Y}}f_*G \otimes \Gamma_{\mathscr{Y}}f_*G \arrow[d, "D^2_{\mathscr{Y}}=\mathrm{id}"] & D_k\Gamma_{\mathscr{Y}}(F) \otimes \Gamma_{\mathscr{Y}}(F) \arrow[ddl, "\mathrm{ev}"] \\
D_k\Gamma_{\mathscr{X}}(G) \otimes \Gamma_{\mathscr{X}}(G) \arrow[r, "\Gamma_{\mathscr{X}}=\Gamma_{\mathscr{Y}}f_*"] \arrow[dr, "\mathrm{ev}"] & D_k\Gamma_{\mathscr{Y}}f_*G \otimes \Gamma_{\mathscr{Y}}f_*G \arrow[d, "\mathrm{ev}"] & \\
 & \Lambda &
\end{tikzcd}$$

one sees that it suffices to show that the diagram

$$\begin{tikzcd}
D_k\Gamma_{\mathscr{Y}} f_*G \otimes \Gamma_{\mathscr{Y}}(F) \arrow[r, "D_k(u)\otimes 1"] \arrow[d, "1\otimes u"] & D_k\Gamma_{\mathscr{Y}}(F) \otimes \Gamma_{\mathscr{Y}}(F) \arrow[d, "\mathrm{ev}"] \\
D_k\Gamma_{\mathscr{Y}} f_*G \otimes \Gamma_{\mathscr{Y}} f_*G \arrow[r, "\mathrm{ev}"] & \Lambda
\end{tikzcd}$$

commutes, which is immediate. □

Now assume in addition that f is quasi-finite and flat, and let F be a constructible sheaf of Λ-modules on $\mathscr{Y}$. Let

$$\alpha : f^*D_{\mathscr{Y}}(F) \to D_{\mathscr{X}}(f^*F)$$

denote the composite map

$$f^*\mathscr{H}om(F, \Omega_{\mathscr{Y}}) \to \mathscr{H}om(f^*F, f^*\Omega_{\mathscr{Y}}) \to \mathscr{H}om(f^*F, f^!\Omega_{\mathscr{Y}}) = \mathscr{H}om(f^*F, \Omega_{\mathscr{X}}),$$

where the second morphism is induced by the map $\widetilde{\mathrm{tr}}_f : f^*\Omega_{\mathscr{Y}} \to f^!\Omega_{\mathscr{Y}} = \Omega_{\mathscr{X}}$. Also define $u_* : f_!D_{\mathscr{X}}(f^*F) \to D_{\mathscr{Y}}(F)$ to be the map defined as above taking $G = f^*F$ and using the identity map $f^*F \to f^*F$.

Lemma 5.23. *The diagram*

$$\begin{tikzcd}
f_!f^*D_{\mathscr{Y}}(F) \arrow[r, "\mathrm{tr}_f"] \arrow[d, "\alpha"] & D_{\mathscr{Y}}(F) \\
f_!D_{\mathscr{X}}(f^*F) \arrow[ur, "u_*"] &
\end{tikzcd}$$

commutes.

Proof. Observe that we have isomorphisms

$$\begin{aligned} R\mathscr{H}om(f_!f^*D_{\mathscr{Y}}(F), D_{\mathscr{Y}}(F)) &\simeq R\mathscr{H}om(F, D_{\mathscr{Y}}f_!f^*D_{\mathscr{Y}}(F)) \\ &\simeq R\mathscr{H}om(F, f_*f^!F). \end{aligned}$$

Since $f^!F \in D_c^{[0,\infty)}(\mathscr{X}, \Lambda)$ by 4.3, this implies that

$$\mathscr{E}xt^i(f_!f^*D_{\mathscr{Y}}(F), D_{\mathscr{Y}}(F)) = 0$$

for $i < 0$. By [13, 2.3.4] this in turn implies that it suffices to prove that the lemma holds locally in the smooth topology on $\mathscr{Y}$. We may therefore assume that $\mathscr{Y}$ is a scheme and that there exists a smooth surjection $p : X \to \mathscr{X}$ of relative dimension d.

To prove the lemma it suffices to show that the diagram obtained by adjunction

(5.23.1)
$$\begin{tikzcd}
f^*D_{\mathscr{Y}}(F) \arrow[r, "\widetilde{\mathrm{tr}}_f"] \arrow[d, "\alpha"] & f^!D_{\mathscr{Y}}(F) \arrow[d, equal] \\
D_{\mathscr{X}}(f^*F) \arrow[r, "\simeq"] & D_{\mathscr{X}}f^*D^2_{\mathscr{Y}}(F)
\end{tikzcd}$$

commutes. By the same calculation as above, the sheaf $\mathscr{E}xt^i(f^*D_{\mathscr{Y}}(F), f^!D_{\mathscr{Y}}(F))$, which is the sheaf associated to the presheaf

$$(V \to \mathscr{X}) \mapsto \mathrm{Ext}^i_{\mathscr{X}}(f^*D_{\mathscr{Y}}(F), f^!D_{\mathscr{Y}}(F)) = \mathrm{Ext}^i_{\mathscr{Y}}(f_!f^*D_{\mathscr{Y}}(F), D_{\mathscr{Y}}(F)),$$

is zero for $i < 0$. It follows from this and [13, 2.3.4] that to prove that 5.23.1 commutes it suffices to show that it commutes after applying p^*. Now using the canonical isomorphism $p^*\Omega_{\mathscr{X}} \simeq \Omega_X(-d)[-2d]$ one is then reduced to showing that the diagram

$$\begin{tikzcd}
(pf)^*D_{\mathscr{Y}}(F) \arrow[r, "\widetilde{\mathrm{tr}}_{pf}"] \arrow[d, "\alpha"] & (pf)^!D_{\mathscr{Y}}(F)(-d)[-2d] \arrow[d, "\simeq"] \\
D_X((fp)^*D)(-d)[-2d] \arrow[r, "\simeq"] & D_X(fp)^*D^2_{\mathscr{Y}}(F)(-d)[-2d]
\end{tikzcd}$$

commutes. This follows from the classical theory. □

Remark 5.24. We will only use 5.23 in the case when $F = \Lambda$, where the result is immediate.

Corollary 5.25. *Let k be an algebraically closed field, and let $f : \mathscr{X} \to \mathscr{X}$ be a quasi-finite flat and proper endomorphism of a smooth k-stack $\mathscr{X}$ of dimension d. Let*

$$A : R\Gamma_c(\mathscr{X}, \mathbb{Q}_\ell) \to R\Gamma_c(\mathscr{X}, \mathbb{Q}_\ell)$$

be the endomorphism induced by the map

$$\begin{tikzcd}
\mathbb{Q}_\ell \arrow[r] & f_*\mathbb{Q}_\ell \arrow[r, "\epsilon^{-1}"] & f_!\mathbb{Q}_\ell.
\end{tikzcd}$$

Then for any i the set of eigenvalues of A on $H^i_c(\mathscr{X}, \mathbb{Q}_\ell)$ is equal to the set $\{\deg(f)/\lambda_i\}_{i\in I}$, where $\{\lambda_i\}_{i\in I}$ is the set of eigenvalues of f^* acting on $H^{2d-i}(\mathscr{X}, \mathbb{Q}_\ell(d))$ in the usual way.

Proof. The isomorphism $\Omega_{\mathscr{X}} \simeq \mathbb{Q}_\ell(d)[2d]$ gives an isomorphism

$$H^i_c(\mathscr{X}, \mathbb{Q}_\ell) \simeq (H^{-i}(\mathscr{X}, \mathbb{Q}_\ell(d)[2d]))^* = H^{2d-i}(\mathscr{X}, \mathbb{Q}_\ell(d))^*.$$

Now by 5.22, the operator f^* on $H^{2d-i}(\mathscr{X}, \mathbb{Q}_\ell(d))$ has adjoint the map $f_* : H^i_c(\mathscr{X}, \mathbb{Q}_\ell) \to H^i_c(\mathscr{X}, \mathbb{Q}_\ell)$, and by 5.23 the map f_* agrees with the map defined by the trace $\mathrm{tr}_f : f_!\mathbb{Q}_\ell \to \mathbb{Q}_\ell$. Since the map

$$\begin{tikzcd}
\mathbb{Q}_\ell \arrow[r] & f_*\mathbb{Q}_\ell \arrow[r, "\epsilon^{-1}"] & f_!\mathbb{Q}_\ell \arrow[r, "\mathrm{tr}_f"] & \mathbb{Q}_\ell
\end{tikzcd}$$

is equal to multiplication by $\deg(f)$, this implies the result. □

Example 5.26. Let k be an algebraically closed field, and let (X, λ) be a principally polarized abelian variety of dimension d over k. Let p be an integer, and let $f : X \to X$ be an endomorphism of degree n such that the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p\lambda"] & X \arrow[d, "\lambda"] \\
X^t & X^t \arrow[l, "f^t"]
\end{tikzcd}$$

commutes. Let ℓ be a prime different from the characteristic of k, and let $V_\ell(X)$ denote the ℓ-adic Tate-module of X. Fix also an embedding $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. If $\alpha \mapsto \alpha^\dagger$ denotes the Rosati involution on $\mathrm{End}^0(X) := \mathrm{End}(X) \otimes \mathbb{Q}$, then we have $f^\dagger = p \cdot f^{-1}$. In particular, $f^\dagger \circ f = p$, and therefore by [18, 19.3] for any eigenvalue λ of f_* acting on $V_\ell(X)$ we have $|\iota(\lambda)| = \sqrt{p}$.

Now let $\mathscr{X}$ denote the classifying stack BX, and let also f denote the endomorphism of $\mathscr{X}$ induced by f. It follows from [4, 6.1.6] that all the odd cohomology groups $H^i(\mathscr{X}, \mathbb{Q}_\ell)$

vanish, and that $H^{2i}(\mathscr{X}, \mathbb{Q}_\ell)$ is canonically isomorphic to $S^i(V_\ell(X)^*)$ (i-th symmetric power). This identification is compatible with the action of f^*, and therefore the eigenvalues of f^* on $H^{2i}(\mathscr{X}, \mathbb{Q}_\ell)$ all have ι-absolute values $p^{i/2}$.

Let $A : H^*_c(\mathscr{X}, \mathbb{Q}_\ell) \to H^*_c(\mathscr{X}, \mathbb{Q}_\ell)$ be the endomorphism defined as in 5.25. Since the degree of $\mathscr{X} \to \mathscr{X}$ is equal to $1/n$, we find that $H^i_c(\mathscr{X}, \mathbb{Q}_\ell)$ is zero for i odd and $i > 2d$, and if τ is an eigenvalue of A acting on $H^{2i}_c(\mathscr{X}, \mathbb{Q}_\ell)$ ($i \leq d$) then $|\iota(\tau)| = 1/(n \cdot p^{(d-i)/2})$.

6. Interlude: Pushing forward Weil complexes.

6.1. Let (S, Λ) be an admissible pair, with S the spectrum of a field k of positive characteristic p, let q be a power of p, and let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of finite type algebraic k-stacks. We then obtain a commutative diagram

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "F_{\mathscr{X}/\mathscr{Y}}"] \arrow[dr, "f"'] & \mathscr{X}' \arrow[r, "\pi"] \arrow[d, "f'"] & \mathscr{X} \arrow[d, "f"] \\
& \mathscr{Y} \arrow[r, "F_{\mathscr{Y}}"] & \mathscr{Y},
\end{tikzcd}$$

where the square is cartesian, $F_{\mathscr{Y}}$ (resp. $F_{\mathscr{X}}$) is the q-power Frobenius morphism on $\mathscr{Y}$ (resp. $\mathscr{X}$) and $F_{\mathscr{X}/\mathscr{Y}}$ is the relative Frobenius.

Proposition 6.2. *For every $M \in D^-_c(\mathscr{X}', \mathbb{Q}_\ell)$ the adjunction map*

$$F_{\mathscr{X}/\mathscr{Y}!}F^!_{\mathscr{X}/\mathscr{Y}}M \to M$$

is an isomorphism.

Proof.

Special Case. Consider first the case when f is representable. Let $y : Y \to \mathscr{Y}$ be a smooth surjection with Y a scheme. We then obtain a commutative diagram

(6.2.1)
$$\begin{tikzcd}
\mathscr{X}_Y \arrow[rr, "F_{\mathscr{X}_Y/Y}"] \arrow[d, "b"] & & \mathscr{X}'_Y \arrow[rr, "\pi_Y"] \arrow[dl, "a"] \arrow[dd] & & \mathscr{X}_Y \arrow[dl] \arrow[dd] \\
\mathscr{X} \arrow[r, "F_{\mathscr{X}/\mathscr{Y}}"] \arrow[ddr] & \mathscr{X}' \arrow[rr] \arrow[dd] & & \mathscr{X} \arrow[dd] & \\
& & Y \arrow[rr, "F_Y"] \arrow[dl] & & Y \arrow[dl] \\
& \mathscr{Y} \arrow[rr] & & \mathscr{Y}, &
\end{tikzcd}$$

where

$$\mathscr{X}_Y := \mathscr{X} \times_{\mathscr{Y}} Y, \quad \mathscr{X}'_Y := \mathscr{X}_Y \times_{Y, F_Y} Y.$$

Note that the square

$$\begin{tikzcd}
\mathscr{X}'_Y \arrow[r] \arrow[d, "a"] & Y \arrow[d, "y"] \\
\mathscr{X}' \arrow[r] & \mathscr{Y}
\end{tikzcd}$$

is cartesian.

Since a is smooth and surjective, it suffices to show that the map

$$a^* F_{\mathscr{X}/\mathscr{Y}!} F^!_{\mathscr{X}/\mathscr{Y}} M \to a^* M \tag{6.2.2}$$

is an isomorphism. Since the square

$$\begin{tikzcd}
\mathscr{X}_Y \arrow[r, "F_{\mathscr{X}_Y/Y}"] \arrow[d, "b"] & \mathscr{X}'_Y \arrow[d, "a"] \\
\mathscr{X} \arrow[r, "F_{\mathscr{X}/\mathscr{Y}}"] & \mathscr{X}'
\end{tikzcd}$$

is cartesian and a is smooth, we also have

$$a^* F_{\mathscr{X}/\mathscr{Y}!} \simeq F_{\mathscr{X}_Y/Y!} b^*$$

and

$$b^* F^!_{\mathscr{X}/\mathscr{Y}} \simeq F^!_{\mathscr{X}_Y/Y} a^*.$$

Thus the morphism 6.2.2 is identified with the adjunction map

$$F_{\mathscr{X}_Y/Y!} F^!_{\mathscr{X}_Y/Y} a^* M \to a^* M.$$

Since f is assumed representable, the stack $\mathscr{X}_Y$ is in fact an algebraic space. This therefore reduces the proof in this case to the case of algebraic spaces, where the result is immediate.

General Case. By the same argument used in the preceding special case, it suffices to consider the case when $\mathscr{Y}$ is a scheme.

Let $q : X \to \mathscr{X}$ be a smooth surjection, and define

$$\widetilde{X} := X \times_{\mathscr{X}, F_{\mathscr{X}}} \mathscr{X}, \quad X' := X \times_{\mathscr{Y}, F_{\mathscr{Y}}} \mathscr{Y}.$$

We then have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "c"] \arrow[rr, bend left, "F_{X/\mathscr{Y}}"] \arrow[dr] & \widetilde{X} \arrow[r, "d"] \arrow[d] & X' \arrow[r] \arrow[d, "\rho"] & X \arrow[d] \\
& \mathscr{X} \arrow[r, "F_{\mathscr{X}/\mathscr{Y}}"] \arrow[dr] & \mathscr{X}' \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\
& & \mathscr{Y} \arrow[r, "F_{\mathscr{Y}}"] & \mathscr{Y}.
\end{tikzcd}$$

Since ρ is smooth it suffices, as in the earlier special case, to show that the natural map

$$d_! d^! \rho^* M \to \rho^* M$$

is an isomorphism. For this consider the commutative diagram

$$\begin{tikzcd}
d_!c_!c^!d^!\rho^*M \arrow[r, "\alpha"] \arrow[d, "\simeq"] & d_!d^!\rho^*M \arrow[r, "\beta"] & \rho^*M \\
(dc)_!(dc)^!\rho^*M, \arrow[urr, "\gamma"'] & &
\end{tikzcd}$$

where α (resp. β, γ) is the adjunction map $c_!c^! \to \mathrm{id}$ (resp. $d_!d^! \to \mathrm{id}$, $(dc)_!(dc)^! \to \mathrm{id}$).

By the representable case already considered applied to $X \to \mathscr{Y}$ (resp. $X \to \mathscr{X}$) the map γ (resp. α) is an isomorphism. We conclude that the map β is also an isomorphism. ☐

Proposition 6.3. *(i) There is a natural isomorphism* $\pi^*\Omega_{\mathscr{X}} \simeq \Omega_{\mathscr{X}'}$.

(ii) The map of functors $\pi^*D_{\mathscr{X}} \to D_{\mathscr{X}'}\pi^*$ *defined as the composite*

$$\pi^*R\mathscr{H}om(-,\Omega_{\mathscr{X}}) \longrightarrow R\mathscr{H}om(\pi^*(-),\pi^*\Omega_{\mathscr{X}}) \xrightarrow{(i)} R\mathscr{H}om(\pi^*(-),\Omega_{\mathscr{X}'})$$

is an isomorphism.

Proof. By the gluing lemma [13, 2.3.3], it suffices to construct the isomorphism $\pi^*\Omega_{\mathscr{X}} \simeq \Omega_{\mathscr{X}'}$ locally in the smooth topology on $\mathscr{X}'$.

Let $y : Y \to \mathscr{Y}$ be a smooth covering as in the proof of 6.2, and form the diagram 6.2.1. Let d be the relative dimension of y (a locally constant function on Y). By [13, 4.6.2], we then have

$$a^*\Omega_{\mathscr{X}'} \simeq \Omega_{\mathscr{X}'_Y}(-d)[-2d],$$

and

$$a^*\pi^*\Omega_{\mathscr{X}} \simeq \pi_Y^*\Omega_{\mathscr{X}_Y}(-d)[-2d].$$

It therefore suffices to construct an isomorphism

$$\pi_Y^*\Omega_{\mathscr{X}_Y} \simeq \Omega_{\mathscr{X}'_Y},$$

which reduces the proof to the case when $\mathscr{Y}$ is a scheme.

From this argument we obtain part (i) of the proposition in the case when f is representable. Using a similar argument one also reduces the proof of (ii) in the representable case to the case of schemes.

For the general case, choose a smooth surjection $q : X \to \mathscr{X}$ with X a scheme, so we have a commutative diagram with cartesian squares

$$\begin{tikzcd}
X' \arrow[r, "\pi_X"] \arrow[d, "\rho"] & X \arrow[d, "q"] \\
\mathscr{X}' \arrow[r, "\pi"] \arrow[d] & \mathscr{X} \arrow[d] \\
\mathscr{Y} \arrow[r, "F_{\mathscr{Y}}"] & \mathscr{Y}.
\end{tikzcd}$$

If d denotes the relative dimension of q, then again by [13, 4.6.2], we have

$$\rho^*\Omega_{\mathscr{X}'} \simeq \Omega_{X'}(-d)[-2d], \quad q^*\Omega_{\mathscr{X}} \simeq \Omega_X(-d)[-2d].$$

From the representable case we therefore obtain an isomorphism

$$\rho^*\Omega_{\mathscr{X}'} \simeq \rho^*\pi^*\Omega_{\mathscr{X}},$$

which again using [13, 2.3.3] descends to an isomorphism $\Omega_{\mathscr{X}'} \simeq \pi^*\Omega_{\mathscr{X}}$. This completes the proof of (i).

Also to verify (ii) it suffices to show that for any $M \in D_c^-(\mathscr{X}, \mathbb{Q}_\ell)$ the map

$$\rho^*\pi^*D_{\mathscr{X}}(M) \to \rho^*D_{\mathscr{X}'}(\pi^*M)$$

is an isomorphism. Using the isomorphisms

$$\rho^*\pi^*D_{\mathscr{X}}(M) \simeq \pi_X^*D_X(q^*M)(-d)[-2d], \quad \rho^*D_{\mathscr{X}'}(\pi^*M) \simeq D_{X'}(\pi_X^*q^*M)(-d)[-2d]$$

statement (ii) is also reduced to the case of algebraic spaces. ☐

Corollary 6.4. *There is a canonical isomorphism $F_{\mathscr{Y}}^*\Omega_{\mathscr{Y}} \simeq \Omega_{\mathscr{Y}}$ and the induced transformation $F_{\mathscr{Y}}^*D_{\mathscr{Y}} \to D_{\mathscr{Y}}F_{\mathscr{Y}}^*$ is an isomorphism.*

Proof. Apply 6.3 with $\mathscr{X} = \mathscr{Y}$ and $f = \mathrm{id}$. ☐

Corollary 6.5. *There is a canonical isomorphism of functors $F^!_{\mathscr{X}/\mathscr{Y}}\pi^* \simeq F^*_{\mathscr{X}}$.*

Proof. Indeed we have

$$\begin{aligned} F^!_{\mathscr{X}/\mathscr{Y}}\pi^* &= D_{\mathscr{X}}F^*_{\mathscr{X}/\mathscr{Y}}D_{\mathscr{X}'}\pi^* \\ &\simeq D_{\mathscr{X}}F^*_{\mathscr{X}/\mathscr{Y}}\pi^*D_{\mathscr{X}} \quad \text{(by 6.3)} \\ &\simeq D_{\mathscr{X}}F^*_{\mathscr{X}}D_{\mathscr{X}} \\ &\simeq F^*_{\mathscr{X}}D^2_{\mathscr{X}} \quad \text{(by 6.4 applied to } \mathscr{X}/k\text{)} \\ &\simeq F^*_{\mathscr{X}} \quad (D^2_{\mathscr{X}} \simeq \mathrm{id}). \end{aligned}$$

☐

Proposition 6.6. *For any $A \in D_c^+(\mathscr{X}, \mathbb{Q}_\ell)$, the base change morphism*

$$F^*_{\mathscr{Y}}f_*A \to f'_*\pi^*A \tag{6.6.1}$$

is an isomorphism.

Proof. Let $y : Y \to \mathscr{Y}$ be a smooth covering as in the proof of 6.2, and consider the resulting commutative diagram as in 6.2.2

$$\begin{tikzcd}
& \mathscr{X}'_Y \arrow[rr, "\pi_Y"] \arrow[dl, "a"'] \arrow[dd, "g'"] & & \mathscr{X}_Y \arrow[dl, "c"'] \arrow[dd, "g"] \\
\mathscr{X}' \arrow[rr, "\pi"] \arrow[dd, "f'"] & & \mathscr{X} \arrow[dd, "f"] & \\
& Y \arrow[rr, "F_Y"] \arrow[dl, "y"] & & Y \arrow[dl, "y"] \\
\mathscr{Y} \arrow[rr, "F_{\mathscr{Y}}"] & & \mathscr{Y}. &
\end{tikzcd}$$

We then have a commutative diagram

$$\begin{tikzcd}
y^*F_{\mathscr{Y}}^*f_*A \arrow[r, "6.6.1"] \arrow[d, "\simeq"] & y^*f'_*\pi^*A \arrow[d, "\simeq"] \\
F_Y^*g_*c^*A \arrow[r] & g'_*\pi_Y^*c^*A,
\end{tikzcd}$$

where the bottom horizontal arrow is the morphism 6.6.1 for $g : \mathscr{X}_Y \to Y$. This therefore reduces the proof to the case when $\mathscr{Y} = Y$ is a scheme.

In this case, let $p : X \to \mathscr{X}$ be a smooth surjection with X a quasi-compact scheme, and let $X_\cdot$ be the associated simplicial space. Let $f_\cdot : X_\cdot \to Y$ be the composite morphism

$$X_\cdot \to \mathscr{X} \to Y.$$

We then obtain a commutative diagram

$$\begin{tikzcd}
X'_\cdot \arrow[r] \arrow[d, "f'_\cdot"] & X_\cdot \arrow[d, "f_\cdot"] \\
Y \arrow[r, "F_Y"] & Y,
\end{tikzcd}$$

where $X'_\cdot := X_\cdot \times_{Y,F_Y} Y$. In this case we have (see for example [8, 5.2.3]) spectral sequences

$$E_1^{st} = R^tf_{s*}M|_{X_s} \implies R^{s+t}f_*M \tag{6.6.2}$$

and

$$E_1^{st} = R^tf'_{s*}\pi^*M|_{X'_s} \implies R^{s+t}f'_*\pi^*M. \tag{6.6.3}$$

Moreover, the map 6.6.1 extends to a morphism of spectral sequences

$$F_Y^*(6.6.2) \to (6.6.3).$$

It therefore suffices to show that each of the maps

$$F_Y^*R^tf_*M|_{X_s} \to R^tf'_*\pi^*M|_{X'_s}$$

is an isomorphism. This reduces the proof to the case when Y is a scheme and X is algebraic space. In this case the result is classical. $\square$

6.7. For $M \in D_c^-(\mathscr{X})$, we therefore get isomorphisms

$$\begin{aligned}
F_{\mathscr{Y}}^*f_!M &= F_{\mathscr{Y}}^*D_{\mathscr{Y}}f_*D_{\mathscr{X}}(M) \\
&\simeq D_{\mathscr{Y}}F_{\mathscr{Y}}^*f_*D_{\mathscr{X}}(M) \text{ (by 6.4)} \\
&\simeq D_{\mathscr{Y}}f'_*\pi^*D_{\mathscr{X}}(M) \text{ (by 6.6)} \\
&\simeq D_{\mathscr{Y}}f'_*D_{\mathscr{X}'}\pi^*M \text{ (by 6.3)} \\
&= f'_!\pi^*M.
\end{aligned}$$

By 6.2 the adjunction map

$$F_{\mathscr{X}/\mathscr{Y}!}F_{\mathscr{X}/\mathscr{Y}}^!\pi^*M \to \pi^*M$$

is an isomorphism, so we obtain an isomorphism

$$f'_!\pi^*M \simeq f'_!F_{\mathscr{X}/\mathscr{Y}!}F_{\mathscr{X}/\mathscr{Y}}^!\pi^*M = f_!F_{\mathscr{X}/\mathscr{Y}}^!\pi^*M,$$

which by 6.5 is isomorphic to

$$f_! F^*_{\mathscr{X}} M.$$

Putting it all together we obtain an isomorphism

$$F^*_{\mathscr{Y}} f_! M \simeq f_! F^*_{\mathscr{X}} M. \tag{6.7.1}$$

6.8. Consider now a commutative diagram of algebraic stacks

$$\begin{tikzcd} & \mathscr{C}_0 \arrow[dl, "c_1"'] \arrow[dd, "q"] \arrow[dr, "c_2"] & \\ \mathscr{X}_0 \arrow[dd, "p"] & & \mathscr{X}_0 \arrow[dd, "p"] \\ & \mathscr{D}_0 \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\ \mathscr{Y}_0 & & \mathscr{Y}_0, \end{tikzcd}$$

where c_2 and d_2 are quasi-finite and representable and c_1 and d_1 are proper with finite diagonals. Let $(\mathcal{F}, \varphi, u)$ be a Weil complex with $\mathscr{C}$-structure on $\mathscr{X}$. Then $p_!\mathcal{F}$ has the structure of a Weil complex with $\mathscr{D}$-structure on $\mathscr{Y}$ as follows.

6.9. Define the Weil structure $p_!\varphi$ on $p_!\mathcal{F}$ to be the composite map

$$\begin{aligned} F^*_{\mathscr{Y}} p_! \mathcal{F} &\simeq p_! F^*_{\mathscr{X}} \mathcal{F} \quad \text{(by 6.7.1)} \\ &\xrightarrow{\varphi} p_! \mathcal{F}. \end{aligned}$$

6.10. Let $\alpha : d_1^* p_! \mathcal{F} \rightarrow q_! c_2^! \mathcal{F}$ denote the composite

$$\begin{aligned} d_1^* p_! \mathcal{F} &\rightarrow d_1^* p_! c_{1*} c_1^* \mathcal{F} \quad \text{(adjunction)} \\ &\simeq d_1^* d_{1*} q_! c_1^* \mathcal{F} \quad (5.17) \\ &\rightarrow q_! c_1^* \mathcal{F} \quad \text{(adjunction for } d_1) \\ &\xrightarrow{u} q_! c_2^! \mathcal{F}. \end{aligned}$$

Then define $p_! u$ to be the composite map

$$\begin{tikzcd} d_{2!} d_1^* p_! \mathcal{F} \arrow[r, "\alpha"] & d_{2!} q_! c_2^! \mathcal{F} \arrow[r, equal] & p_! c_{2!} c_2^! \mathcal{F} \arrow[r] & p_! \mathcal{F}. \end{tikzcd}$$

We call the Weil complex with $\mathscr{D}$-structure $(p_!\mathcal{F}, p_!\varphi, p_!u)$ the *pushforward of* $(\mathcal{F}, \varphi, u)$.

Remark 6.11. We leave to the reader the verification that for any integer $n \geq 0$ we have $(p_! u)^{(n)} = p_!(u^{(n)})$.

7. THE CLASSIFYING STACK OF A CONNECTED GROUP

7.1. In the following three sections we prove 1.19 in the case when $\mathscr{X}$ is the classifying stack of a finite type group scheme. For technical reasons it will be useful to prove a slightly stronger statement than 1.19.

7.2. Let $\mathscr{X}_0/\mathbb{F}_q$ be a finite type algebraic stack, and let $c = (c_1, c_2) : \mathscr{C}_0 \to \mathscr{X}_0 \times \mathscr{X}_0$ be a correspondence with c_1 proper and c_2 quasi-finite and representable. Fix also an embedding $\iota : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$.

Definition 7.3. A Weil complex with $\mathscr{C}$-structure $(\mathcal{F}, \varphi, u)$ is *ι-convergent* (or just *convergent* if the reference to ι is clear) if there exists an integer n_0 such that for every $n \geq n_0$ and $(x, \lambda) \in \mathrm{Fix}(\mathscr{C}^{(n)})(k)$ the pair $(\mathcal{F}_{c_2(x)}, u^{(n)}_{(x,\lambda)})$ is ι-convergent in the sense of 1.14. Here $u^{(n)}_{(x,\lambda)}$ is defined as in 1.17.

Remark 7.4. A bounded Weil complex with $\mathscr{C}$-structure $(\mathcal{F}, \varphi, u)$ is convergent.

Remark 7.5. Note that for a convergent Weil complex with $\mathscr{C}$-structure $(\mathcal{F}, \varphi, u)$ we can still define local terms $\mathrm{LT}_\iota(\beta, (\mathcal{F}, \varphi, u))$ as in 1.18.

7.6. Let $G_0/\mathbb{F}_q$ be a group scheme of finite type, and let $\alpha : G_0 \to G_0$ be an endomorphism. We write $B\alpha : BG_0 \to BG_0$ for the induced endomorphism of the classifying stack. For $n \geq 0$ let $\alpha^{(n)}$ denote $F_G^{(n)} \circ \alpha$. Let $c : \mathscr{C}_0 \to BG_0 \times BG_0$ be the correspondence $(B\alpha, \mathrm{id}) : BG_0 \to BG_0 \times BG_0$. Note that $c^{(n)} : \mathscr{C}_0 \to BG_0 \times BG_0$ is the correspondence $(B\alpha^{(n)}, \mathrm{id})$.

Theorem 7.7. *Let $(\mathcal{F}, \varphi, u)$ be a convergent Weil complex with $\mathscr{C}$-structure on $\mathscr{X}$. Then there exists an integer n_0, independent of $(\mathcal{F}, \varphi, u)$, such that for every $n \geq n_0$ we have:*

(i) The complex $R\Gamma_c(\mathscr{X}, \mathcal{F}) \in D_c^-(\mathbb{Q}_\ell)$ with the endomorphism $R\Gamma_c(u^{(n)})$ is convergent.

(ii) $\mathrm{Fix}(\mathscr{C}^{(n)})$ *is pseudo-finite over* $\mathrm{Spec}(k)$*, and*

$$\mathrm{tr}_\iota(R\Gamma_c(u^{(n)})|R\Gamma_c(\mathscr{X}, \mathcal{F})) = \sum_{\beta \subset \mathrm{Fix}(\mathscr{C}^{(n)})} \mathrm{LT}_\iota(\beta, (\mathcal{F}, \varphi, u^{(n)})).$$

For the remainder of this section we prove 7.7 in the case when G_0 is geometrically connected. The proof in the general case will be given in section 9.

7.8. For $n \geq 0$, let $\rho^{(n)}$ denote the action of the group scheme G_0 on the scheme G_0 (so $\rho^{(n)}$ does not act through homomorphisms) given by

$$h * g = \alpha^{(n)}(h)^{-1} g h.$$

Lemma 7.9. *For $n \geq 0$, the stack* $\mathrm{Fix}(\mathscr{C}^{(n)})$ *is isomorphic to the stack-theoretic quotient $[G/\rho^{(n)}]$ of the scheme G by the action of G given by $\rho^{(n)}$.*

Proof. Replacing α by $\alpha^{(n)}$ it suffices to consider the case $n = 0$.

Let S be a scheme. An object of $\mathrm{Fix}(\mathscr{C})(S)$ is a pair (P, ι), where P is a G-torsor over S and

$$\iota : P \times^{G,\alpha} G \to P$$

is an isomorphism of G-torsors. In the case when P is a trivial torsor and we fix a trivialization $e \in P$, then ι is specified by an element $g \in G$ characterized by the condition $\iota(e) = g \cdot e$. Now if (P', ι', e') is a second object together with a trivialization of P', then an isomorphism $\tau : (P, \iota) \to (P', \iota')$ is given by an isomorphism of G-torsors $\tau : P \to P'$ such that the diagram

(7.9.1)
$$\begin{tikzcd}
P \times^{G,\alpha} G \arrow[r, "\iota"] \arrow[d, "\alpha_*\tau"] & P \arrow[d, "\tau"] \\
P' \times^{G,\alpha} G \arrow[r, "\iota'"] & P'
\end{tikzcd}$$

commutes. The map τ is determined by an element $h \in G$ such that $\tau(e) = he'$, and then the commutativity of 7.9.1 is equivalent to the equality

$$\alpha(h)g' = gh,$$

or equivalently $g' = \alpha(h)^{-1}gh$ (here $g' \in G'$ is characterized by $\iota'(e') = g'e'$). This implies the lemma. ☐

Remark 7.10. Note that the proof of 7.9 does not use the fact that G is connected, and 7.9 holds without this assumption.

7.11. By 2.2 for any $g \in G(k)$ and $n \geq 1$ the morphism of schemes

$$G \to G, \quad h \mapsto \alpha^{(n)}(h)^{-1}gh$$

is étale, and therefore surjective (since G is geometrically connected). It follows that for $n \geq 1$ the category $\mathrm{Fix}(\mathscr{C}^{(n)})(k)$ has up to isomorphism only one object, and that the automorphism group scheme of this object is equal to the scheme of fixed points $\mathrm{Fix}(\alpha^{(n)})$ of $\alpha^{(n)}$ (note that this is a group scheme). Note that taking $g = e$ it follows that for all $n \geq 1$ the group scheme $\mathrm{Fix}(\alpha^{(n)})$ is finite and étale over $\mathrm{Spec}(k)$.

7.12. Let $G_0' \subset G_0$ be the maximal reduced closed subscheme, and write $\beta : G_0' \to G_0'$ for the endomorphism induced by α. Let H (resp. H') denote the inverse image of the identity under the map $G \to G$ (resp. $G' \to G'$) sending h to $\alpha^{(n)}(h)^{-1}h$ (resp. $\beta^{(n)}(h)^{-1}h$). We then have a cartesian diagram

$$\begin{tikzcd}
H' \arrow[r, hook] \arrow[d, hook] & G' \arrow[d, hook] \\
H \arrow[r, hook] & G.
\end{tikzcd}$$

On the other hand, H is étale so $H \subset G'$ which implies that $H = H'$. It follows that if $\mathscr{C}_0' \to BG_0' \times BG_0'$ is the correspondence induced by β, then for $n \geq 1$ the functor

$$\mathrm{Fix}(\mathscr{C}'^{(n)}) \to \mathrm{Fix}(\mathscr{C}^{(n)})$$

is an equivalence of categories.

Lemma 7.13. *The morphism $Bj : BG' \to BG$ induced by the inclusion $j : G' \hookrightarrow G$ is representable and radicial.*

Proof. That Bj is representable is clear since j is a closed immersion so Bj is faithful.

To verify that Bj is radicial, note that since $\mathrm{Spec}(k) \to BG$ is flat and surjective, it suffices to show that the fiber product $P := BG' \times_{BG} \mathrm{Spec}(k)$ is radicial over $\mathrm{Spec}(k)$. This is clear because $P(k)$ is the set of G'-invariant closed subschemes $Z \subset G$ such that the action of G' on Z is torsorial. Since k is algebraically closed, $Z(k) \neq \emptyset$ and therefore Z is reduced. Thus $Z \subset G'$ and since both are G'-torsors $Z = G'$. We conclude that $P(k)$ consists of one element. Since P is of finite type over k it follows that P is equal to the spectrum of an artinian local ring with residue field k. ☐

7.14. It follows that

$$Bj^* : D_c^-(BG, \mathbb{Q}_\ell) \to D_c^-(BG', \mathbb{Q}_\ell)$$

is an equivalence of categories and that $Bj_! = Bj_*$. This gives a canonical isomorphism

$$R\Gamma_c(BG, \mathcal{F}) \simeq R\Gamma_c(BG', Bj^*\mathcal{F})$$

compatible with the actions of the correspondences. We conclude that to prove 7.7 in the present situation it suffices to consider the case when G is reduced, and hence smooth.

7.15. To prove the theorem for smooth G, consider first the case when $\mathcal{F} = \mathbb{Q}_\ell$. Let d be the dimension of G, and let M_n denote the degree of $\alpha^{(n)} : G \to G$. Then the degree of the map $B\alpha^{(n)} : BG \to BG$ is equal to $1/M_n$. As in 5.25 the space $H_c^{-i}(BG, \mathbb{Q}_\ell)$ is isomorphic to the dual of $H^{i-2d}(BG, \mathbb{Q}_\ell)(-d)$, and the trace of $\alpha^{(n)}$ on $H_c^{-i}(BG, \mathbb{Q}_\ell)$ is equal to

$$\frac{1}{M_n}\mathrm{tr}((\alpha^{(n)*})^{-1}|H^{i-2d}(BG, \mathbb{Q}_\ell(-d))),$$

where $\alpha^{(n)*} : H^{i-2d}(BG, \mathbb{Q}_\ell(-d)) \to H^{i-2d}(BG, \mathbb{Q}_\ell(-d))$ is the usual pullback on cohomology.

7.16. By Borel's theorem [4, 6.1.6], there exists a graded vector space $N = \oplus_{q\geq 1}N^q$ concentrated in even degrees and a canonical surjection of graded $\mathbb{Q}_\ell$-vector spaces

$$\pi : H^*(BG, \mathbb{Q}_\ell) \to N$$

such that any section $s : N \to H^*(BG, \mathbb{Q}_\ell)$ of π induces an isomorphism

$$\rho(s) : \mathrm{Sym}^{\cdot} N \to H^*(BG, \mathbb{Q}_\ell).$$

Fix an integer i, and let I_i denote the set of tuples $(q_1, \ldots, q_r)$ of even positive integers such that $q_1 \geq q_2 \geq \cdots \geq q_r$, $q_1 + \cdots + q_r = i$, and such that $N^{q_j} \neq 0$ for all $j = 1, \ldots, r$. Note that I_i is a finite set. The set I_i becomes an ordered set with the lexicographical order. For $\underline{q} \in I$, let $G^{\underline{q}} \subset \mathrm{Sym}^i(N)$ denote the subpace generated by monomials

$$n_1 \otimes \cdots \otimes n_r$$

with $n_j \in N^{w(n_j)}$ and $(w(n_1), \ldots, w(n_r)) \leq \underline{q}$. This gives an I-graded filtration on $\mathrm{Sym}^i N$.

This also defines a filtration on $H^i(BG, \mathbb{Q}_\ell)$. Namely, choose a section s of π, and for $\underline{q} \in I$ define $F^{\underline{q}}$ on $H^i(BG, \mathbb{Q}_\ell)$ to be the image under $\rho(s)$ of $G^{\underline{q}}$. Then this filtration on $H^i(BG, \mathbb{Q}_\ell)$ is independent of the choice of section s, since for a second section s' we have

$$(\rho(s) - \rho(s'))(G^{\underline{q}}) \subset F^{\underline{q}'}$$

for some $\underline{q}' < \underline{q}$. In particular we obtain a canonical isomorphism

$$\mathrm{gr}_G(\mathrm{Sym}^i(N)) \simeq \mathrm{gr}_F(H^i(BG, \mathbb{Q}_\ell)).$$

7.17. Let $A : N \to N$ denote the endomorphism induced by the automorphism $(\alpha^*)^{-1}$ on $H^*(BG, \mathbb{Q}_\ell)$, and let $\Phi : N \to N$ denote the map induced by arithmetic Frobenius. Then we see from the above that the eigenvalues of $\alpha^{(n)}$ acting on $H_c^{-i}(BG, \mathbb{Q}_\ell) \otimes_\iota \mathbb{C}$ are equal to $1/M_n$ times the eigenvalues of $\Phi^{n+d} \circ A$ acting on the dual space $\mathrm{Sym}^{i+2d}(N)^* \otimes_\iota \mathbb{C}$.

Since the endomorphism $B\alpha$ of BG is defined over $\mathbb{F}_q$, the endomorphisms A and Φ acting on $\mathrm{Sym}^{i+2d}(N)^*$ commute. By [7, 3.3.5], there exists a collection of negative integers $\{w_i\}$ such that any eigenvalue of Φ acting on $N \otimes_\iota \mathbb{C}$ has absolute value $p^{w_i/2}$ for some i. Therefore the eigenvalues of $\Phi^{n+d} \circ A$ acting on N all have absolute value $p^{w/2}|\lambda|$, where w is a negative integer and λ is an eigenvalue of A acting on $N \otimes_\iota \mathbb{C}$.

It follows that there exists a collection of triples $\{(t_j, w_j, \lambda_j)\}_{j\in J}$ with t_j a positive integer, w_j a negative integer, and $\lambda_j \in \mathbb{C}$, such that the eigenvalues of $\alpha^{(n)}$ acting on $H^{-i}_c(BG, \mathbb{Q}_\ell)\otimes_\iota\mathbb{C}$ all have absolute value

$$\{\frac{1}{M_n}\prod_{\underline{m}} q^{w_jm_j(n+d)/2}|\lambda_j|\}$$

where the product is taken over collections $\underline{m} = (m_1, \dots, m_r)_{j\in J}$ of natural numbers with $\sum m_jt_j = i$.

In particular if we choose n so that $q^{w_jm_j(n+d)/2}|\lambda_j| < 1$, then the sequence of sums

$$S_p(H^*_c(BG, \mathbb{Q}_\ell)) = \frac{1}{M_n}\sum_{k\geq p}\sum_{\lambda\in \mathrm{Eg}^k(\Phi^n\circ A)} |\lambda| \tag{7.17.1}$$

of the absolute values of the eigenvalues of $\alpha^{(n)}$ on $H^*_c(BG, \mathbb{Q}_\ell)$ converges to

$$\frac{1}{M_n}\prod\frac{1}{1-q^{w_j(n+d)/2}|\lambda_j|}.$$

In particular, $R\Gamma_c(BG, \mathbb{Q}_\ell)$ with the endomorphism $\alpha^{(n)}$ is convergent.

The same argument shows that if $\gamma^{(n)}_1, \dots, \gamma^{(n)}_r$ denote the eigenvalues of $\Phi^n \circ A$ acting on $N \otimes_\iota \mathbb{C}$, then

$$\mathrm{tr}_\iota(\alpha^{(n)}|H^*_c(BG, \mathbb{Q}_\ell)) = \frac{1}{M_n}\prod_i\frac{1}{1-\gamma_i}.$$

Now recall also (again by Borel's theorem [4, 6.1.6]) that $H^*(G, \mathbb{Q}_\ell) \simeq \Lambda^\cdot(N[1])$.

Lemma 7.18. *Let W be a vector space of finite dimension over an algebraically closed field K, and let $B : W \to W$ be an endomorphism. Let $\alpha_1, \dots, \alpha_r \in K$ be the eigenvalues of B. Then the trace of $\Lambda^\cdot(B)$ acting on $\Lambda^\cdot(W)$ is equal to*

$$\prod(1-\alpha_j).$$

Proof. This is an elementary exercise. □

7.19. From this we conclude that

$$\mathrm{tr}_\iota(\Phi^n \circ A|H^*(G, \mathbb{Q}_\ell)) = \prod_i(1-\gamma^{(n)}_i).$$

On the other hand, applying 5.25 to $\alpha^{(n)} : G \to G$ and Fujiwara's theorem 1.1, there exists an integer n_0 such that

$$\#\mathrm{Fix}(\alpha^{(n)}) = M_n\mathrm{tr}_\iota(\Phi^n \circ (\alpha^*)^{-1}|H^*(G, \mathbb{Q}_\ell)) = M_n\prod_i(1-\gamma^{(n)}_i).$$

for $n \geq n_0$. Therefore for $n \geq n_0$ we have by 7.18

$$\mathrm{tr}_\iota(\alpha^{(n)}|H^*_c(BG, \mathbb{Q}_\ell)) = \frac{1}{\#\mathrm{Fix}(\alpha^{(n)})}.$$

It follows from 2.2 that this completes the proof in the case $F = \mathbb{Q}_\ell$.

More generally, if F is concentrated in degree 0, then since G is connected the sheaf $\mathcal{F}$ is isomorphic to the constant sheaf associated to a finite-dimensional $\mathbb{Q}_\ell$-vector space V, and u is induced by an automorphism $U : V \to V$ of this vector space.

The proof in this case then proceeds by a similar argument to the one for $\mathcal{F} = \mathbb{Q}_\ell$. If $\delta_1, \dots, \delta_s$ denotes the eigenvalues of U acting on $V \otimes_\iota \mathbb{C}$, then the sum of the absolute values of the eigenvalues in 7.17.1 gets replaced by

$$S_p(H^*_c(BG, \mathcal{F})) = \frac{1}{M_n} \sum_e |\delta_e| (\sum_{k \geq p} \sum_{\lambda \in \mathrm{Eg}^k(\Phi^n \circ A)} |\lambda|),$$

and we have

$$\mathrm{tr}_\iota(\alpha^{(n)} | H^*_c(BG, \mathcal{F})) = \frac{1}{M_n} \mathrm{tr}_\iota(U|V) \cdot \prod \frac{1}{1-\gamma_i} = \sum_{\beta \subset \mathrm{Fix}(\mathscr{C}^{(n)})} \mathrm{LT}_\iota(\beta, \mathcal{F}).$$

This completes the proof in the case when $\mathcal{F}$ is concentrated in a single degree.

Now consider the case of a general Weil complex with $\mathscr{C}$-structure $(\mathcal{F}, \varphi, u)$.

Lemma 7.20. *Let $p : \mathscr{X} \to \mathrm{Spec}(k)$ be an algebraic stack of finite type. Then there exists an integer t such that for all $k \in \mathbb{Z}$ we have*

$$p_! : D_c^{[-\infty,k]}(\mathscr{X}, \mathbb{Q}_\ell) \to D_c^{[-\infty,k-t]}(\mathbb{Q}_\ell).$$

Proof. Since the dualizing complex of $\mathscr{X}$ has finite quasi-injective dimension, there exists an integer t such that

$$D_{\mathscr{X}} : D_c^{[-\infty,k]}(\mathscr{X}, \mathbb{Q}_\ell) \to D_c^{[-k+t,\infty]}(\mathscr{X}, \mathbb{Q}_\ell).$$

Therefore we have a commutative diagram

$$\begin{tikzcd}
D_c^{[-\infty,k]}(\mathscr{X}, \mathbb{Q}_\ell) \arrow[r, "D_{\mathscr{X}}"] \arrow[rrd, "p_!"] & D_c^{[-k+t,\infty]}(\mathscr{X}, \mathbb{Q}_\ell) \arrow[r, "p_*"] & D_c^{[-k+t,\infty]}(\mathbb{Q}_\ell) \arrow[d, "D_{\mathrm{Spec}(k)}"] \\
 & & D_c^{[-\infty,k-t]}(\mathbb{Q}_\ell).
\end{tikzcd}$$

□

Proposition 7.21. *Suppose that the sum*

(7.21.1)
$$\sum_i \sum_k \sum_{\lambda \in \mathrm{Eg}^k(u^{(n)} | H^*_c(BG, \mathscr{H}^i(\mathcal{F})))} |\lambda|$$

converges. Then the sum

$$S(p_!\mathcal{F}) = \sum_k \sum_{\lambda \in \mathrm{Eg}^k(u^{(n)} | H^*_c(BG, \mathcal{F}))} |\lambda|$$

converges and

$$\mathrm{tr}_\iota(u^{(n)} | p_!\mathcal{F}) = \sum_i \mathrm{tr}_\iota(u^{(n)} | p_!\mathscr{H}^i(\mathcal{F})).$$

Proof. As before, let

$$S_q(p_!\mathcal{F}) := \sum_{k\geq q} \sum_{\lambda \in \mathrm{Eg}^k(p_!\mathcal{F})} |\lambda|,$$

and define similarly $S_q(p_!\tau_{\geq k}\mathcal{F})$.

By 7.20 there exists an integer t such that

$$S_{k-t}(p_!\mathcal{F}) = S_{k-t}(p_!\tau_{\geq k}\mathcal{F}).$$

By induction on k and using the distinguished triangles

$$\mathscr{H}^k(\mathcal{F})[-k] \to \tau_{\geq k}\mathcal{F} \to \tau_{\geq k+1}\mathcal{F} \to \mathscr{H}^k(\mathcal{F})[-k+1]$$

one sees that for all q we have

$$S_q(p_!\tau_{\geq k}\mathcal{F}) \leq \sum_{i\geq q} S_{q-i}(p_!\mathscr{H}^i_c(\tau_{\geq k}\mathcal{F})).$$

It follows that

$$S_{k-t}(p_!\mathcal{F}) \leq \sum_{i\geq k-t} S_{k-t-i}(p_!\mathscr{H}^i(\mathcal{F})),$$

and therefore $S(p_!\mathcal{F})$ converges. Set

$$\epsilon_k := \sum_{i<k} (-1)^i \mathrm{tr}_\iota(u^{(n)}|H^i_c(\tau_{\geq k-t}\mathcal{F})).$$

Lemma 7.22. *The sequence ϵ_k converges to zero absolutely as $k \to -\infty$.*

Proof. For a convergent complex (K, φ) with $K \in D^-_c(\mathbb{Q}_\ell)$, set

$$T_q(K) := \sum_{i<q} \sum_{\lambda \in \mathrm{Eg}^i(K)} |\lambda|.$$

Then certainly $|\epsilon_k| < T_k(\tau_{\geq k-t}\mathcal{F})$, so it suffices to show that the sequence $T_k(\tau_{\geq k-t}\mathcal{F})$ converges to zero.

For this note that by a similar argument to the above using the distinguished triangles

$$\mathscr{H}^s(\mathcal{F})[-s] \to \tau_{\geq s}\mathcal{F} \to \tau_{\geq s+1}\mathcal{F} \to \mathscr{H}^s(\mathcal{F})[-s+1],$$

one sees that

$$T_k(\tau_{\geq k-t}\mathcal{F}) \leq \sum_{i<k} T_{k-i}(p_!\mathscr{H}^i(\tau_{\geq k-t}\mathcal{F})).$$

From this and the convergence of 7.21.1 the result follows. □

We then have

$$\begin{aligned}
\mathrm{tr}_\iota(u^{(n)}|p_!\mathcal{F}) &= \lim_{k\to-\infty}\sum_{i\geq k}(-1)^i\mathrm{tr}_\iota(u^{(n)}|H^i_c(BG,\mathcal{F}))\\
&= \lim_{k\to-\infty}\sum_{i\geq k}(-1)^i\mathrm{tr}_\iota(u^{(n)}|H^i_c(BG,\tau_{\geq k-t}\mathcal{F}))\\
&= \lim_{k\to-\infty}\sum_{i\in\mathbb{Z}}(-1)^i\mathrm{tr}_\iota(u^{(n)}|H^i_c(BG,\tau_{\geq k-t}\mathcal{F}))-\epsilon_k\\
&= \lim_{k\to-\infty}\sum_{i\in\mathbb{Z}}(-1)^i\mathrm{tr}_\iota(u^{(n)}|H^i_c(BG,\tau_{\geq k-t}\mathcal{F}))\\
&= \sum_i \mathrm{tr}_\iota(u^{(n)}|p_!\mathscr{H}^i(\mathcal{F})).
\end{aligned}$$

□

Next let us verify that the sum 7.21.1 indeed converges. Let $F\in D^-_c(\mathbb{Q}_\ell)$ denote the pullback of $\mathcal{F}$ along $\mathrm{Spec}(k)\to BG$, and for $i\in\mathbb{Z}$ let $\mathrm{Eg}^i(u^{(n)}|F)$ denote the set of eigenvalues of $u^{(n)}$ acting on $H^i(F)$. By assumption there exists an integer n_0 such that the sum

$$\sum_i\sum_{\gamma\in\mathrm{Eg}^i(u^{(n)}|F)}|\gamma|$$

converges. After possibly replacing n_0 by a bigger integer, we may also assume that for $n\geq n_0$ the sum

$$\sum_{i\in\mathbb{Z}}\sum_{\lambda\in\mathrm{Eg}^i(u^{(n)}|p_!\mathbb{Q}_\ell)}|\lambda|$$

converges, that $\mathrm{Fix}(\mathscr{C}^{(n)})$ consists of a single component which is an étale gerbe over $\mathrm{Spec}(k)$, and that

$$\mathrm{tr}_\iota(u^{(n)}|p_!\mathbb{Q}_\ell)=\sum_{\beta\subset\mathrm{Fix}(\mathscr{C}^{(n)})}\mathrm{LT}_\iota(\beta,\mathbb{Q}_\ell).$$

Then

$$\begin{aligned}
&\sum_i\sum_k\sum_{\lambda\in\mathrm{Eg}^k(u^{(n)}|H^k_c(BG,\mathscr{H}^i(\mathcal{F})))}|\lambda|\\
=\ &\sum_i\sum_{\gamma\in\mathrm{Eg}^i(u^{(n)}|F)}|\gamma|(\sum_k\sum_{\lambda\in\mathrm{Eg}^k(u^{(n)}|H^k_c(BG,\mathbb{Q}_\ell))}|\lambda|)\\
=\ &S(u^{(n)}|p_!\mathbb{Q}_\ell)\cdot(\sum_i\sum_{\gamma\in\mathrm{Eg}^i(u^{(n)}|F)}|\gamma|).
\end{aligned}$$

We therefore find that $p_!\mathcal{F}$ is convergent and that

$$\begin{aligned}
\mathrm{tr}_\iota(u^{(n)}|p_!\mathcal{F}) &= \textstyle\sum_i \mathrm{tr}_\iota(u^{(n)}|p_!\mathscr{H}^i(\mathcal{F})) \quad \text{(by 7.21)} \\
&= \textstyle\sum_i \sum_{\beta\subset \mathrm{Fix}(\mathscr{C}^{(n)})} \mathrm{LT}_\iota(\beta, (\mathscr{H}^i(\mathcal{F}), \varphi, u^{(n)})) \\
&= \textstyle\sum_{\beta\subset \mathrm{Fix}(\mathscr{C}^{(n)})} \sum_i \mathrm{LT}_\iota(\beta, (\mathscr{H}^i(\mathcal{F}), \varphi, u^{(n)})) \\
&= \textstyle\sum_{\beta\subset \mathrm{Fix}(\mathscr{C}^{(n)})} \mathrm{LT}_\iota(\beta, (\mathcal{F}, \varphi, u^{(n)})).
\end{aligned}$$

This completes the proof of 7.7 in the case when G_0 is geometrically connected. □

8. Classifying stacks of finite groups

Throughout this section we work with $\mathbb{Q}_\ell$-coefficients.

8.1. Let G be a finite group and consider the classifying stack BG over $\mathrm{Spec}(k)$. The category of constructible sheaves of $\mathbb{Q}_\ell$-modules on BG is then equivalent to the category of finite type representations of G over $\mathbb{Q}_\ell$. Let $p : \mathrm{Spec}(k) \to BG$ be the étale projection corresponding to the trivial torsor. Since p is étale, we have $p^* = p^!$ and this functor is given by the functor sending a representation V of G to the underlying $\mathbb{Q}_\ell$-module. This functor has left adjoint the functor sending a $\mathbb{Q}_\ell$-module F to $\mathbb{Q}_\ell[G] \otimes_{\mathbb{Q}_\ell} F$ and right adjoint the functor sending F to $\mathrm{Hom}(G, F)$. It follows that if F is a finite type $\mathbb{Q}_\ell$-module then the map $\epsilon : p_!F \to p_*F$ is given by a map

$$\epsilon^\dagger : \mathbb{Q}_\ell[G] \otimes_{\mathbb{Q}_\ell} F \to \mathrm{Hom}(G, F).$$

Lemma 8.2. *Let $g_0 \in G$ and $f \in F$ be elements. Then $\epsilon^\dagger(g_0 \otimes f)$ is the function $G \to F$ sending g_0 to f and $g \neq g_0$ to 0.*

Proof. It suffices to consider the case when $F = \mathbb{Q}_\ell$. By construction of the map ϵ in the proper representable case [13, 5.2.1], the map $\epsilon^\dagger$ is induced by the canonical isomorphism $f_!\mathbb{Q}_\ell \to f_*\mathbb{Q}_\ell$, where $f : G \to \mathrm{Spec}(k)$ is the structure morphism. We leave to the reader the verification this classically constructed isomorphism is the one indicated in the lemma. □

8.3. Now consider a morphism of finite groups $\alpha : H \to G$, which induces a diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, "s"] & BH \arrow[r, "\alpha"] & BG, \end{tikzcd}$$

where s is the projection defined by the trivial H-torsor. Let M be an H-representation, with underlying $\mathbb{Q}_\ell$-module M_0 and associated sheaf $\mathcal{M}$ on BG. Let $\mathcal{M}_0$ denote $s^*\mathcal{M}$ (the constant sheaf defined by M_0). We then have a commutative diagram

$$\begin{tikzcd}
\alpha_*\mathcal{M} \arrow[d, hook, "\mathrm{id}\to s_*s^*"] & \alpha_!\mathcal{M} \arrow[l, "\epsilon^{\alpha}"'] \arrow[d, hook, "\mathrm{id}\to s_*s^*"] \\
\alpha_*s_*\mathcal{M}_0 & \alpha_!s_*\mathcal{M}_0 \arrow[l, "\epsilon^{\alpha}"'] \\
(\alpha s)_!\mathcal{M}_0 \arrow[u, "\epsilon^{\alpha s}"] \arrow[r, "\simeq"] & \alpha_!s_!\mathcal{M}_0. \arrow[u, "\epsilon^{s}"']
\end{tikzcd}$$

In terms of representations, this diagram can be rewritten as

$$\begin{tikzcd}
\mathrm{Hom}_H(G, M) \arrow[d, hook, "a"] & \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \arrow[l, "\epsilon^{\alpha\dagger}"'] \arrow[d, hook, "b"] \\
\mathrm{Hom}(G, M_0) & \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \mathrm{Hom}(H, M_0) \arrow[l, "\epsilon^{\alpha\dagger}"'] \\
\mathbb{Z}[G] \otimes_{\mathbb{Z}} M_0 \arrow[u, "\epsilon^{\alpha s\dagger}"] \arrow[r, "\simeq"] & \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} (\mathbb{Z}[H] \otimes M_0), \arrow[u, "1\otimes\epsilon^{s\dagger}"']
\end{tikzcd}$$

where a is the natural inclusion and b is the map induced by the map $M \to \mathrm{Hom}(H, M_0)$ sending m to the function $h \mapsto h \cdot m$.

Lemma 8.4. *(i) Let $g_0 \in G$ and $f \in M$ be elements, and let $g_0 \otimes f \in \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M$ be the resulting tensor. Then $\epsilon^{\alpha\dagger}(g_0 \otimes f)$ is equal to the function*

$$\psi_{g_0\otimes f} : G \to M, \quad g \mapsto \sum_{\{h\in H | g=\alpha(h)\cdot g_0\}} h \cdot m,$$

with the convention that a sum over the empty set is 0.

(ii) Let $\mathcal{V}$ be a constructible sheaf of $\mathbb{Q}_\ell$-modules on BG corresponding to a G-representation V. Then the map

$$V \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} V$$

corresponding to the composite

$$\mathcal{V} \xrightarrow{\mathrm{id}\to\alpha_*\alpha^*} \alpha_*\alpha^*\mathcal{V} \xrightarrow{\epsilon^{\alpha-1}} \alpha_!\alpha^*\mathcal{V}$$

sends $v \in V$ to the element

$$\frac{1}{|H|}\sum_{g\in G} g \otimes (gv).$$

Proof. Statement (ii) follows immediately from (i).

To prove (i), it suffices to show that the image of $\epsilon^{\alpha\dagger}(g_0 \otimes f)$ in $\mathrm{Hom}(G, M_0)$ is equal to $\psi_{g_0\otimes f}$. Now observe that $b(g_0 \otimes f)$ is by 8.2 equal to

$$g_0 \otimes \epsilon^{s\dagger}(\sum_{h\in H} h \otimes hm).$$

Therefore

$$\epsilon^{\alpha^\dagger} b(g_0 \otimes f) = \epsilon^{\alpha s\dagger}(\sum_{h\in H} \alpha(h)\cdot g_0 \otimes hm).$$

By 8.2 this implies the result. □

8.5. Let H and G be finite groups and let $\alpha, \beta : H \to G$ be two homomorphisms with β injective. Let $\mathscr{C}_0$ (resp. $\mathscr{X}_0$) denote the classifying stack BH (resp. BG) over $\mathbb{F}_p$, and let $c_1 : \mathscr{C}_0 \to \mathscr{X}_0$ (resp. $c_2 : \mathscr{C}_0 \to \mathscr{X}_0$) be the map induced by α (resp. β).

Theorem 8.6. *Let $(\mathcal{F}, \varphi, u)$ be a convergent Weil complex with $\mathscr{C}$-structure on $\mathscr{X}$. Then there exists an integer n_0, independent of $(\mathcal{F}, \varphi, u)$, such that for every $n \geq n_0$ we have:*

(i) The complex $R\Gamma_c(\mathscr{X}, \mathcal{F}) \in D_c^-(\mathbb{Q}_\ell)$ with the endomorphism $R\Gamma_c(u^{(n)})$ is convergent.

(ii) $\mathrm{Fix}(\mathscr{C}^{(n)})$ is pseudo-finite over $\mathrm{Spec}(k)$, and

$$\mathrm{tr}_\iota(R\Gamma_c(u^{(n)})|R\Gamma_c(\mathscr{X}, \mathcal{F})) = \sum_{\beta\subset \mathrm{Fix}(\mathscr{C}^{(n)})} \mathrm{LT}_\iota(\beta, (\mathcal{F}, \varphi, u^{(n)})).$$

8.7. The proof occupies the remainder of this section.

As mentioned above, the category $D_c^-(\mathscr{X}, \mathbb{Q}_\ell)$ is canonically equivalent to the bounded above derived category $D_c^-(\mathrm{Rep}_{\mathbb{Q}_\ell}(G))$ of complexes of G-representations over $\mathbb{Q}_\ell$ with finite-dimensional cohomology groups, and with this identification the functor

$$R\Gamma_c : D_c^-(\mathscr{X}, \mathbb{Q}_\ell) \to D_c^-(\mathbb{Q}_\ell)$$

is identified with the coinvariants functor

$$(-)_G : D_c^-(\mathrm{Rep}_{\mathbb{Q}_\ell}(G)) \to D_c^-(\mathbb{Q}_\ell).$$

8.8. Consider first the case of a Weil complex $\mathcal{F}$ concentrated in degree 0. Let $F \in \mathrm{Rep}_{\mathbb{Q}_\ell}(G)$ denote the object corresponding to $\mathcal{F}$, and let F_α (resp. F_β) denote the object of $\mathrm{Rep}_{\mathbb{Q}_\ell}(H)$ obtained by pulling back along α (resp. β). Then the $\mathscr{C}$-structure u on $\mathcal{F}$ corresponds to a morphism $F_\alpha \to F_\beta$ in $\mathrm{Rep}_{\mathbb{Q}_\ell}(H)$. The endomorphism $R\Gamma_c(u^{(n)})$ on $R\Gamma_c(\mathcal{F})$ corresponds to the map

$$F_G \xrightarrow{a} F_{\alpha,H} \xrightarrow{u^{(n)}} F_{\beta,H} \xrightarrow{\pi} F_G, \tag{8.8.1}$$

where $\pi : F_{\beta,H} \to F_G$ is the canonical projection, and by 8.4 (ii) the map a is the map induced by the map

$$\frac{1}{|H|}\sum_{g\in G} g : F \to F.$$

Consideration of the commutative diagram

$$\begin{tikzcd}
F^G \arrow[r, "\frac{1}{|H|}\sum_{g\in G} g"] \arrow[d, hook] & F_\alpha^H \arrow[r, "u^{(n)}"] \arrow[d, hook] & F_\beta^H \arrow[d, hook] & \\
F \arrow[r, "\frac{1}{|H|}\sum_{g\in G} g"] \arrow[d, two heads] \arrow[ur, dashed] & F \arrow[r, "u^{(n)}"] \arrow[d, two heads] & F \arrow[d, two heads] \arrow[dr, two heads] & \\
F_G \arrow[r, "a"] & F_{\alpha,H} \arrow[r, "u^{(n)}"] & F_{\beta,H} \arrow[r, "\pi"] & F_G
\end{tikzcd}$$

then shows that the trace of 8.8.1 is equal to the trace of the map

$$\frac{1}{|H|}\sum_{g\in G} u^{(n)}\circ g : F \to F.$$

8.9. The fixed point stack $\mathrm{Fix}(\mathscr{C}^{(n)})$ can be described as follows. Since any torsor over $\mathrm{Spec}(k)$ is trivial, a similar argument to the one proving 7.9 shows that $\mathrm{Fix}(\mathscr{C}^{(n)})$ is isomorphic to the quotient of G by the action of H given by

$$h * g = \beta(h)g\alpha(h)^{-1}.$$

From this it follows that

$$\sum_{x\in\mathrm{Fix}(\mathscr{C}^{(n)})(k)} \mathrm{tr}_\iota(u^{(n)}, \mathcal{F}_x) = \mathrm{tr}_\iota(\frac{1}{|H|}\sum_{g\in G} u^{(n)}\circ g, F).$$

This proves 8.6 in the case when F is concentrated in degree 0.

8.10. Next consider a general bounded Weil complex with $\mathscr{C}$-structure $(\mathcal{F},\varphi,u)$ on $\mathscr{X}_0$, and let $F\in D^b_c(\mathrm{Rep}_{\mathbb{Q}_\ell}(G))$ denote the corresponding complex of G-representations. The condition that $(\mathcal{F},\varphi,u)$ is convergent means that there exists an integer n_0 such that for any $n\geq n_0$ and any $g\in G$ the sum of the eigenvalues

$$\sum_k \sum_{\lambda\in\mathrm{Eg}^k(u^{(n)}\circ g|F)} |\lambda| \tag{8.10.1}$$

converges. The following proposition now completes the proof of 8.6.

Proposition 8.11. *For any $n\geq n_0$ the sum*

$$\sum_k \sum_{\lambda\in\mathrm{Eg}^k(u^{(n)}|p_!\mathcal{F})} |\lambda| \tag{8.11.1}$$

converges and

$$\mathrm{tr}_\iota(u^{(n)}|p_!\mathcal{F}) = \sum_{\beta\subset\mathrm{Fix}(\mathscr{C}^{(n)})} \mathrm{LT}_\iota(\beta,(\mathcal{F},\varphi,u)).$$

Proof. For the convergence of 8.11.1, note that

$$\begin{aligned}\sum_{\lambda\in\mathrm{Eg}^k(u^{(n)}|p_!\mathcal{F})}|\lambda| &= \sum_{\lambda\in\mathrm{Eg}(u^{(n)}|H^k(F)_G)}|\lambda| \\ &\leq \sum_{\lambda\in\mathrm{Eg}(u^{(n)}|H^k(F))}|\lambda| \\ &= \sum_{\lambda\in\mathrm{Eg}^k(u^{(n)}|F)}|\lambda|.\end{aligned}$$

The convergence of 8.11.1 therefore follows from the convergence of 8.10.1. The statement about the traces follows from the case of a complex concentrated in a single degree and from noting that

$$\mathrm{tr}_\iota(u^{(n)}|p_!\mathcal{F}) = \sum_k \mathrm{tr}_\iota(u^{(n)}|p_!\mathscr{H}^k(\mathcal{F})),$$

and

$$\sum_{\beta\subset \mathrm{Fix}(\mathscr{C}^{(n)})} \mathrm{LT}_\iota(\beta,(\mathcal{F},\varphi,u)) = \sum_k \sum_{\beta\subset \mathrm{Fix}(\mathscr{C}^{(n)})} \mathrm{LT}_\iota(\beta,(\mathscr{H}^k(\mathcal{F}),\varphi,u)).$$

□

9. The classifying stack of a general group scheme

In this section we prove 7.7 in general.

9.1. As in 7.7, let $G_0/\mathbb{F}_q$ be a finite type group scheme, and let $\alpha : G_0 \to G_0$ be a finite endomorphism. Let $\mathscr{X}_0$ denote BG_0.

9.2. Let

$$1 \to G_0^0 \to G_0 \to H_0 \to 1$$

be the connected-étale sequence of G_0. To verify 7.7 we may by 1.20 make a finite extension $\mathbb{F}_q \to \mathbb{F}_{q^r}$, and therefore we may assume that H_0 is a constant group scheme. After perhaps making a further extension we may also assume that the map $G_0(\mathbb{F}_q) \to H_0$ is surjective. Fix a set $\{g_h\}_{h\in H}$ of elements of $G_0(\mathbb{F}_q)$, with g_h a lifting of h. We further assume that g_e is the identity in G_0.

Let α_H (resp. α_{G^0}) be the endomorphism of H_0 (resp. G_0^0) induced by α.

Recall that by 7.10 the stack $\mathrm{Fix}(\mathscr{C}^{(n)})$ is isomorphic to the stack quotient of G by the action of G given by

$$h * g = \alpha^{(n)}(h)gh^{-1}.$$

For $h \in H$ let P_h denote the inverse image in G of h so that P_h is a G^0-torsor.

Lemma 9.3. *For $n \geq 1$ and $h \in H$, the map*

$$G^0 \to P_h, \quad z \mapsto \alpha^{(n)}(z)g_hz^{-1} \tag{9.3.1}$$

is surjective.

Proof. The element g_h defines an isomorphism $G^0 \to P_h$. Under this identification the map 9.3.1 becomes identified with the map

$$G^0 \to G^0, \quad z \mapsto \alpha^{(n)}(z)g_hz^{-1}g_h^{-1}.$$

The result therefore follows from 2.2 (applied with j the map $z \mapsto g_hzg_h^{-1}$) which implies that 9.3.1 is étale, and the fact that G^0 is connected. □

It follows that every connected component of $\mathrm{Fix}(\mathscr{C}^{(n)})$ can be represented by the point $[g] \in \mathrm{Fix}(\mathscr{C}^{(n)})(\mathbb{F}_q)$ defined by an element $g \in G(\mathbb{F}_q)$ (since by assumption the map $G(\mathbb{F}_q) \to H$ is surjective), and that any two liftings $g \in G(\mathbb{F}_q)$ of a given $h \in H$ define the same connected component.

For any element $c \in G$ let $c_g : G \to G$ and $c_g : G^0 \to G^0$ be the maps given by conjugation by g.

The induced functor $Bc_g : BG \to BG$ is canonically isomorphic to the identity. Indeed if P is a G-torsor then translation by g defines an isomorphism $P \times^{G,c_g} G \to P$. We therefore have a 2-commutative diagram

$$\begin{tikzcd}
BG^0 \arrow[rr, "c_g"] \arrow[dr] & & BG^0 \arrow[dl] \\
& BG. &
\end{tikzcd}$$

In particular, if $\mathcal{F} \in D_c^-(BG, \mathbb{Q}_\ell)$ and if $\mathcal{F}^0 \in D_c^-(BG^0, \mathbb{Q}_\ell)$ denotes the restriction, then there is a canonical isomorphism $t_g : c_g^* \mathcal{F}^0 \to \mathcal{F}^0$.

For $h \in H$ let $\mathscr{C}_0^h$ denote the correspondence on BG_0^0 given by

$$(c_{g_h} \circ \alpha_{G^0}, \mathrm{id}) : BG_0^0 \to BG_0^0 \times BG_0^0.$$

If $(\mathcal{F}, \varphi, u)$ is a Weil complex with $\mathscr{C}$-structure on BG, then we obtain for every $h \in H$ a Weil complex with $\mathscr{C}^h$-structure $(\mathcal{F}^0, \varphi, u \circ t_{g_h})$ on BG_0.

Lemma 9.4. *The Weil complex with $\mathscr{C}$-structure $(\mathcal{F}, \varphi, u)$ on BG is convergent if and only if for every $h \in H$ the Weil complex with $\mathscr{C}^h$-structure $(\mathcal{F}^0, \varphi, u \circ t_{g_h})$ is convergent on BG^0.*

Proof. As mentioned in 9.2, for $n \geq 1$ the points of $\mathrm{Fix}(\mathscr{C}^{(n)})$ are all represented by the points defined by the g_h. Now if $\mathcal{F}_{g_h}$ denotes the pullback along $g_h : \mathrm{Spec}(k) \to \mathscr{C}^{(n)}$ of $\mathcal{F}$ to $D_c^-(\mathbb{Q}_\ell)$ and if $\mathcal{F}_e \in D_c^-(\mathbb{Q}_\ell)$ denotes the pullback along $e : \mathrm{Spec}(k) \to \mathscr{C}^{(n)}$, then the action of $u^{(n)}$ on $\mathcal{F}_{g_h}$ is given by the composite

$$\mathcal{F}_{g_h} \xrightarrow{\simeq} \mathcal{F}_e \xrightarrow{t_{g_h}} \mathcal{F}_e \xrightarrow{u^{(n)}} \mathcal{F}_e \simeq \mathcal{F}_{g_h},$$

where we have used the canonical isomorphism $\mathcal{F}_{g_h} \simeq \mathcal{F}_e$. Thus $(\mathcal{F}, \varphi, u)$ is convergent if and only if there exists an integer n_0 such that the complexes with endomorphisms $(\mathcal{F}_e, u^{(n)} \circ t_{g_h})$ are convergent. This implies the lemma. □

Let $p : BG \to BH$ be the projection and consider the diagram

$$\begin{tikzcd}
& BG \arrow[dl, "\alpha"'] \arrow[dd, "p"] \arrow[dr, "\mathrm{id}"] & \\
BG \arrow[dd, "p"] & & BG \arrow[dd, "p"] \\
& BH \arrow[dl, "\alpha_H"'] \arrow[dr, "\alpha_H"] & \\
BH & & BH.
\end{tikzcd}$$

Let $h \in H$ be an element defining a fixed point $[h]$ of $\alpha_H^{(n)}$. Then the local term of $(p_! \mathcal{F}, p_! \varphi, p_! u^{(n)})$ at $[h] \in \mathrm{Fix}(\alpha^{(n)})$ is given by the trace of the map

$$u^{(n)} \circ t_{g_h} : R\Gamma_c(BG^0, \mathcal{F}^0) \to R\Gamma_c(BG^0, \mathcal{F}^0).$$

From the case of a connected group, we conclude that $(p_! \mathcal{F}, p_! \varphi, p_! u)$ is convergent, and then by the case of a finite group we conclude that $(R\Gamma_c \mathcal{F}, R\Gamma_c \varphi, R\Gamma_c u)$ is also convergent.

To prove the statement about local terms, observe

$$\begin{aligned}
\mathrm{tr}_\iota(u^{(n)}|R\Gamma_c(BG,\mathcal{F})) &= \mathrm{tr}_\iota(p_!u^{(n)}|R\Gamma_c(BH,p_!\mathcal{F}))\\
&= \frac{1}{|H|}\sum_{h\in H}\mathrm{tr}_\iota(p_!u^{(n)}\circ h|(p_!\mathcal{F})_h) \qquad \text{(case of finite group)}\\
&= \frac{1}{|H|}\sum_{h\in H}\mathrm{tr}_\iota(u^{(n)}\circ t_{g_h}|R\Gamma_c(BG^0,\mathcal{F}^0))\\
&= \frac{1}{|H|}\sum_{h\in H}\frac{1}{|G^0(\mathbb{F}_q)|}\sum_{g\in G^0(\mathbb{F}_q)}\mathrm{tr}_\iota(u^{(n)}\circ g_h\cdot g|\mathcal{F}_{g_hg}) \quad \text{(connected case)}\\
&= \frac{1}{|G(\mathbb{F}_q)|}\sum_{g\in G(\mathbb{F}_q)}\mathrm{tr}_\iota(u^{(n)}\circ g|\mathcal{F}_g).
\end{aligned}$$

This completes the proof of 7.7.

10. Proof of 1.24

Lemma 10.1. *The map $c_1 : \mathscr{C}_0 \to \mathscr{X}_0$ is proper with finite diagonal and $c_2 : \mathscr{C}_0 \to \mathscr{X}_0$ is quasi-finite.*

Proof. That c_2 is quasi-finite is clear since the diagram

$$\begin{tikzcd}
C_0 \arrow[r] \arrow[d, "c_2"] & \mathscr{C}_0 \arrow[d, "c_2"] \\
X_0 \arrow[r] & \mathscr{X}_0
\end{tikzcd}$$

is cartesian.

For the properness of c_1, note that the fiber product $\mathscr{P}_0$ of the diagram

$$\begin{tikzcd}
 & {[C_0/G_0]} \arrow[d, "c_1"] \\
X_0 \arrow[r] & {[X_0/G_0],}
\end{tikzcd}$$

is isomorphic to

$$[C_0\times G_0/G_0],$$

where $h\in G_0$ acts on $C_0\times G_0$ by

$$(x,g)\mapsto (hx, g\alpha(h)^{-1}).$$

To verify that the diagonal of $\mathscr{P}_0$ over X_0 is finite it therefore suffices to show that the map

$$(C_0\times G_0)\times G_0 \to (C_0\times G_0)\times_{X_0}(C_0\times G_0),\quad (x,g,h)\mapsto (x,g)\times(hx, g\alpha(h)^{-1})$$

is a finite morphism, which is clear since α is finite.

To verify the properness of $\mathscr{P}_0 \to X_0$ we may base change to the algebraic closure k of $\mathbb{F}_q$, in which case there exist points $x_1, \dots, x_r \in G(k)$ such that the map

$$\coprod_{i=1}^{r} G \to G,$$

which on the i-th component is given by $g \mapsto x_i \cdot \alpha(g)$, is surjective.

The induced map

$$\coprod_{i=1}^{r} C \to [C \times G/G],$$

which on the i-th component sends $z \in C$ to (z, x_i), is then a surjection. We therefore obtain a commutative diagram

$$\begin{tikzcd}
\coprod_{i=1}^{r} C \arrow[r, two heads, "a"] \arrow[rr, bend left, "b"] & \mathscr{P} \arrow[r, "c"] & X,
\end{tikzcd}$$

where a is surjective, b is proper, and c is separated. It follows that the map c is also proper. □

Write $B\alpha^{(n)} : BG_0 \to BG_0$ for the endomorphism defined by $\alpha^{(n)}$.

Let $p : X_0 \to \mathscr{X}_0$ (resp. $q : C_0 \to \mathscr{C}_0$) be the projection. Choose n_0 as in 1.19 for convergent Weil-complexes with α-structure on BG. By 1.20 it suffices to prove 1.19 after making a finite extension $\mathbb{F}_q \to \mathbb{F}_{q^r}$. We may therefore assume that if $G_0 \to H_0$ is the maximal étale quotient of G_0, then H_0 is a constant group scheme and the map $G_0(\mathbb{F}_q) \to H_0$ is surjective. This implies that there exists $g_1, \dots, g_r \in G(\mathbb{F}_q)$ such that the components of $\mathrm{Fix}(B\alpha^{(n)})$ are represented by the components $\{[g_i]\}$ of $\mathrm{Fix}(B\alpha^{(n)})$ corresponding to these elements, and the same holds after replacing n by a larger integer (see the discussion after 9.3).

For $i = 1, \dots, r$, let $c^i : C_0^i \to X_0 \times X_0$ denote the correspondence

$$(t_{g_i} \circ c_1, c_2) : C_0 \to X_0 \times X_0,$$

where $t_{g_i} : X_0 \to X_0$ denotes the action of g_i on X_0. Let $\alpha^i : G_0 \to G_0$ denote $c_{g_i} \circ \alpha$, where c_{g_i} denotes conjugation by g_i. Then $c_1^i : C_0^i \to X_0$ is compactible with the G_0-actions in the sense that for any $z \in C_0^i$ and $h \in G$ we have

$$c_1^i(h * z) = \alpha^i(h) \cdot c_1^i(z).$$

We therefore obtain a correspondence

$$c^i : \mathscr{C}_0^i \to \mathscr{X}_0 \times \mathscr{X}_0.$$

After possible replacing n_0 by a larger integer, we may assume that Deligne's conjecture holds for any bounded Weil complex with C_0^i-structure on X_0. We claim that with these assumptions the conclusions (i) and (ii) in 1.19 hold for any Weil complex $(\mathcal{F}, \varphi, u)$ with $\mathscr{C}$-structure on $\mathscr{X}$ and $\mathcal{F}$ bounded.

Let $F \in D_c^b(X, \mathbb{Q}_\ell)$ be the restriction of $\mathcal{F}$ to X, and let $\sigma : \mathscr{X}_0 \to BG_0$ be the canonical morphism.

Lemma 10.2. *The map*

$$\sigma_! u^{(n)} : (\sigma_!\mathcal{F})_{[g_i]} \to (\sigma_!\mathcal{F})_{[g_i]}$$

is canonically identified with the map

$$R\Gamma_c(u^{i,(n)}) : R\Gamma_c(X, F) \to R\Gamma_c(X, F),$$

where $[g_i] : \mathrm{Spec}(k) \to \mathrm{Fix}(\alpha_G^{(n)})$ *is the point defined by* g_i*. In particular* $(\sigma_!\mathcal{F}, \sigma_!\varphi, \sigma_! u)$ *is convergent (since* $R\Gamma_c(X, F)$ *is a bounded complex).*

Proof. This follows from the definition of the local terms. □

Let $G_{g_i}^{(n)}$ denote the group scheme

$$G_{g_i}^{(n)} := \{h \in G | h^{-1} g_i \alpha_G^{(n)}(h) = g_i\}.$$

Then the connected components of $\mathrm{Fix}(B\alpha^{(n)})$ are all of the form $BG_{g_i}^{(n)}$ for some i.

Fix an integer n_0 such that Fujiwara's theorem holds for F with respect to each of the C^i-structures. From 10.2 we also find that for $n \geq n_0$

$$\begin{aligned} \mathrm{tr}_\iota(u^{(n)}|R\Gamma_c(\mathscr{X}, \mathcal{F})) &= \sum\nolimits_{[g_i]\in \mathrm{Fix}(\alpha_G^{(n)})} \frac{1}{|G_{g_i}^{(n)}|}\mathrm{tr}_\iota(u^{i,(n)}|R\Gamma_c(X, F)) \\ &= \sum\nolimits_{[g_i]\in \mathrm{Fix}(B\alpha_G^{(n)})} \frac{1}{|G_{g_i}^{(n)}|} \sum\nolimits_{x\in \mathrm{Fix}(C^{i,(n)})} \mathrm{tr}_\iota(u^{i,(n)}|F_{c_2(x)}). \end{aligned} \tag{10.2.1}$$

Let

$$\pi : \mathrm{Fix}(\mathscr{C}^{(n)}) \to \mathrm{Fix}(B\alpha_G^{(n)})$$

be the projection. For any i, let $\mathscr{P}_i$ denote the fiber product

$$\mathrm{Fix}(\mathscr{C}^{(n)}) \times_{\mathrm{Fix}(B\alpha_G^{(n)})} BG_{g_i}^{(n)}.$$

This fiber product can be described as follows. There is an action of $G_{g_i}^{(n)}$ on $\mathrm{Fix}(C^{i,(n)})$ for which an element $h \in G_{g_i}^{(n)}$ sends a fixed point $x \in \mathrm{Fix}(C^{i,(n)})$ to the point hx. Note that this is again a fixed point as

$$c_1^{i,(n)}(hx) = g_i c_1^{(n)}(hx) = g_i \alpha_G^{(n)}(h) c_1^{(n)}(x) = h(h^{-1} g_i \alpha_G^{(n)}(h)) c_1^{(n)}(x) = h(g_i c_1^{(n)}(x)) = c_2(hx).$$

Lemma 10.3. *There is a canonical isomorphism* $\mathscr{P}_i \simeq [\mathrm{Fix}(C^{i,(n)})/G_{g_i}^{(n)}]$.

Proof. The stack $\mathscr{P}_i$ is equal to the stack associated to the prestack which to any scheme T associates the following category:

Objects: Pairs (x, g), where $x : T \to C$ is a morphism and $g \in G(T)$ is an element in the $\rho_{\alpha^{(n)}}$-orbit of g_i, such that the diagram

$$\begin{tikzcd} T \arrow[rr, "x"] \arrow[dr, "c_2"'] & & C \arrow[dl, "t_g\circ c_1^{(n)}"'] \\ & X & \end{tikzcd}$$

commutes.

Morphisms: A morphism $(x, g) \to (x', g')$ of objects over T is an element $h \in G(T)$ such that

$$g' = \alpha_G^{(n)}(h) g h^{-1}, \quad x' = h \cdot x.$$

It follows that every object of $\mathscr{P}_i(T)$ is locally on T represented by a pair (x, g_i), where $x \in \mathrm{Fix}(C^{i,(n)})$. From this the lemma follows. ☐

Lemma 10.3 implies that

$$\sum_{\beta \subset \mathscr{P}_i} \frac{1}{|\mathrm{Stab}(\beta)|} \mathrm{tr}_\iota(\beta|\mathcal{F}) = \sum_{x \in \mathrm{Fix}(C^{i,(n)})} \frac{1}{|G_{g_i}^{(n)}|} \mathrm{tr}_\iota(u^{i,(n)}|R\Gamma_c(X, F)).$$

Combining this with 10.2.1 we obtain 1.24. ☐

11. Automorphisms of algebraic spaces

11.1. In general the validity of 1.19 is additive in the following sense. Consider a correspondence $c : \mathscr{C}_0 \to \mathscr{X}_0 \times \mathscr{X}_0$ as in 1.19. If $\mathscr{Z}_0 \to \mathscr{X}_0$ is any morphism of algebraic stacks, then as discussed in 1.5 the correspondence c induces a correspondence on $\mathscr{Z}_0$ given by

$$c_{\mathscr{Z}} : \mathscr{C}_{\mathscr{Z},0} := \mathscr{C}_0 \times_{(\mathscr{X}_0 \times \mathscr{X}_0)} (\mathscr{Z}_0 \times \mathscr{Z}_0) \xrightarrow{\mathrm{pr}} \mathscr{Z}_0 \times \mathscr{Z}_0.$$

If $\mathscr{Z}_0 \to \mathscr{X}_0$ is the inclusion of a locally closed substack, then we say that $\mathscr{Z}_0$ is *c-invariant* if the maximal reduced closed substacks of $c_1^{-1}(\mathscr{Z})$ and $c_2^{-1}(\mathscr{Z})$ in $\mathscr{C}$ are equal.

Lemma 11.2. *Let $\mathscr{Z}_0 \subset \mathscr{X}_0$ be a c-invariant substack, and let $\mathscr{C}_{\mathscr{Z},0} \subset \mathscr{C}_0$ denote $c_1^{-1}(\mathscr{Z}_0)_{\mathrm{red}} = c_2^{-1}(\mathscr{Z}_0)_{\mathrm{red}}$. Then $c_{\mathscr{Z},1} : \mathscr{C}_{\mathscr{Z},0} \to \mathscr{Z}_0$ is proper and $c_{\mathscr{Z},2} : \mathscr{C}_{\mathscr{Z},0} \to \mathscr{Z}_0$ is quasi-finite.*

Proof. The quasi-finiteness follows from noting that there is a commutative diagram

$$\begin{tikzcd}
\mathscr{C}_{\mathscr{Z},0} \arrow[r, "c_{\mathscr{Z},2}"] \arrow[d, hook] & \mathscr{Z}_0 \arrow[d, hook] \\
\mathscr{C}_0 \arrow[r, "c_2"] & \mathscr{X}_0,
\end{tikzcd}$$

where the vertical arrows are immersions and c_2 is quasi-finite (note also that this doesn't require $\mathscr{Z}_0$ to be c-invariant).

The properness of $c_{\mathscr{Z},1}$ can be seen by consider the commutative diagram

$$\begin{tikzcd}
\mathscr{C}_{\mathscr{Z},0} \arrow[d, "k"] & & \\
\mathscr{C}_0 \times_{\mathscr{X}_0 \times \mathscr{X}_0} (\mathscr{Z}_0 \times \mathscr{X}_0) \arrow[r] \arrow[d] \arrow[rr, bend left, "c_{\mathscr{Z},1}"] & \mathscr{Z}_0 \times \mathscr{X}_0 \arrow[r, "\mathrm{pr}_1"] \arrow[d, hook] & \mathscr{Z}_0 \arrow[d, hook] \\
\mathscr{C}_0 \arrow[r] \arrow[rr, bend right, "c_1"'] & \mathscr{X}_0 \times \mathscr{X}_0 \arrow[r, "\mathrm{pr}_1"] & \mathscr{X}_0,
\end{tikzcd}$$

where the squares are cartesian and k is a closed immersion defined by a nilpotent ideal since $\mathscr{Z}_0$ is c-invariant. $\square$

11.3. Let $j : \mathscr{Z}_0 \hookrightarrow \mathscr{X}_0$ be a c-invariant substack. We then have a commutative diagram

$$\begin{tikzcd}
\mathscr{C}_{0,\mathscr{Z}} \arrow[r, "k"] \arrow[rr, bend left, "c_{\mathscr{Z},2}"] & \mathscr{C}_0 \times_{c_1,\mathscr{X}_0,j} \mathscr{Z}_0 \arrow[r] \arrow[d, "q"] & \mathscr{Z}_0 \arrow[d, hook, "j"] \\
 & \mathscr{C}_0 \arrow[r, "c_2"] & \mathscr{X}_0,
\end{tikzcd}$$

where the square is cartesian and the map k is a closed immersion defined by a nilpotent ideal. By the base change theorem [14, 12.1], for any $F \in D^-(\mathscr{C}, \mathbb{Q}_\ell)$ we have a canonical base change isomorphism

$$j^*c_{2!}F \simeq c_{\mathscr{Z},2!}k^*q^*F.$$

In particular, if $F \in D^-(\mathscr{X}, \mathbb{Q}_\ell)$ is a complex with a $\mathscr{C}$-structure $u : c_{2!}c_1^*F \to F$, then the pullback j^*F to $\mathscr{Z}$ has a $\mathscr{C}_{\mathscr{Z}}$-structure given by

$$j^*u : c_{\mathscr{Z}2!}c^*_{\mathscr{Z}1}j^*F \simeq c_{\mathscr{Z}2!}k^*q^*c_1^*F \simeq j^*c_{2!}c_1^*F \overset{u}{\to} j^*F.$$

Since $j : \mathscr{Z}_0 \hookrightarrow \mathscr{X}_0$ is defined over $\mathbb{F}_q$ it is also clear that if φ is a Weil-structure on $F \in D^-(\mathscr{X}, \mathbb{Q}_\ell)$ then j^*F has a natural Weil structure $j^*\varphi$. Finally for every n there is a natural immersion $\mathrm{Fix}(\mathscr{C}^{(n)}_{\mathscr{Z}}) \hookrightarrow \mathrm{Fix}(\mathscr{C}^{(n)})$. It follows that if (F, φ, u) is a convergent Weil complex with $\mathscr{C}$-structure on $\mathscr{X}$ then $(j^*F, j^*\varphi, j^*u)$ is a convergent Weil complex with $\mathscr{C}_{\mathscr{Z}}$-structure on $\mathscr{Z}$.

Proposition 11.4. *Let $j : \mathscr{U}_0 \hookrightarrow \mathscr{X}_0$ be a c-invariant open substack, and let $i : \mathscr{Z}_0 \hookrightarrow \mathscr{X}_0$ be the complement (with the reduced structure). Let (F, φ, u) be a convergent Weil complex with $\mathscr{C}$-structure on $\mathscr{X}$. If 1.19 holds for $(j^*F, j^*\varphi, j^*u)$ on $\mathscr{U}$ and $(i^*F, i^*\varphi, i^*u)$ on $\mathscr{Z}$, then 1.19 also holds for (F, φ, u).*

Proof. Let $n_{0,\mathscr{Z}}$ (resp. $n_{0,\mathscr{U}}$) be an integer as in 1.19 for $\mathscr{Z}$ (resp. $\mathscr{U}$), and let n_0 denote the maximum of the two. Taking cohomology of the distinguished triangle

$$j_!j^*F \to F \to i_*i^*F \to j_!j^*F[1]$$

we obtain a distinguished triangle

$$R\Gamma_c(\mathscr{U}, j^*F) \to R\Gamma_c(\mathscr{X}, F) \to R\Gamma_c(\mathscr{Z}, i^*F) \to R\Gamma_c(\mathscr{U}, j^*F)[1].$$

Applying the following 11.5, it follows that for every $n \geq n_0$ the complex $R\Gamma_c(\mathscr{X}, F)$ with the endomorphism $R\Gamma_c(u^{(n)})$ is convergent.

For the equality of $\mathrm{tr}_\iota(u^{(n)}|R\Gamma_c(\mathscr{X}, F))$ with the sum of local terms, note that $\mathrm{Fix}(\mathscr{C}^{(n)}) = \mathrm{Fix}(\mathscr{C}^{(n)}_{\mathscr{U}}) \coprod \mathrm{Fix}(\mathscr{C}^{(n)}_{\mathscr{Z}})$. Therefore the equality 1.19.1 follows from the corresponding equality for $\mathscr{U}$ and $\mathscr{Z}$ and the following lemma. $\square$

Lemma 11.5. *Let*

$$\begin{tikzcd}
K' \arrow[r] \arrow[d, "\varphi'"'] & K \arrow[r] \arrow[d, "\varphi"] & K'' \arrow[r] \arrow[d, "\varphi''"] & K'[1] \arrow[d, "\varphi'[1]"] \\
K' \arrow[r] & K \arrow[r] & K'' \arrow[r] & K'[1]
\end{tikzcd}$$

be an endomorphism of a distinguished triangle $K' \to K \to K'' \to K'[1]$ in $D_c^-(\mathbb{Q}_\ell)$. If (K', φ') and (K'', φ'') are ι-convergent, then so is (K, φ) and

$$\operatorname{tr}_\iota(\varphi'|K') + \operatorname{tr}_\iota(\varphi''|K'') = \operatorname{tr}_\iota(\varphi|K). \tag{11.5.1}$$

Proof. For $p \in \mathbb{Z}$ define

$$S_p(K) := \sum_{k \geq p} \sum_{\lambda \in \mathrm{Eg}^k(\varphi)} |\lambda|.$$

For every p we have an exact sequence

$$H^p(K') \to H^p(K) \to H^p(K'') \to H^{p+1}(K') \to \cdots,$$

which implies that

$$S_p(K) = S_p(K') + S_p(K'') + \epsilon_p,$$

where $0 \leq \epsilon_p \leq \sum_{\lambda \in \mathrm{Eg}^p(\varphi')} |\lambda|$. Since the sequence of partial sums $\{S_p(K')\}$ converges we have $\varinjlim \epsilon_p = 0$. It follows that the sequence $\{S_p(K)\}$ also converges.

The equality of traces 11.5.1 follows from noting that the sum $\sum_p (-1)^p \iota(\operatorname{tr}(\varphi|H^p(K)))$ converges absolutely and therefore we can rearrange the terms, and write

$$\sum_p (-1)^p \iota(\operatorname{tr}(\varphi|H^p(K))) = \sum_p (-1)^p \iota(\operatorname{tr}(\varphi|H^p(K'))) + \sum_p (-1)^p \iota(\operatorname{tr}(\varphi|H^p(K''))).$$

□

Now consider a separated algebraic space X_0 of finite type over $\mathbb{F}_q$ and an automorphism $\sigma : X_0 \to X_0$ defining a correspondence $c : C_0 \to X_0 \times X_0$ (so $C_0 = X_0$ with $c_1 = \sigma$ and $c_2 = \mathrm{id}$).

Theorem 11.6. *Conjecture 1.19 holds for any bounded Weil complex with C-structure (F, φ, u) on X.*

Proof. Let $U_0 \subset X_0$ denote the maximal open subspace of X_0 which is a scheme, and let $Z_0 \subset X_0$ be the complement of U_0 (with the reduced structure). Then $\sigma(U_0) = U_0$. The result therefore follows from 11.4, the case of schemes, and noetherian induction. □

12. DELIGNE-MUMFORD STACKS

12.1. Let $c : \mathscr{C}_0 \to \mathscr{X}_0 \times \mathscr{X}_0$ be a correspondence over $\mathbb{F}_q$ with $\mathscr{C}_0$ and $\mathscr{X}_0$ Deligne-Mumford stacks, and c_1 (resp. c_2) proper (resp. quasi-finite). Let $p : \mathscr{X}_0 \to X_0$ and $q : \mathscr{C}_0 \to C_0$ be the coarse moduli spaces so we have a commutative diagram

$$\begin{tikzcd}
 & \mathscr{C}_0 \arrow[dl, "c_1"'] \arrow[d, "q"] \arrow[dr, "c_2"] & \\
\mathscr{X}_0 \arrow[dd, "p"] & C_0 \arrow[dl, "\bar{c}_1"'] \arrow[dr, "\bar{c}_2"] & \mathscr{X}_0 \arrow[dd, "p"] \\
 & & \\
X_0 & & X_0,
\end{tikzcd}$$

where $\bar{c}_1$ is proper and $\bar{c}_2$ is quasi-finite. Let

$$\pi : \mathrm{Fix}(\mathscr{C}^{(n)}) \to \mathrm{Fix}(C^{(n)})$$

be the projection.

12.2. Let $\bar{x} \in \mathrm{Fix}(C^{(n)})(k)$ be a point and let $x \in \mathscr{C}(k)$ be a lifting of $\bar{x}$. Note that x is unique up to noncanonical isomorphism. The fiber $\pi^{-1}(\bar{x})$ can be described as follows. Giving x the structure of an object of $\mathrm{Fix}(\mathscr{C}^{(n)})$ is equivalent to specifying an isomorphism

$$\sigma : c_1^{(n)}(x) \to c_2(x)$$

in $\mathscr{X}(k)$. Fix one such isomorphism σ_0 (which exists since $c_1^{(n)}(x)$ and $c_2(x)$ map to the same element of $X(k)$), and let

$$\alpha^{(n)} : G_x \to G_{c_2(x)}$$

denote the map induced by $\sigma_0 \circ c_1$, where G_x (resp. $G_{c_2(x)}$) denotes the stabilizer group of x (resp. $c_2(x)$). Let $\beta : G_x \to G_{c_2(x)}$ be the map induced by c_2. An isomorphism between two pairs (x, σ) and (x, σ') in $\mathrm{Fix}(\mathscr{C}^{(n)})$ is an element $h \in G_x$ such that the diagram

$$\begin{tikzcd}
c_1^{(n)}(x) \arrow[r, "\sigma"] \arrow[d, "c_1^{(n)}(h)"] & c_2(x) \arrow[d, "c_2(h)"] \\
c_1^{(n)}(x) \arrow[r, "\sigma'"] & c_2(x)
\end{tikzcd}$$

commutes. From this it follows that $\pi^{-1}(\bar{x})$ is isomorphic to the stack quotient of $G_{c_2(x)}$ by the action of G_x given by

$$h * g = \alpha^{(n)}(h) g \beta(h)^{-1}.$$

It follows that

$$\sum_{[x,\sigma] \in \pi^{-1}(\bar{x})} \mathrm{LT}_\iota([x,\sigma], (\mathcal{F}, u^{(n)})) = \frac{1}{|G_x|} \sum_{\gamma \in G_{c_2(x)}} \mathrm{tr}_\iota(u^{(n)} | \mathcal{F}_{x, \gamma \circ \sigma_0}),$$

where the left sum is over isomorphism classes of objects in $\pi^{-1}(\bar{x})(k)$. By the discussion in section 8 this is also equal to

$$\mathrm{tr}_\iota(\bar{u}^{(n)} | (p_! \mathcal{F})_{\bar{x}}).$$

From this we obtain the following.

Theorem 12.3. *Conjecture 1.19 holds for a bounded convergent Weil complex* $(\mathcal{F}, \varphi, u)$ *on* $\mathscr{X}$ *if and only if 1.19 holds for* $(p_! \mathcal{F}, p_! \varphi, p_! u)$*. In particular, 1.19 holds for* $\mathscr{X}$ *if either*

(i) X_0 *is a scheme;*

(ii) $\mathscr{C}_0$ *is induced by an automorphism of* $\mathscr{X}_0$*.*

13. An example from toric varieties

13.1. Fix a prime number p.

Let X be a free abelian group of rank r, and let $Q \subset X_{\mathbb{R}}$ be an integral polytope. Let P denote the integral points of $C_Q := \mathrm{Cone}(1, Q) \subset \mathbb{R} \times X_{\mathbb{R}}$, and let

$$V_0 := \mathrm{Proj}(\mathbb{F}_p[P])$$

be the corresponding toric variety over $\mathbb{F}_p$, with torus $T_0 := \mathrm{Spec}(\mathbb{F}_p[X])$.

Let $D \in \mathrm{GL}(X)$ be an automorphism of X such that the induced automorphism of $X_{\mathbb{R}}$ takes Q to Q. This map D then induces automorphisms

$$\alpha : T_0 \to T_0, \quad \delta : V_0 \to V_0$$

which are compatible in the sense that the diagram

$$\begin{tikzcd} T_0 \times V_0 \arrow[r, "\rho"] \arrow[d, "\alpha\times\delta"] & V_0 \arrow[d, "\delta"] \\ T_0 \times V_0 \arrow[r, "\rho"] & V_0 \end{tikzcd}$$

commutes, where ρ denotes the action of T_0 on V_0.

Let $\mathscr{V}_0$ denote the stack $[V_0/T_0]$, and let

$$c = (\delta, \mathrm{id}) : \mathscr{C} := \mathscr{V}_0 \to \mathscr{V}_0 \times \mathscr{V}_0$$

be the correspondence obtained by passing to the quotient by the T_0-action.

For a face $F \subset Q$ of Q let $P_F \subset P$ denote the integral points of $C_F := \mathrm{Cone}(1, F) \subset \mathbb{R} \times X_{\mathbb{R}}$. Since F is a face there is a map of graded rings

$$\mathbb{F}_p[P] \to \mathbb{F}_p[P_F]$$

sending an element $e_p \in \mathbb{F}_p[P]$ (the generator defined by p) to 0 if $p \notin P_F$ and e_p otherwise. This defines a closed subscheme $V_{F,0} \subset V_0$ which is T_0-invariant. Let $T_F \subset T$ denote the stabilizer of the generic point of $V_{F,0}$. The inclusion $T_F \subset T$ corresponds to a surjection of free abelian groups $X \to Y_F$. The kernel M_F of this map is the subgroup of degree 0 elements in P_F^{gp}. Let $U_F := D(M_F)$ be the torus corresponding to M_F.

If furthermore F is stable under D, then D induces an automorphism $D_F : M_F \to M_F$. Let $A_F(t), A(t) \in \mathbb{Z}[t]$ denote the absolute values of the characteristic polynomials

$$A_F(t) := |\det(1 - tD_F|M_F)|, \quad A(t) := |\det(1 - tD|X)|.$$

Theorem 13.2. *There exists an integer* n_0 *such that for all* $n \geq n_0$ *the complex* $R\Gamma_c(\mathscr{V}, \mathbb{Q}_\ell)$ *with the endomorphism* $\delta^{(n)}$ *is convergent, and*

$$\sum_{F \subset Q, D(F)=F} \frac{A_F(p^n)}{A(p^n)} = \mathrm{tr}_\iota(\delta^{(n)} | R\Gamma_c(\mathscr{V}, \mathbb{Q}_\ell)).$$

Proof. Let $\mathscr{V}_{F,0} \subset \mathscr{V}_0$ denote the locally closed substack $[V_{F,0}/T_0] \subset [V_0/T_0]$, and let $\mathscr{V}^0_{F,0} \subset \mathscr{V}_{F,0}$ denote the open substack defined by the maximal torus orbit $V^0_{F,0} \subset V_{F,0}$.

By 10.3, we have $\mathrm{Fix}(\mathscr{C}^{(n)}) \simeq [\mathrm{Fix}(\delta^{(n)})/H^{(n)}]$, where $H^{(n)} \subset T$ is the kernel of the map

$$T \rightarrow T, \quad m \mapsto \alpha^{(n)}(m) \cdot m^{-1}.$$

In particular, for any object $x \in \mathrm{Fix}(\mathscr{C}^{(n)})(k)$, there exists a unique face $F \subset Q$ such that $D(F) = F$ and such that x can be represented by an element $\tilde{x} \in \mathrm{Fix}(\delta^{(n)})$ with $\tilde{x} \in V^0_F$.

Let $\delta^{(n)}_F : V_F \rightarrow V_F$ be the map induced by $\delta^{(n)}$, let

$$\alpha^{(n)}_F : U_F \rightarrow U_F$$

be the map induced by $\alpha^{(n)}$, and let $H^{(n)}_F$ be the kernel of the map

$$U_F \rightarrow U_F, \quad u \mapsto \alpha^{(n)}_F(u)u^{-1}.$$

If we fix a point $z_0 \in V^0_F(k)$, then any other element of $V^0_F(k)$ can be written uniquely as uz_0, with $u \in U_F(k)$. Write $\delta^{(n)}(z_0) = w \cdot z_0$, where $w \in U_F(k)$

An element $uz_0 \in V^0_F(k)$ is in $\mathrm{Fix}(\delta^{(n)}) \cap V^0_F$ if and only if

$$\delta^{(n)}(uz_0) = \alpha^{(n)}_F(u)\delta^{(n)}(z_0) = \alpha^{(n)}_F(u)wz_0$$

is equal to uz_0. Equivalently, if and only if

$$w = \alpha^{(n)}(u)^{-1}u.$$

Since the map

$$U_F \rightarrow U_F, \quad u \mapsto \alpha^{(n)}_F(u)^{-1}u$$

is étale and surjective, it follows that the number of points in $\mathrm{Fix}(\delta^{(n)}) \cap V^0_F$ is equal to the order of $H^{(n)}_F$.

From this it follows that

$$\sum_{\beta \subset \mathrm{Fix}(\mathscr{C}^{(n)})(k)} \mathrm{LT}_\iota(\beta, (\mathbb{Q}_\ell, \mathrm{can}, \alpha^{(n)})) = \sum_{F \subset Q, D(F)=F} \frac{\# H^{(n)}_F}{\# H^{(n)}}.$$

On the other hand, $H^{(n)}_F$ (resp. $H^{(n)}$) is equal to the diagonalizable group scheme associated to the cokernel of the map

$$1 - p^n D_F : M_F \rightarrow M_F, \quad (\text{resp.} \quad 1 - p^n D : X \rightarrow X),$$

and therefore

$$\# H^{(n)}_F = A_F(p^n), \quad \# H^{(n)} = A(p^n).$$

This implies the theorem. □

Remark 13.3. A more detailed analysis of the proof of 1.24 shows that in 13.2 we can take $n_0 = 1$.

13.4. If V_0 is a smooth variety over $\mathbb{F}_q$ then we can rewrite the theorem as follows. Note first of all that Frobenius on V_0 is flat, and therefore $\delta^{(n)} : V \to V$ is also a flat morphism of degree equal to p^{nr}. This number is also equal to the order of $K^{(n)} := \mathrm{Ker}(\alpha^{(n)} : T \to T)$. Since we have a cartesian diagram

$$\begin{tikzcd}
{[V/K^{(n)}]} \arrow[r] \arrow[d, "\delta^{(n)}"] & \mathscr{V} \arrow[d, "\delta^{(n)}"] \\
V \arrow[r] & \mathscr{V},
\end{tikzcd}$$

it follows that the degree of $\delta^{(n)} : \mathscr{V} \to \mathscr{V}$ is equal to 1. From this and 5.25 we obtain the following:

Theorem 13.5. *With notation as in 13.2, assume in addition that* V *is smooth over* k*. Then the complex* $R\Gamma(\mathscr{V}, \mathbb{Q}_\ell)$ *with the endomorphism* $(\delta^{(n)*})^{-1}$ *is convergent and*

$$\sum_{F \subset Q, D(F)=F} \frac{A_F(p^n)}{A(p^n)} = \mathrm{tr}_\iota((\delta^{(n)*})^{-1} | R\Gamma(\mathscr{V}, \mathbb{Q}_\ell)). \tag{13.5.1}$$

Remark 13.6. In the case when V is smooth, there is a well-known description of $H^*(\mathscr{V}, \mathbb{Q}_\ell)$ in terms of the so-called *Stanley-Reisner* ring R_Q (see for example [5, §4]). This ring R_Q is defined as follows. Let S denote the set of vertices of Q (so Q is the convex hull of the set of points $s \in S$). Then R_Q is defined as a quotient of the free polynomial algebra on S

$$R_Q := \mathbb{Q}_\ell[x_s]_{s \in S} / I_Q,$$

where I_Q is the ideal generated by $\prod_{s \in S} x_s$ and the monomials $x_{s_1} \cdots x_{s_t}$ for which the simplex spanned by $s_1, \ldots, s_t$ is not a face of Q.

Any element $s \in S$ is a fact of Q and the corresponding closed subscheme $V_s \subset V$ is a T-equivariant divisor. There is a map

$$\mathbb{Q}_\ell[x_s]_{s \in S} \to H^*(\mathscr{V}, \mathbb{Q}_\ell)$$

which sends x_s to the equivariant Chern class of the divisor V_s. It is shown for example in [5, Theorem 8] that this map induces an isomorpishm

$$R_Q \xrightarrow{\simeq} H^*(\mathscr{V}, \mathbb{Q}_\ell).$$

Since the Frobenius pullback of a line bundle is its p-th power, it follows that under this isomorphism the action of $(\delta^{(n)*})^{-1}$ is given by the automorphism

$$\rho^{(n)} : R_Q \to R_Q$$

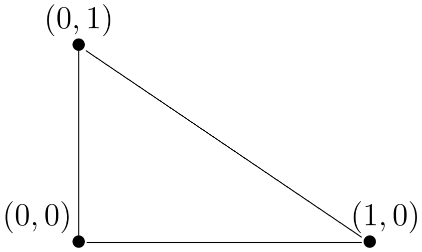
induced by the map

$$x_s \mapsto (1/p^n) x_{D(s)}.$$

Thus the formula 13.5.1 can be rewritten as

$$\sum_{F \subset Q, D(F)=F} \frac{A_F(p^n)}{A(p^n)} = \mathrm{tr}_\iota(\rho^{(n)} | R_Q). \tag{13.6.1}$$

Example 13.7. Let $Q \subset \mathbb{R}^2$ be the polytope



and let $D : \mathbb{Z}^2 \to \mathbb{Z}^2$ be the endomorphism

$$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The associated toric variety V_0 is then $\mathbb{P}^2$.

Write x_{00} (resp. x_{01}, x_{10}) for $x_{(0,0)}$ (resp. $x_{(0,1)}$, $x_{(1,0)}$), so the Stanley-Reisner ring in this case is equal to

$$R_Q = \mathbb{Q}_\ell[x_{00}, x_{01}, x_{10}]/(x_{00}x_{01}x_{10}).$$

The endomorphism $\rho^{(n)}$ is given by

$$x_{00} \mapsto \frac{1}{p^n}x_{00}, \quad x_{01} \mapsto \frac{1}{p^n}x_{10}, \quad x_{10} \mapsto \frac{1}{p^n}x_{01}.$$

Let $\tilde{\rho}^{(n)}$ be the endomorphism of the polynomial ring $\mathbb{Q}_\ell[x_{00}, x_{01}, x_{10}]$ given by the same formula. Setting $y = x_{01} + x_{10}$ and $z = x_{01} - x_{10}$ we can also view this as $\mathbb{Q}_\ell[x_{00}, y, z]$ with $\tilde{\rho}^{(n)}$ acting on x_{00} and y by multiplication by $1/p^n$ and on z by multiplication by $-1/p^n$. Using this one sees that

$$\mathrm{tr}(\tilde{\rho}^{(n)}|\mathbb{Q}_\ell[x_{00}, x_{01}, x_{10}]) = \frac{1}{(1-(1/p^n))^2(1+(1/p^n))}.$$

Consideration of the exact sequence

$$0 \longrightarrow \mathbb{Q}_\ell[x_{00}, x_{01}, x_{10}] \xrightarrow{x_{00}x_{01}x_{10}} \mathbb{Q}_\ell[x_{00}, x_{01}, x_{10}] \longrightarrow R_Q \longrightarrow 0$$

and the fact that $\tilde{\rho}^{(n)}$ acts on $x_{00}x_{01}x_{10}$ by multiplication by $1/p^{3n}$ gives

$$\mathrm{tr}(\rho^{(n)}|R_Q) = \frac{1}{(1-(1/p^n))^2(1+(1/p^n))} - \frac{(1/p^{3n})}{(1-(1/p^n))^2(1+(1/p^n))}.$$

On the other hand, let $L \subset Q$ denote the line segment connecting $(0, 1)$ and $(1, 0)$. Then

$$\{F \subset Q | D(F) = F\} = \{Q, L, (0, 0)\},$$

with corresponding $A_F(t)$'s equal to

$$|1 - t^2|, |1 + t|, 1.$$

It follows that

$$\sum_{F \subset Q, D(F) = F} \frac{A_F(p^n)}{A(p^n)} = 1 + \frac{1 + p^n}{p^{2n} - 1} + \frac{1}{p^{2n} - 1}.$$

Thus the formula 13.6.1 in this cases amounts to the elementary identity

$$1+\frac{1+p^n}{p^{2n}-1}+\frac{1}{p^{2n}-1}=\frac{1}{(1-(1/p^n))^2(1+(1/p^n))}-\frac{(1/p^{3n})}{(1-(1/p^n))^2(1+(1/p^n))}.$$

Remark 13.8. The toric variety V_0 has an integral model given by

$$V_{\mathbb{Z}} := \mathrm{Proj}(\mathbb{Z}[P]).$$

Let $V_{\mathbb{C}}$ denote the base change to $\mathbb{C}$. For any integer m, multiplication by m on the monoid P induces an endomorphism $\gamma_m : V_{\mathbb{Z}} \to V_{\mathbb{Z}}$. If $m = p^t$ for some prime p, then the reduction modulo p of γ_m is equal to the t-power of the Frobenius morphism on V_0. Similarly, T_0 and $\mathscr{V}_0$ have integral models $T_{\mathbb{Z}}$ and $\mathscr{V}_{\mathbb{Z}}$. By a standard argument using base change for compactly supported cohomology, one can deduce that for $n \geq 1$ we have

$$\sum_{F\subset Q, D(F)=F}\frac{A_F(p^n)}{A(p^n)}=\mathrm{tr}_\iota(\gamma_{p^n}\circ\delta|R\Gamma_c(\mathscr{V}_{\mathbb{C}},\mathbb{Q}_\ell)).$$

From this one might speculate that for arbitrary $m > 1$ we have

$$\sum_{F\subset Q, D(F)=F}\frac{A_F(m)}{A(m)}=\mathrm{tr}_\iota(\gamma_{m}\circ\delta|R\Gamma_c(\mathscr{V}_{\mathbb{C}},\mathbb{Q}_\ell)),$$

and in particular the right side is convergent.

This is in fact not hard to show using the same argument as above. First let p be a prime dividing m, and write $m = p \cdot m'$. Then look at the reduction modulo p, but instead of δ consider the endomorphism $\gamma_{m'} \circ \delta$. We leave the details to the reader.

14. TRACES OF HECKE OPERATORS

14.1. Fix pairwise distinct primes p, ℓ, and m. Let $\mathscr{A}_{g,\Gamma_0(m)}$ be the moduli stack over $\mathbb{F}_p$ classifying principally polarized abelian varieties of dimension g with $\Gamma_0(m)$-level structure. The fiber of the stack $\mathscr{A}_{g,\Gamma_0(m)}$ over a scheme S is by definition the groupoid of collections

$$\{(A,\lambda),(B,\tau),f\},$$

where (A,λ) and (B,τ) are principally polarized abelian schemes of dimension g over S and $f : A \to B$ is an isogeny such that the diagram

(14.1.1)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m\lambda"'] & B \arrow[d, "\tau"] \\ A^t & B^t \arrow[l, "f^t"'] \end{tikzcd}$$

commutes. Note that the degree of f is equal to m^g.

If $\mathscr{A}_g/\mathbb{F}_p$ denotes the moduli stack of principally polarized abelian varieties, we obtain a correspondence

$$c : \mathscr{A}_{g,\Gamma_0(m)} \to \mathscr{A}_g \times \mathscr{A}_g$$

given by

$$c_1((A,\lambda),(B,\tau),f) = (B,\tau), \quad c_2((A,\lambda),(B,\tau),f) = (A,\lambda).$$

Lemma 14.2. *The map c_1 is proper and quasi-finite, and c_2 is quasi-finite and representable.*

Proof. If S is a scheme and $((A,\lambda),(B,\tau),f) \in \mathscr{A}_{g,\Gamma_0(m)}(S)$ is an object, then an automorphism of this object is given by an automorphism α of (A,λ) such that if $H_f \subset A[m]$ denotes the kernel of f then $\alpha(H_f) = H_f$. From this it follows that c_2 is representable. Moreover, if $S = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field k of characteristic p, and $x = [(A,\lambda)] \in \mathscr{A}_g(k)$ is a point, then $c_2^{-1}(x)(k)$ is in bijection with the set of rank m^g subgroups $H \subset A[m]$ which are isotropic with respect to the Weil pairing induced by λ (see for example [9, 1.7]). From this it follows that c_2 is also quasi-finite.

To verify the properness of c_1, we check the valuative criterion. Let $S = \mathrm{Spec}(V)$ be the spectrum of a discrete valuation ring with generic point $\eta \in S$ and closed point $s \in S$. Let (B,τ) be a principally polarized abelian scheme over S, and assume given a principally polarized abelian scheme (A_η,λ_η) over η, and a morphism $f : A_\eta \to B_\eta$ defining an object

$$((A_\eta,\lambda_\eta),(B_\eta,\tau_\eta),f_\eta) \in \mathscr{A}_{g,\Gamma_0(m)}(\eta).$$

We need to show that after possible replacing V by a finite extension, this objects extends to an object of $\mathscr{A}_{g,\Gamma_0(m)}(S)$ (which since $\mathscr{A}_g$ is separated will automatically map to (B,τ) under c_2).

After replacing V by a finite extension, we may assume that $A_\eta[m]$ is a constant group scheme. Let $H \subset A_\eta[m]$ be the kernel of f_η, and fix an isomorphism $H \simeq (\mathbb{Z}/(m))^g$. This identifies A_η with a $(\mathbb{Z}/(m))^g$-torsor over B_η. If $g : B \to S$ denotes the structure morphism, then since the étale cohomology sheaf $R^1g_*(\mathbb{Z}/(m))^g$ is locally constant on S, we can, after possibly replacing V by another finite extension, extend A_η to a $(\mathbb{Z}/(m))^g$-torsor $P \to B$. Since P is proper, the identity section $e : \eta \to A_\eta$ extends uniquely to a section $S \to P$. This gives P the structure of an abelian scheme over S, which we denote by A. This then gives an extension $f : A \to B$ of $f_\eta : A_\eta \to B_\eta$. Furthermore, since the relative Picard scheme $\underline{\mathrm{Pic}}_{A/S}$ is proper, we get also a unique extension λ of λ_η to a principal polarization over S (at least after making another extension of S). Furthermore, the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m\lambda"] & B \arrow[d, "\tau"] \\ A^t & B^t \arrow[l, "f^t"'] \end{tikzcd}$$

commutes, as this can be verified over η. This completes the verification that c_1 is proper.

For the quasi-finiteness of c_1, let k be an algebraically closed field and let $x = [(B,\tau)] \in \mathscr{A}_g(k)$ be an object. If $((A,\lambda),(B,\tau),f) \in c_1^{-1}(x)(k)$ is an object, then after choosing an isomorphism

$$\mathrm{Ker}(f) \simeq (\mathbb{Z}/(m))^g$$

we obtain a $(\mathbb{Z}/(m))^g$-torsor $\pi : P \to B$ together with an element $p \in \pi^{-1}(e)$. Since $H^1(B,(\mathbb{Z}/(m))^g)$ is a finite set, the set of isomorphism classes of such pairs $(\pi : P \to B, p)$ is a finite set. It follows that there are only finitely many possibilities for the pair (A,f). Furthermore, given $f : A \to B$, the set of possible λ's such that 14.1.1 commutes is equal to the set of dotted arrows filling in the diagram

$$\begin{tikzcd}
 & A \arrow[dl, dashed] \arrow[d, "f^t \circ \tau \circ f"] \\
A^t \arrow[r, "\times m"] & A^t.
\end{tikzcd}$$

This set is a torsor under the finite set

$$\mathrm{Hom}(A, A^t[m]),$$

which implies that $c_1^{-1}(x)(k)$ is finite. □

14.3. For an object $(B, \tau) \in \mathscr{A}_g(S)$ over a scheme S, let $(B^{(n)}, \tau^{(n)})$ be the pullback of (B, τ) along the n-th power Frobenius morphism $S \to S$. For any $n \geq 0$ the fixed point stack $\mathrm{Fix}(c^{(n)})$ has fiber over a scheme S the category of quadruples

$$\Delta = ((A, \lambda), (B, \tau), f, \iota),$$

where $((A, \lambda), (B, \tau), f) \in \mathscr{A}_{g,\Gamma_0(m)}(S)$ and

$$\iota : (B^{(n)}, \tau^{(n)}) \simeq (A, \lambda)$$

is an isomorphism of principally polarized abelian schemes over S. For such a fixed point let

$$\chi_\Delta : A \to A$$

be the composite morphism

$$A \xrightarrow{f} B \xrightarrow{F_B^n} B^{(n)} \xrightarrow{\iota} A,$$

where $F_B^n : B \to B^{(n)}$ is the map induced by the n-th power Frobenius morphism on B (the relative Frobenius of B/S). The map χ_Δ has degree $(mp^n)^g$ and the diagram

$$\begin{tikzcd}
A \arrow[r, "\chi_\Delta"] \arrow[d, "mp^n\lambda"'] & A \arrow[d, "\lambda"] \\
A^t & A^t \arrow[l, "\chi_\Delta^t"']
\end{tikzcd}$$

commutes.

Remark 14.4. Note that $\mathrm{Aut}(\Delta)$ is an étale group scheme since it is a subgroup scheme of $\mathrm{Aut}(A, \lambda)$, and $\mathrm{Ker}(\chi_\Delta - \mathrm{id})$ is an étale group scheme when $n \geq 1$ by 2.2 and 2.5.

14.5. Let $\mathbb{V}$ denote the smooth sheaf on $\mathscr{A}_g$ given by $R^1 h_* \mathbb{Q}_\ell$, where $h : X \to \mathscr{A}_g$ is the universal principally polarized abelian scheme, and for $k \geq 0$ let $S^k\mathbb{V}$ denote the k-th symmetric power of $\mathbb{V}$.

Let

$$\begin{tikzcd}
X_2 \arrow[rr, "g"] \arrow[dr, "h_2"] & & X_1 \arrow[dl, "h_1"'] \\
 & \mathscr{A}_{g,\Gamma_0(m)} &
\end{tikzcd}$$

denote the universal isogeny over $\mathscr{A}_{g,\Gamma_0(m)}$. By the proper base change theorem we have

$$c_i^* \mathbb{V} \simeq R^1 h_{i*} \mathbb{Q}_\ell.$$

The finite morphism g therefore induces a map

$$\tilde{u} : c_1^*\mathbb{V} \to c_2^*\mathbb{V},$$

which when composed with the trace map $c_2^*\mathbb{V} \to c_2^!\mathbb{V}$ gives a map $u : c_1^*\mathbb{V} \to c_2^!\mathbb{V}$.

The sheaf $\mathbb{V}$ also has a natural Weil-sheaf structure as it is defined over $\mathbb{F}_p$. For every $n \geq 0$ let

$$T_m^{(n)} : R\Gamma_c(\mathscr{A}_g, S^k\mathbb{V}) \to R\Gamma_c(\mathscr{A}_g, S^k\mathbb{V})$$

be the endomorphism defined by the twist $u^{(n)}$.

Theorem 14.6. *There exists an integer n_0 such that for every $n \geq n_0$ the series*

$$\sum_{k\geq 0} \frac{1}{(mp^n)^{g(k+1)}} \mathrm{tr}(T_m^{(n)} | R\Gamma_c(\mathscr{A}_g, S^k\mathbb{V})) \tag{14.6.1}$$

converges absolutely to

$$\sum_{\Delta\in|\mathrm{Fix}(c^{(n)})(k)|} \frac{1}{(\#\mathrm{Aut}(\Delta)(k))\cdot \#\mathrm{Ker}(\chi_\Delta - \mathrm{id})(k)},$$

where the sum is taken over isomorphism classes of objects $\Delta = ((A,\lambda),(B,\tau),f,\iota)$ in $\mathrm{Fix}(c^{(n)})(k)$ and $\mathrm{Aut}(\Delta)$ denotes the automorphism group scheme of Δ.

Proof. First note that for $\Delta = ((A,\lambda),(B,\tau),f,\iota)$ we have

$$\frac{1}{\#\mathrm{Ker}(\chi_\Delta - \mathrm{id})(k)} = \mathrm{tr}(B\chi_\Delta | R\Gamma_c(BA, \mathbb{Q}_\ell)),$$

where we write $B\chi_\Delta : BA \to BA$ for the map defined by χ_Δ.

The map $B\chi_\Delta$ has degree equal to $1/(mp^n)^g$, and therefore by 5.25 we have

$$\mathrm{tr}(B\chi_\Delta | R\Gamma_c(BA,\mathbb{Q}_\ell)) = (1/(mp^n)^g)\mathrm{tr}(B\chi_\Delta^{*-1} | H^*(BA,\mathbb{Q}_\ell)).$$

On the other hand, by Borel's theorem [4, 5.6], we have

$$H^*(BA,\mathbb{Q}_\ell) \simeq S^{\cdot}H^1(A,\mathbb{Q}_\ell).$$

Therefore we have

$$\mathrm{tr}(B\chi_\Delta | R\Gamma_c(BA,\mathbb{Q}_\ell)) = (1/(mp^n)^g)\sum_{k\geq 0}\mathrm{tr}(S^k\chi_\Delta^{*-1} | S^kH^1(A,\mathbb{Q}_\ell)).$$

As in 5.25 and 5.26, one sees that we have

$$\mathrm{tr}(B\chi_\Delta | R\Gamma_c(BA,\mathbb{Q}_\ell)) = \sum_{k\geq 0}\frac{1}{(mp^n)^g}\mathrm{tr}(S^k(\chi_\Delta^*)^{-1} | S^kH^1(A,\mathbb{Q}_\ell)),$$

where $\chi_\Delta^* : H^1(A,\mathbb{Q}_\ell) \to H^1(A,\mathbb{Q}_\ell)$ is the map induced by pullback. Furthermore, by the discussion in 5.26 this sum converges absolutely.

Now recall (see for example [15, Theorem 4, page 180]) that the characteristic polynomial of χ_Δ^* on $H^1(A,\mathbb{Q}_\ell)$ lies in $\mathbb{Q}[X]$, and all the eigenvalues of χ_Δ^* acting on $H^1(A,\mathbb{Q}_\ell)\otimes_\iota\mathbb{C}$ have absolute value $(mp^n)^{g/2}$. It follows that the set of eigenvalues of χ_Δ^* acting on $H^1(A,\mathbb{Q}_\ell)\otimes_\iota\mathbb{C}$

is equal to $(1/p^n m)^g$ times the set of eigenvalues of χ_Δ^{*-1} acting on $H^1(A, \mathbb{Q}_\ell) \otimes_\iota \mathbb{C}$. We conclude that

$$\sum_{k\geq 0} \frac{1}{(mp^n)^g} \mathrm{tr}(S^k(\chi_\Delta^*)^{-1}|S^k H^1(A, \mathbb{Q}_\ell)) = \sum_{k\geq 0} \frac{1}{(mp^n)^{k+1}} \mathrm{tr}(S^k(\chi_\Delta^*)|S^k H^1(A, \mathbb{Q}_\ell)).$$

We therefore get

$$\mathrm{tr}(B\chi_\Delta|R\Gamma_c(BA, \mathbb{Q}_\ell)) = \sum_{k\geq 0} \frac{1}{(mp^n)^{g(k+1)}} \mathrm{tr}(S^k\chi_\Delta^*|S^k H^1(A, \mathbb{Q}_\ell)),$$

and the right side converges absolutely.

Now choose n_0 big enough so that Deligne's conjecture holds for $\mathscr{A}_g$ and $n \geq n_0$. We then have

$$\mathrm{tr}(T_m^{(n)}|R\Gamma(\mathscr{A}_g, S^k\mathbb{V})) = \sum_{\Delta\in|\mathrm{Fix}(c^{(n)})(k)|} \frac{1}{\#\mathrm{Aut}(\Delta)(k)} \mathrm{tr}(S^k\chi^*|S^k H^1(A, \mathbb{Q}_\ell)).$$

From this the absolutely convergence of 14.6.1 follows, and in addition we get

$$\begin{aligned} &\sum_{k\geq 0} \frac{1}{(mp^n)^{g(k+1)}} \mathrm{tr}(T_m^{(n)}|R\Gamma_c(\mathscr{A}_g, S^k\mathbb{V})) \\ = \sum_{\Delta\in|\mathrm{Fix}(c^{(n)})(k)|} &\frac{1}{\#\mathrm{Aut}(\Delta)(k)} \sum_{k\geq 0} \frac{1}{(mp^n)^{g(k+1)}} \mathrm{tr}(S^k\chi^*|S^k H^1(A, \mathbb{Q}_\ell)) \\ &= \sum_{\Delta\in|\mathrm{Fix}(c^{(n)})(k)|} \frac{1}{\#\mathrm{Aut}(\Delta)(k)} \mathrm{tr}(B\chi_\Delta|R\Gamma_c(BA, \mathbb{Q}_\ell)) \\ &= \sum_{\Delta\in|\mathrm{Fix}(c^{(n)})(k)|} \frac{1}{(\#\mathrm{Aut}(\Delta)(k)) \cdot \#\mathrm{Ker}(\chi_\Delta - \mathrm{id})(k)}. \end{aligned}$$

□

Appendix A. Rf_* for unbounded complexes

Let (S, Λ) be an admissible pair as in 1.28, and let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of finite type S-stacks. In general, the functor

$$Rf_* : D_c^+(\mathscr{X}, \Lambda) \to D_c^+(\mathscr{Y}, \Lambda)$$

does not extend to a well-behaved functor on the whole $D_c(\mathscr{X}, \Lambda)$. As we now explain, however, it does extend if the following finiteness condition holds:

(*) There exists an integer n_0 such that for every constructible sheaf of Λ-modules F on $\mathscr{X}$ we have $R^n f_* F = 0$ for $n \geq n_0$.

Lemma A.1. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of finite type S-stacks, and assume condition (*) holds.*

(i) If $F \in D_c^{[a,b]}(\mathscr{X}, \Lambda)$, then $Rf_* F \in D_c^{[a,b+n_0]}(\mathscr{Y}, \Lambda)$.

(ii) Suppose $\rho : F \to G$ is a morphism in $D_c^+(\mathscr{X}, \Lambda)$ such that for some integer r the map $\mathscr{H}^j(F) \to \mathscr{H}^j(G)$ is an isomorphism for $j \geq r$. Then the map

$$\mathscr{H}^j(Rf_*F) \to \mathscr{H}^j(Rf_*G)$$

is an isomorphism for $j > r + n_0$.

Proof. For (i) we proceed by induction on $e := b - a$. The case $e = 1$ is by assumption. For the inductive step, consider the distinguished triangle

$$\mathscr{H}^a(F)[-a] \to F \to \tau_{\geq a+1}F \to \mathscr{H}^a(F)[-a+1],$$

which induces a long exact sequence

$$\cdots \to R^{j-a}f_*\mathscr{H}^a(F) \to R^jf_*F \to R^j\tau_{\geq a+1}F \to \cdots .$$

By induction we have

$$R^j\tau_{\geq a+1}F = 0$$

for $j > b + n_0$, and

$$R^{j-a}f_*\mathscr{H}^a(F) = 0$$

for $j - a > n_0$. It follows that $R^jf_*F = 0$ for $j > b + n_0$.

For (ii), let C be the cone of ρ. Then $\mathscr{H}^j(C) = 0$ for $j \geq r$, and therefore by (i) we have $Rf_*C \in D_c^{(-\infty, r+n_0-1]}$. Consideration of the long exact sequence

$$\cdots \to R^{j-1}f_*C \to R^jf_*F \to R^jf_*G \to \cdots$$

then gives (ii). □

Theorem A.2. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of finite type S-stacks, and assume condition (*) holds. Then the functor*

$$f^* : D_c(\mathscr{Y}, \Lambda) \to D_c(\mathscr{X}, \Lambda)$$

has a right adjoint

$$Rf_* : D_c(\mathscr{X}, \Lambda) \to D_c(\mathscr{Y}, \Lambda),$$

and for every integer j and $M \in D_c(\mathscr{X}, \Lambda)$ the natural map

$$R^jf_*M := \mathscr{H}^j(Rf_*M) \to R^jf_*\tau_{\geq -n}M$$

is an isomorphism for $n >> 0$.

Proof. The key point is the following lemma. Recall [6, 2.3] that if

$$\cdots M_{n+1} \to M_n \to M_{n-1} \to \cdots ,$$

is a projective system in the derived category $D_c(\mathscr{X}, \Lambda)$ of a finite type S-stack, then the homotopy limit holim $M.$ is by definition the mapping fiber of the map

$$1 - \text{shift} : \prod_n M_n \to \prod_n M_n.$$

Lemma A.3. *(i) Let*

$$\cdots M_{n+1} \to M_n \to M_{n-1} \to \cdots ,$$

be a projective system of objects of $D_c^+(\Lambda)$ *(the derived category of bounded below complexes of* Λ*-modules with cohomology groups finitely generated over* Λ*). Assume that for every* j *there exists an integer* n_0 *such that the map*

$$H^j(M_{n+1}) \to H^j(M_n)$$

is an isomorphism for every $n \geq n_0$*. Then for every* j *there exists an integer* n_0 *such that the natural map*

$$H^j(\mathrm{holim}_n M_n) \to H^j(M_n)$$

is an isomorphism for all $n \geq n_0$*.*

(ii) Let $\mathscr{X}/S$ *be an algebraic stack of finite type, and suppose given a projective system of objects* $M_n \in D_c^+(\mathscr{X}, \Lambda)$

$$\cdots M_{n+1} \to M_n \to M_{n-1} \to \cdots ,$$

such that for every j *there exists an integer* n_0 *such that the map*

$$\mathscr{H}^j(M_{n+1}) \to \mathscr{H}^j(M_n)$$

is an isomorphism for every $n \geq n_0$*. Then for every* j *there exists an integer* n_0 *such that the natural map*

$$\mathscr{H}^j(\mathrm{holim}_n M_n) \to \mathscr{H}^j(M_n)$$

is an isomorphism for all $n \geq n_0$*.*

Proof. For (i), note that by the definition of $\mathrm{holim}_n M_n$ there is a distinguished triangle

$$\mathrm{holim}_n M_n \longrightarrow \textstyle\prod_n M_n \xrightarrow{1-s} \textstyle\prod_n M_n \longrightarrow \mathrm{holim}_n M_n[1],$$

where $s : \prod_n M_n \to \prod_n M_n$ is the shifting map induced by the maps $M_{n+1} \to M_n$. Now an elementary calculation (using the assumptions on the M_n) shows that the map

$$1 - s : \prod_n H^j(M_n) \to \prod_n H^j(M_n)$$

is surjective, and that the kernel is isomorphic to $\varprojlim H^j(M_n)$. This implies (i).

For (ii), observe first that for any smooth morphism $U \to \mathscr{X}$ of finite type with U a scheme there exists an integer n_0 such that for any $F \in D_c^{[a,b]}(U, \Lambda)$ we have

$$R\Gamma(U, F) \in D_c^{[a,b+n_0]}(\Lambda).$$

This follows from A.1 (i) applied to $U \to S$ and the definition of an admissible pair in 1.28. Note that by our assumptions on (S, Λ), we can choose n_0 such for any étale quasi-compact morphism $j : V \to U$ and $G \in D_c^{[a,b]}(V, \Lambda)$ we also have

$$R\Gamma(V, G) \in D_c^{[a,b+n_0]}(\Lambda),$$

as the bound n_0 can be chosen to be a function of the dimension of U.

As in A.1 (ii) this implies that for any smooth morphism $U \to \mathscr{X}$ with U a quasi-compact scheme, the system $R\Gamma(U, M_{n,U})$ satisfies the assumptions of (i), where $M_{n,U}$ is the restriction of M_n to U_{et}. Therefore applying $R\Gamma(U, -)$ to the distinguished triangle

$$\text{holim}_n M_n \longrightarrow \textstyle\prod_n M_n \xrightarrow{1-s} \textstyle\prod_n M_n \longrightarrow \text{holim}_n M_n[1]$$

and using (i) together with the fact that $R\Gamma(U, -)$ commutes with products and hence homotopy limits, we find that for every j the sequence

$$0 \to H^j(U, (\text{holim}_n M_n)_U) \to H^j(U, \prod_n M_n) \overset{1-s}{\to} H^j(U, \prod_n M_n) \to 0$$

is exact, and that there exists an integer n_0 depending only on j, the system $\{M_n\}$, and the dimension of U, such that the projection map

$$H^j(U, (\text{holim}_n M_n)_U) \to H^j(U, M_n)$$

is an isomorphism for $n \geq n_0$. Sheafifying the presheaves

$$U \mapsto H^j(U, \text{holim}_n M_n)$$

and

$$U \mapsto H^j(U, M_n)$$

we obtain (ii). □

For $M \in D_c(\mathscr{X}, \Lambda)$ let M_n denote $\tau_{\geq -n} M$. Then by A.3 the natural map

$$M \to \text{holim}_n M_n$$

is an isomorphism.

Now by A.1 (ii) the system $Rf_* M_n \in D_c^+(\mathscr{Y}, \Lambda)$ satisfies the assumptions of A.3 (ii). We define

$$Rf_* M := \text{holim}_n Rf_* M_n.$$

By A.3 for any $j \in \mathbb{Z}$ there exists an integer n_0 such that the map

$$R^j f_* M \to R^j f_* M_n$$

is an isomorphism for all $n \geq n_0$. In particular, $Rf_* M \in D_c(\mathscr{Y}, \Lambda)$ and the last statement in A.2 holds.

Now fix $G \in D_c(\mathscr{Y}, \Lambda)$. Since the functor $R\text{Hom}(G, -)$ commutes with products, one deduces from the distinguished triangle

$$Rf_* M \to \prod_n Rf_* M_n \overset{1-s}{\to} \prod_n Rf_* M_n \to Rf_* M[1]$$

that

$$\begin{aligned} R\text{Hom}(G, Rf_* M) &\simeq \text{holim}_n R\text{Hom}(G, Rf_* M_n) \\ &\simeq \text{holim}_n R\text{Hom}(f^* G, M_n) \quad \text{(adjunction)} \\ &\simeq R\text{Hom}(f^* G, \text{holim}_n M_n) \\ &\simeq R\text{Hom}(f^* G, M) \quad \text{(since } M \simeq \text{holim}_n M_n). \end{aligned}$$

These isomorphisms realize Rf_* as a right adjoint to f^*. □

A.4. With assumptions as in A.2, note that if $M \in D_c^-(\mathscr{X}, \Lambda)$, then also $Rf_*M \in D_c^-(\mathscr{Y}, \Lambda)$, and that for every integer m there exists an integer n_0 such that

$$\tau_{\geq m} Rf_*M = \tau_{\geq m} Rf_* \tau_{\geq -n} M$$

for every $n \geq n_0$. This also implies that if $A \in D_c^+(\mathscr{Y}, \Lambda)$ then for every m there exists an integer n_0 such that

$$\tau_{\leq m} R\mathscr{H}om(Rf_*M, A) \simeq \tau_{\leq m} R\mathscr{H}om(Rf_*\tau_{\geq -n}M, A)$$

for all $n \geq n_0$.

Since by [13, 4.3.2]

$$R\mathscr{H}om(Rf_*M, A) \simeq \mathrm{hocolim}_m \tau_{\leq m} R\mathscr{H}om(Rf_*M, A)$$

this implies that

$$R\mathscr{H}om(Rf_*M, A) \simeq \mathrm{hocolim}_n R\mathscr{H}om(Rf_*\tau_{\geq -n}M, A). \tag{A.4.1}$$

A.5. Now suppose $B \in D_c^+(\mathscr{X}, \Lambda)$ is an object and $M \in D_c^-(\mathscr{X}, \Lambda)$. Then

$$R\mathscr{H}om(M, B) \in D_c^+(\mathscr{X}, \Lambda)$$

and we can define $Rf_*R\mathscr{H}om(M, B)$ in the usual way. Again by [13, 4.3.2] we have

$$R\mathscr{H}om(M, B) \simeq \mathrm{hocolim}_n R\mathscr{H}om(\tau_{\geq -n}M, B).$$

If $B \in D_c^{[a,\infty)}(\mathscr{X}, \Lambda)$ then the cone of the morphism

$$R\mathscr{H}om(\tau_{\geq -n}M, B) \to R\mathscr{H}om(\tau_{\geq -n-1}M, B)$$

is isomorphic to

$$R\mathscr{H}om(\mathscr{H}^{-n}(M)[n], B) \simeq R\mathscr{H}om(\mathscr{H}^{-n}(M), B)[-n]$$

which is in $D_c^{[a+n,\infty)}(\mathscr{X}, \Lambda)$. It follows that that for every integer j, there exists an integer n_0 such that the natural map

$$R^jf_*R\mathscr{H}om(\tau_{\geq -n}M, B) \to R^jf_*R\mathscr{H}om(\tau_{\geq -n-1}M, B)$$

is an isomorphism for all $n \geq n_0$. From this it follows that the natural map

$$Rf_*R\mathscr{H}om(M, B) \leftarrow \mathrm{hocolim}_n Rf_*R\mathscr{H}om(\tau_{\geq -n}M, B) \tag{A.5.1}$$

is an isomorphism.

Corollary A.6. *With assumptions as in A.2, for any $M \in D_c^-(\mathscr{X}, \Lambda)$ and $B \in D_c^+(\mathscr{X}, \Lambda)$ there is a canonical map*

$$Rf_*R\mathscr{H}om(M, B) \to R\mathscr{H}om(Rf_*M, Rf_*B)$$

defined as the composite

$$\begin{aligned} Rf_*R\mathscr{H}om(M, B) &\simeq \mathrm{hocolim}_n Rf_*R\mathscr{H}om(\tau_{\geq -n}M, B) \ \ \text{(A.5.1)} \\ &\to \mathrm{hocolim}_n R\mathscr{H}om(Rf_*\tau_{\geq -n}M, Rf_*B) \ \ \text{(canonical map)} \\ &\simeq R\mathscr{H}om(Rf_*M, Rf_*B) \ \ \text{(A.4.1).} \end{aligned}$$

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