MOTIVIC COHOMOLOGY, LOCALIZED CHERN CLASSES, AND LOCAL TERMS

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Abstract. Let $c : C \to X \times X$ be a correspondence with C and X quasi-projective schemes over an algebraically closed field k. We show that if $u_\ell : c_1^*\mathbb{Q}_\ell \to c_2^!\mathbb{Q}_\ell$ is an action defined by the localized Chern classes of a c_2-perfect complex of vector bundles on C, where ℓ is a prime invertible in k, then the local terms of u_ℓ are given by the class of an algebraic cycle independent of ℓ. We also prove some related results for quasi-finite correspondences. The proofs are based on the work of Cisinski and Deglise on triangulated categories of motives.

Contents

1. Introduction

The motivation for this work comes from our study of local terms arising from actions of correspondences defined by local Chern classes of complexes of vector bundles in [18]. The purpose of the present paper is to elucidate the motivic nature of these local terms using the machinery developed by Cisinski and Deglise in [4].

The basic problem we wish to address is the following. Fix an algebraically closed field k of characteristic p (possibly 0), and let $\mathscr{S}$ denote the category of finite type separated k-schemes. Let $c : C \to X \times X$ be a correspondence with $C, X \in \mathscr{S}$. A c_2-perfect complex $E^\cdot$ on C defines for any prime ℓ invertible in k an action $u_\ell : c_1^*\mathbb{Q}_\ell \to c_2^!\mathbb{Q}_\ell$, and therefore by the general machinery of SGA 5 a class $\mathrm{Tr}(u_\ell) \in H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)})$, where $\mathrm{Fix}(c)$ denotes the scheme of fixed points $\mathrm{Fix}(c) := C \times_{c, X\times X, \Delta_X} X$ and $\Omega_{\mathrm{Fix}(c)}$ is the ℓ-adic dualizing complex (see [13, III, §4] for further discussion). Recall from loc. cit. that for any proper connected component $Z \subset \mathrm{Fix}(c)$ the local term of u_ℓ is given by the proper pushforward of the restriction of $\mathrm{Tr}(u_\ell)$ to Z, and consequently in good situations can be used via the Grothendieck-Lefschetz trace formula [13, III, 4.7] to calculate the trace of the induced action of u_ℓ on global cohomology.

On the other hand, $H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)})$ is the ℓ-adic Borel-Moore homology of $\mathrm{Fix}(c)$ and there is a cycle class map

$$\mathrm{cl}_\ell : A_0(\mathrm{Fix}(c)) \to H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)}),$$

where $A_0(\mathrm{Fix}(c))$ denotes the group of 0-cycles on $\mathrm{Fix}(c)$ modulo rational equivalence.

The main result about local terms in this paper is the following:

Theorem 1.1 (Theorem 8.7)**.** *There exists a zero-cycle $\Sigma \in A_0(\mathrm{Fix}(c))_{\mathbb{Q}}$ such that for any prime ℓ invertible in k the class $\mathrm{Tr}(u_\ell)$ is equal to $\mathrm{cl}_\ell(\Sigma)$.*

As we explain, this theorem is a fairly formal consequence of a suitable theory of derived categories of motives and six operations for such categories. The fact that such a theory exists is due to Cisinski and Deglise [4]. They developed a notion of triangulated motivic categories with a six operations formalism realizing a vision of Beilinson. Roughly speaking such a category is a fibered category $\mathscr{M}$ over $\mathscr{S}$ such that for every $X \in \mathscr{S}$ the fiber $\mathscr{M}(X)$ is a monoidal triangulated category and for every morphism $f : X \to Y$ in $\mathscr{S}$ we have functors

$$f_!, f_* : \mathscr{M}(X) \to \mathscr{M}(Y), \quad f^*, f^! : \mathscr{M}(Y) \to \mathscr{M}(X)$$

satisfying the usual properties. In addition there should be a suitable notion of Chern classes. Already in this context we can define localized Chern classes of complexes of vector bundles as well as analogous of the classes $\mathrm{Tr}(u_\ell)$, which are functorial in $\mathscr{M}$. In particular, we can consider the category $\mathscr{M}_B$ of Beilinson motives defined in [4, §14]. This category not only has a good six functor formalism, but is also closely related to algebraic cycles as one would expect from a good motivic theory.

The connection with cycles (discussed in more detail in sections 4 and 6) is established by developing the basic theory of Borel-Moore homology, discussed in the étale setting in [18], to a rather general context of triangulated motivic categories with a six operations formalism. Let $\mathscr{M}$ be such a motivic category, and for quasi-projective $X \in \mathscr{S}$ let $\Omega_X^{\mathscr{M}} \in \mathscr{M}(X)$ (or sometimes we just write Ω_X if the reference to $\mathscr{M}$ is clear) denote $f^! 1_{\mathrm{Spec}(k)}$, where $1_{\mathrm{Spec}(k)} \in \mathscr{M}(\mathrm{Spec}(k))$ is the unit object for the monoidal structure and $f : X \to \mathrm{Spec}(k)$ is the structure morphism. For an integer i the *i-th $\mathscr{M}$-valued Borel-Moore homology of X* is defined to be

$$H_{i,BM}^{\mathscr{M}}(X) := \mathrm{Ext}^{-2i}_{\mathscr{M}(X)}(1_X, \Omega_X^{\mathscr{M}}(-i)),$$

where the notation $\Omega_X^{\mathscr{M}}(-i)$ denotes a suitable Tate twist of $\Omega_X^{\mathscr{M}}$. Then there is a natural cycle class map

$$A_i(X)_{\mathbb{Q}} \to H_{i,BM}^{\mathscr{M}}(X), \qquad (1.1.1)$$

where $A_i(X)_{\mathbb{Q}}$ denotes the i-th Chow homology groups (as defined in [11, §1.8]) tensored with $\mathbb{Q}$.

Theorem 1.2 (Special case of 6.2)**.** *If $\mathscr{M}$ is the motivic category $\mathscr{M}_B$ of Beilinson motives and X is quasi-projective then the map (1.1.1) is an isomorphism for all i.*

The idea behind the proof of 1.1 is to lift the construction of local terms to the category of Beilinson motives $\mathscr{M}_B$, where the local term is by 1.2 given by an algebraic cycle, and then show that the étale realizations of the motivic local term is equal to $\mathrm{Tr}(u_\ell)$.

The proof of 1.1 can essentially be phrased as saying that actions arising from c_2-perfect complexes are motivic. In general it seems a difficult question to prove that a given action of a correspondence is motivic. There is one other case, however, where one can fairly easily detect if an action is motivic. Namely, for a quasi-finite morphism $f : Y \to X$ there is a natural necessary condition for a section $u_\ell \in H^0(Y, f^!\mathbb{Q}_\ell)$ to be the étale realization of a morphism $u : 1_Y \to f^!1_X$ in the triangulated category of Beilinson motives over Y. In theorem 9.4 we show that this condition is also sufficient. This also has global consequences. In particular, a special case of theorem 9.24 is the following:

Theorem 1.3. *Let k be an algebraically closed field and let X/k be a separated Deligne-Mumford stack. Let $f : X \to X$ be a finite morphism (as a morphism of stacks). Then the alternating sum of traces*
$$\sum_i (-1)^i \mathrm{tr}(f^*|H^i(X, \mathbb{Q}_\ell))$$
is in $\mathbb{Q}$ and independent of ℓ.

Remark 1.4. Following standard conventions we usually write $\mathrm{tr}(f^*|R\Gamma(X, \mathbb{Q}_\ell))$ for the alternating sum of traces $\sum_i (-1)^i \mathrm{tr}(f^*|H^i(X, \mathbb{Q}_\ell))$.

Remark 1.5. It is tempting to try to prove 1.3 using Fujiwara's theorem and naive local terms as in [14, 3.5 (c)]. The cohomology $R\Gamma(X, \mathbb{Q}_\ell)$ is dual to the compactly supported cohomology $R\Gamma_c(X, \Omega_X)$ of the dualizing complex Ω_X of X, and the dual operator to f^* is the map
$$f_* : R\Gamma_c(X, \Omega_X) \to R\Gamma_c(X, \Omega_X)$$
induced by the map $f_*\Omega_X \to \Omega_X$ arising from the identification $f_! \simeq f_*$ (since f is proper) and adjunction. In the finite field case, one can then apply Fujiwara's theorem to the complex Ω_X with this action of the correspondence $(\mathrm{id}, f) : X \to X \times X$ to relate the trace on $R\Gamma_c(X, \Omega_X)$ to the so-called naive local terms of this action on Ω_X. However, the calculation of these naive local terms of Ω_X is not immediate and they are not formally rational and independent of ℓ.

Remark 1.6. Since the trace appearing 1.3 is in $\mathbb{Z}_\ell$ it follows that the alternating sum of traces is in $\mathbb{Z}[1/p]$, where p is the characteristic of k. In fact, notice that since $R\Gamma(X, \mathbb{Z}_\ell)$ is a perfect complex we can define $\mathrm{tr}(u^*|R\Gamma(X, \mathbb{Z}_\ell)) \in \mathbb{Z}_\ell$, which by the above is an element of $\mathbb{Z}[1/p]$ which reduces mod ℓ to $\mathrm{tr}(u^*|R\Gamma(X, \mathbb{F}_\ell))$, thereby yielding ℓ-independence for mod ℓ traces as well.

Remark 1.7. One might hope more generally to use the techniques of this paper to study motivic local terms with $\mathbb{Z}$ coefficients to obtain cycles in $A_0(\mathrm{Fix}(c))$ before tensoring with $\mathbb{Q}$. However, the theory at present seems restricted to $\mathbb{Q}$-coefficients as the six operations on a suitable triangulated category of motives is not known to exist integrally. Work in preparation by Cisinski and Deglise on integral motives may, however, lead to integral results.

Remark 1.8. In this paper we discuss étale cohomology and local terms defined in the étale theory. However, with a suitable theory of p-adic local terms and p-adic realization functors one would also get rationality of p-adic local terms and compatility with the étale local terms.

Remark 1.9. Theorem 1.3 has also been obtained by Bondarko using variant motivic methods [3, Discussion following 8.4.1].

Remark 1.10. Many of the foundational results obtained in this paper hold not just over a field but over more general base schemes and we develop the theory in greater generality. For the applications to local terms, however, it suffices to work over an algebraically closed field.

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2. Motivic categories and the six operations

Let B be a regular separated scheme of finite dimension, and let $\mathscr{S}$ denote the category of finite type separated B-schemes.

2.1. Recall from [4, Section 1] that a *triangulated premotivic category* $\mathscr{M}$ is a fibered category over $\mathscr{S}$ satisfying the following five conditions (a good summary is given in [5, A.1.1]):

(PM1) For every $S \in \mathscr{S}$ the fiber category $\mathscr{M}(S)$ is a well-generated (in the sense of [16]) triangulated category with a closed monoidal structure.

(PM2) For every morphism $f : X \to Y$ in $\mathscr{S}$ the functor (well-defined up to unique isomorphism)
$$f^* : \mathscr{M}(Y) \to \mathscr{M}(X)$$
is triangulated, monoidal, and admits a right adjoint f_*.

(PM3) For every smooth morphism $f : X \to Y$ in $\mathscr{S}$ the functor $f^* : \mathscr{M}(Y) \to \mathscr{M}(X)$ admits a left adjoint $f_\sharp$.

(PM4) For every cartesian square with p smooth
$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ T & \xrightarrow{p} & S \end{array}$$
there is a canonical isomorphism of functors
$$\mathrm{Ex}(\Delta^*_\sharp) : q_\sharp g^* \simeq f^* p_\sharp.$$

(PM5) For every smooth morphism $p : T \to S$, $M \in \mathscr{M}(T)$, and $N \in \mathscr{M}(S)$ there is a canonical isomorphism
$$\mathrm{Ex}(p^*_\sharp, \otimes) : p_\sharp(M \otimes_T p^*N) \simeq p_\sharp(M) \otimes_S N.$$

Remark 2.2. Note that for any category $\mathscr{S}$ we can talk about a *triangulated fibered category over* $\mathscr{S}$. By this we mean a fibered category $p : \mathscr{M} \to \mathscr{S}$ satisfying axioms (PM1) and (PM2).

Remark 2.3. In [4, Section 1.4] there is a notion of a premotivic triangulated category over a general base category, but the above suffices for our purposes.

2.4. For every $X \in \mathscr{S}$, the monoidal structure on $\mathscr{M}(X)$ gives a unit object $1_X \in \mathscr{M}(X)$. For a smooth morphism $f : X \to S$ in $\mathscr{S}$ define $M_S(X) \in \mathscr{M}(S)$ to be $f_\sharp(1_X)$. Because the pullback functor f^* is monoidal we have $f^* 1_S = 1_X$ and therefore by adjunction a morphism

$$M_S(X) = f_\sharp f^* 1_S \xrightarrow{f_\sharp f^* \to \mathrm{id}} 1_S,$$

which we denote by $a_{X/S}$.

A *Tate motive* for $\mathscr{M}$ is a cartesian section $\tau : \mathscr{S} \to \mathscr{M}$ with $\tau(S)$ fitting into a distinguished triangle

$$\tau(S)[-2] \longrightarrow M_S(\mathbb{P}^1_S) \xrightarrow{a_{\mathbb{P}^1_S/S}} 1_S \longrightarrow \tau(S)[-1]$$

functorial in S. We usually write just $1_S(1)$ for $\tau(S)$.

2.5. We can consider various other natural axioms on a triangulated premotivic category with a Tate object:

(Semi-separation) For any finite surjective radical morphism $f : X \to Y$ the functor $f^* : \mathscr{M}(Y) \to \mathscr{M}(X)$ is conservative.

(Homotopy axiom) For every $S \in \mathscr{S}$ the map

$$a_{\mathbb{A}^1_S/S} : M_S(\mathbb{A}^1_S) \to 1_S$$

is an isomorphism.

(Stability property) The Tate motive $1_S(1)$ is $\otimes$-invertible. In this case we get motives $1_S(n)$ for all $n \in \mathbb{Z}$, and for any F in $\mathscr{M}$ and integer n we can define $F(n)$.

2.6. Given a triangulated premotivic category $\mathscr{M}$ with a Tate motive satisfying the stability property we define motivic cohomology, a bigraded cohomology theory on $\mathscr{S}$, by

$$H^{i,n}_{\mathscr{M}}(S) := \mathrm{Ext}^i_{\mathscr{M}(S)}(1_S, 1_S(n)).$$

2.7. A morphism between two triangulated premotivic categories $\mathscr{M}$ and $\mathscr{M}'$ is a cartesian functor $\varphi^* : \mathscr{M} \to \mathscr{M}'$ such that the following hold:

(i) For every $S \in \mathscr{S}$ the functor $\varphi^*_S : \mathscr{M}(S) \to \mathscr{M}'(S)$ is a triangulated monoidal functor which admits a right adjoint φ_{S*}.
(ii) For every smooth morphism $p : T \to S$ in $\mathscr{S}$ there is a canonical isomorphism

$$\mathrm{Ex}(p_\sharp, \varphi^*) : p_\sharp \varphi^*_T \to \varphi^*_S p_\sharp.$$

In fact triangulated premotivic categories form a 2-category in which the above morphisms are the 1-morphisms, and 2-morphisms are given by morphisms of cartesian functors $\epsilon : \varphi^* \to \psi^*$ compatible with the structures in (i) and (ii).

Remark 2.8. Similarly we can consider the 2-category of triangulated fibered categories over any base category $\mathscr{S}$.

2.9. Let $\mathscr{S}$ be as above, and let $\mathrm{Ar}(\mathscr{S})$ be the category of morphisms in $\mathscr{S}$. We have two functors

$$s, t : \mathrm{Ar}(\mathscr{S}) \to \mathscr{S}$$

given by the source and target respectively. For a triangulated premotivic category $\mathscr{M}$ over $\mathscr{S}$ let $\mathscr{M}^s$ (resp. $\mathscr{M}^t$) denote $s^*\mathscr{M}$ (resp. $t^*\mathscr{M}$), a triangulated fibered category over $\mathrm{Ar}(\mathscr{S})$.

A *six functor formalism for* $\mathscr{M}$ consists of the following data (see [4, A.5] for more details):

(1) 2-functors $f \mapsto f_*$ and $f \mapsto f_!$ from $\mathscr{M}^s \to \mathscr{M}^t$ and $f \mapsto f^*$ and $f \mapsto f^!$ from $\mathscr{M}^t$ to $\mathscr{M}^s$ such that for every $f : X \to Y \in \mathrm{Ar}(\mathscr{S})$ the functors f_* and f^* are as previously defined, and $f_!$ is left adjoint to $f^!$.

(2) There exists a morphism of 2-functors $\alpha : f_! \to f_*$ which is an isomorphism if f is proper.

(3) For any smooth morphism $f : X \to S$ in $\mathscr{S}$ of relative dimension d there are isomorphisms $\mathfrak{p}_f : f_\sharp \to f_!(d)[2d]$ and $\mathfrak{p}'_f : f^* \simeq f^!(-d)[-2d]$. These are given by isomorphisms of 2-functors on the category of smooth morphisms of relative dimension d.

(4) For every cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{q} & X' \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{p} & X \end{array}$$

there are natural isomorphisms of functors

$$p^* f_! \simeq g_! q^*,$$

$$g_* q^! \simeq p^! f_*.$$

In the case when f, and hence also g, is proper the induced isomorphism

$$g_! q^! \xrightarrow{\alpha} g_* q^! \simeq p^! f_* \xrightarrow{\alpha^{-1}} p^! f_!$$

is the map induced by the adjunction

$$q^! \xrightarrow{\mathrm{id} \to f^! f_!} q^! f^! f_! \xrightarrow{\simeq} g^! p^! f_!.$$

(5) For every $f : Y \to X$ there are natural isomorphisms

$$\mathrm{Ex}(f_!^*, \otimes) : (f_! K) \otimes_X L \simeq f_!(K \otimes_Y f^* L),$$

$$\mathscr{H}om_X(f_! L, K) \simeq f_* \mathscr{H}om_Y(L, f^! K),$$

and

$$f^! \mathscr{H}om_X(L, M) \simeq \mathscr{H}om_Y(f^* L, f_! M).$$

(Loc) Let $X \in \mathscr{S}$ be an object, $i : Z \hookrightarrow X$ a closed imbedding, and let $j : U \hookrightarrow X$ be the complementary open set. Then there exists a map of functors $\partial : i_* i^* \to j_! j^![1]$ such for every $F \in M(X)$ the induced triangle

$$j_! j^! F \longrightarrow F \longrightarrow i_* i^* F \xrightarrow{\partial} j_! j^! F[1]$$

is distinguished, where the first two maps are those induced by adjunction.

Finally Deglise and Cisinski consider purity and duality properties:

(Relative Purity) For a closed immersion $i : Z \hookrightarrow X$ of smooth separated B-schemes there is a canonical isomorphism

$$1_Z(-c)[-2c] \simeq i^!(1_X),$$

where c is the codimension of Z in X.

(Duality) For $X \in \mathscr{S}$ with structure morphism $f : X \to B$ we write $\Omega_X^{\mathscr{M}}$ (or just Ω_X if no confusion seems likely to arise) for $f^! 1_B \in \mathscr{M}(X)$. Define $D_X : \mathscr{M}(X)^{\mathrm{op}} \to \mathscr{M}(X)$ to be the functor $M \mapsto \mathscr{H}om_X(M, \Omega_X^{\mathscr{M}})$.

(a) For every $M \in \mathscr{M}(X)$ the natural map

$$M \to D_X(D_X(M))$$

is an isomorphism.

(b) For every X and $M, N \in \mathscr{M}(X)$ we have a canonical isomorphism

$$D_X(M \otimes D_X(N)) \simeq \mathscr{H}om_X(M, N).$$

(c) For every $f : Y \to X$ in $\mathscr{S}$, $M \in \mathscr{M}(X)$, and $N \in \mathscr{M}(Y)$ we have natural isomorphisms

$$D_Y(f^*(M)) \simeq f^!(D_X(M)),$$
$$f^* D_X(M) \simeq D_Y(f^!(M)),$$
$$D_X(f_!(N)) \simeq f_*(D_Y(N)),$$
$$f_!(D_Y(N)) \simeq D_X(f_*(N)).$$

These isomorphisms interchange the base change isomorphisms in (4).

We say that a triangulated premotivic category $\mathscr{M}$ is a *triangulated motivic category* over $\mathscr{S}$ if all of the above conditions hold.

Remark 2.10. This is stronger than what is in [4, 2.4.45] but we will not need their slightly weaker notion.

Remark 2.11. The relative purity property follows from property 2.9 (3), but we state it explicitly for later use.

Remark 2.12. If R is a ring we can also consider a notion of an *R-linear triangulated motivic category* over $\mathscr{S}$. By definition this means that each $\mathscr{M}(X)$ is an R-linear symmetric monoidal triangulated category, and that all the above structure respects this R-linear structure.

Remark 2.13. It is shown in [4, 2.1.9] that our assumptions on $\mathscr{M}$ (in particular 2.9 (2) and (4) and the semi-separation) imply that for any finite surjective radicial morphism $f : X \to Y$ the pullback functor $f^* : \mathscr{M}(Y) \to \mathscr{M}(X)$ is an equivalence of categories. Since f_* is right adjoint to f^* it is also an equivalence and for any $K \in \mathscr{M}(Y)$ the adjunction map $K \to f_* f^* K$ is an isomorphism. In particular, the adjunction map $1_Y \to f_* 1_X$ is an isomorphism.

Since this point is crucial for what follows we sketch for the convenience of the reader the proof given in [4, 2.1.9] of the statement that $f^* : \mathscr{M}(Y) \to \mathscr{M}(X)$ is an equivalence for f finite surjective radicial.

In the case when f is also a closed immersion the square

$$\begin{array}{ccc} Y & = & Y \\ \| & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{f} & X \end{array}$$

is cartesian and by 2.9 (2) and (4) the adjunction map $f^*f_* \to \mathrm{id}$ is an isomorphism. This implies that the adjunction $\mathrm{id} \to f_*f^*$ is also an isomorphism as this can be verified after applying f^*, since f is semi-separated, and the induced map $f^* \to f^*f_*f^*$ is a section of the isomorphism induced by the adjunction $f^*f_* \to \mathrm{id}$.

For the general case consider the commutative diagram

$$\begin{array}{ccc} Y & & \\ & \searrow^{\delta} \quad \xrightarrow{\mathrm{id}} & \\ & Y \times_X Y & \xrightarrow{g_1} Y \\ {\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle g_2} & \downarrow{\scriptstyle f} \\ & Y & \xrightarrow{f} X, \end{array}$$

where δ is the diagonal imbedding and g_1 (resp. g_2) is the first (resp. second) projection. Since f is finite surjective and radicial the morphism δ is a closed imbedding which is surjective and radicial. By the first case considered the functor $\delta^* : \mathscr{M}(Y \times_X Y) \to \mathscr{M}(Y)$ is therefore an equivalence with quasi-inverse δ_*. Since the compositions $\delta^* \circ g_i^*$ are isomorphic to the identity functor on $\mathscr{M}(Y)$ it follows that $g_i^* : \mathscr{M}(Y) \to \mathscr{M}(Y \times_X Y)$ is also an equivalence with quasi-inverse g_{i*} for $i = 1, 2$. This also implies that g_{i*} is a quasi-inverse for δ^*.

Now to verify that the morphism of functors $\mathrm{id} \to f_*f^*$ is an isomorphism it suffices by semi-separation to show that the induced map $f^* \to f^*f_*f^*$ is an isomorphism. For this note by 2.9 (2) and (4) and the fact that δ^* is an equivalence we have

$$f^*f_*f^* \simeq g_{1*}g_2^*f^* \simeq g_{1*}\delta_*\delta^*g_2^*f^* \simeq f^*$$

and composing $f^* \to f^*f_*f^*$ with this isomorphism we get the identity map $f^* \to f^*$. This also implies that the adjunction map $f^*f_* \to \mathrm{id}$ is an isomorphism as we have

$$f^*f_* \simeq g_{1*}g_2^* \simeq g_{1*}\delta_*\delta^*g_{2*} \simeq \mathrm{id}.$$

Now the fact that the adjunctions $\mathrm{id} \to f_*f^*$ and $f^*f_* \to \mathrm{id}$ are isomorphisms implies that f^* is an equivalence of categories.

Remark 2.14. Assume that B is the spectrum of a field k and let $X \in \mathscr{S}$ be an object. Then, using for example Noether normalization, we can find a nonempty open subset $U \subset X$ of some pure dimension d and a factorization of the structure morphism

$$U \xrightarrow{a} V \xrightarrow{b} Y \longrightarrow \mathrm{Spec}(k),$$

where a is étale, b is finite radicial, and Y is smooth of relative dimension d over k. Combining the preceding remark with property (3) we find that if $u : U \to \mathrm{Spec}(k)$ is the structure morphism then $u^!1_{\mathrm{Spec}(k)} \simeq 1_U(d)[2d]$. More generally, for any other object $T \in \mathscr{S}$ we get a factorization of the second projection $\mathrm{pr}_2 : U \times T \to T$ as

$$U \times T \to V \times T \to Y \times T \to T,$$

from which it follows that if $f : U \times T \to \mathrm{Spec}(k)$ and $g : T \to \mathrm{Spec}(k)$ are the structure morphisms, then $f^!1_{\mathrm{Spec}(k)} \simeq \mathrm{pr}_2^* g^! 1_{\mathrm{Spec}(k)}(d)[2d]$.

We will use this in various devissage arguments that follow.

3. Chern classes

3.1. The key ingredient in our study of étale Borel-Moore homology in [19] is the theory of local Chern classes. In order to develop a good theory of Borel-Moore homology in the motivic setting we need to define local Chern classes also in this more general setup. Our approach to local Chern classes follows the method of Iversen [15] which uses the calculation of the cohomology of certain relative flag varieties. In this section we summarize the basic theory of Chern classes in the motivic setting and explain the calculations (3.7 and 3.10) of cohomology that are necessary for Iversen's method. In the following section we then explain how to define local Chern classes.

As in the previous section let B be a regular separated scheme of finite dimension, and let $\mathscr{S}$ denote the category of finite type separated B-schemes.

3.2. Fix an R-linear triangulated motivic category $\mathscr{M}$ over $\mathscr{S}$. For $n, m \in \mathbb{Z}$ let

$$H^{n,m}_{\mathscr{M}} : \mathscr{S}^{\mathrm{op}} \to \mathrm{Mod}_R$$

be the functor sending $X \in \mathscr{S}$ to $H^{n,m}_{\mathscr{M}}(X) := \mathrm{Ext}^n_{\mathscr{M}(X)}(1_X, 1_X(m))$.

Let Pic (resp. Vec, K^0) be the functor on $\mathscr{S}$ sending X to the Picard group $\mathrm{Pic}(X)$ of X (resp. the set of isomorphism classes of finite rank vector bundles on X, the Grothendieck group of vector bundles on X). A *pre-orientation* on $\mathscr{M}$ is a morphism of functors

$$c_1 : \mathrm{Pic} \to H^{2,1}_{\mathscr{M}}.$$

Let $X \in \mathscr{S}$ be an object smooth over B and let $i : Z \hookrightarrow X$ be a Cartier divisor smooth over B. By relative purity, we get a canonical isomorphism

$$i^!1_X \simeq 1_Z(-1)[-2].$$

Applying a shift, Tate twist, and i_*, this isomorphism defines an isomorphism

$$i_*1_Z \to i_*i^!1_X(1)[2],$$

which upon composition with the adjunction $i_*i^!1_X \to 1_X$ gives a morphism

$$i_*1_Z \to 1_X(1)[2].$$

Applying $\mathrm{Hom}_{\mathscr{M}(X)}(1_X, -)$ we get a map

$$H^{0,0}_{\mathscr{M}}(Z) \to H^{2,1}_{\mathscr{M}}(X).$$

We say that a pre-orientation c_1 is an *orientation* if this map sends the identity class in $H^{0,0}_{\mathscr{M}}(Z)$ to $c_1(\mathscr{O}_X(Z))$ for every such closed imbedding $i : Z \hookrightarrow X$.

For the remainder of this section we fix an orientation c_1 on $\mathscr{M}$.

3.3. A *theory of Chern classes for $\mathscr{M}$* is a collection of morphisms of functors

$$c_n : \mathrm{Vec} \to H^{2n,n}_{\mathscr{M}}, \quad n \geq 0$$

such that the following conditions hold:

(i) c_0 is the constant 1 and c_1 is the given orientation.
(ii) (*Vanishing*) For a vector bundle E on X of rank r we have $c_i(E) = 0$ for $i > r$.
(iii) (*Commutativity*) For vector bundles E and F on $X \in \mathscr{S}$ and integers $i, j \in \mathbb{Z}$ we have

$$c_i(E) \cdot c_j(F) = c_j(F) \cdot c_i(E).$$

(iv) (*Whitney sum*) For a short exact sequence of vector bundles on $X \in \mathscr{S}$

$$0 \to E'' \to E \to E' \to 0$$

we have

$$c_k(E) = \sum_{i+j=k} c_i(E'')c_j(E').$$

Remark 3.4. When R is a $\mathbb{Q}$-algebra, we can define as usual the Chern character which defines a morphism of functors

$$\mathrm{ch} : K^0 \to \prod_n H^{2n,n}_{\mathscr{M}}.$$

as well as Todd classes.

3.5. Assume given a theory of Chern classes for $\mathscr{M}$. The classical computations of cohomology for flag varieties can then be carried out in our cohomology theory as well. Let us briefly recall the statement and construction. For $X \in \mathscr{S}$ define

$$A^{n,m}_{\mathscr{M}}(X) := H^{2n,m}_{\mathscr{M}}(X),$$

and set

$$A^{*,*}_{\mathscr{M}}(X) := \oplus_{n,m\in\mathbb{Z}} A^{n,m}_{\mathscr{M}}(X).$$

Then $A^{*,*}_{\mathscr{M}}(X)$ is a bigraded ring. For $F \in \mathscr{M}(X)$ define

$$A^{n,m}_{\mathscr{M}}(X, F) := \mathrm{Ext}^{2n}_{\mathscr{M}(X)}(1_X, F(m)),$$

and set

$$A^{*,*}_{\mathscr{M}}(X, F) := \oplus_{n,m} A^{n,m}_{\mathscr{M}}(X, F),$$

a module over $A^{*,*}_{\mathscr{M}}(X)$. The main case of interest is when $F = 1_X(m)[2n]$ for some n and m, in which case $A^{*,*}_{\mathscr{M}}(X, F)$ is a free module of rank 1 over $A^{*,*}_{\mathscr{M}}(X)$ with generator in degree $A^{-n,-m}_{\mathscr{M}}(X, F)$.

3.6. Let $X \in \mathscr{S}$ be a scheme, let E be a vector bundle on X. Fix a sequence of integers $(r_1, \dots, r_m)$ and let $p : \mathcal{F} \to X$ be the flag variety classifying flags

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m = E$$

such that the rank of F_i/F_{i-1} is equal to r_i. Over $\mathcal{F}$ there is a universal flag $F^u_\cdot$ on p^*E. Set $E_i := F^u_i/F^u_{i-1}$ $(i = 1, \dots m)$, so E_i is a locally free sheaf of rank r_i on $\mathcal{F}$.

Consider the polynomial ring $A^{*,*}_{\mathscr{M}}(X)[T_{i,j_i}]_{1\leq i\leq m, 1\leq j_i\leq r_i}$ with variable T_{i,j_i} of bidegree (j_i, j_i), and let J be the bigraded ideal in this ring generated by the homogeneous elements (note that $c_t(E) \in A^{t,t}_{\mathscr{M}}(X)$)

$$c_t(E) - \sum_{i_1+\cdots+i_m=t} T_{1,i_1}\cdots T_{m,i_m}, \quad t \geq 1.$$

There is a map of bigraded rings

$$\alpha : A^{*,*}_{\mathscr{M}}(X)[T_{i,j_i}] \to A^{*,*}_{\mathscr{M}}(\mathcal{F}), \quad T_{i,j_i} \mapsto c_{j_i}(E_i).$$

Proposition 3.7. *The map α induces an isomorphism*

$$A^{*,*}_{\mathscr{M}}(X)[T_{i,j_i}]/J \simeq A^{*,*}_{\mathscr{M}}(\mathcal{F}).$$

Proof. This follows from the argument of [13, Exposé VII, §5]. Let us just indicate the necessary modifications here.

Let $f : D \to X$ be the flag variety classifying flags of type $(1, 1, \ldots, 1)$ on E. There is a natural map $g : D \to \mathcal{F}$ realizing D as the fiber product $D_1 \times_{\mathcal{F}} D_2 \cdots \times_{\mathcal{F}} D_m$, where D_i is the variety over $\mathcal{F}$ classifying full flags in E_i.

Lemma 3.8. *Let $S \in \mathscr{S}$ be an object and E a locally free sheaf of rank r on S with associated projective bundle $q : \mathbb{P}E \to S$. Let $c_1 \in H^{2,1}_{\mathscr{M}}(\mathbb{P}E)$ denote the first Chern class of the universal quotient, which induces a map $1_S(-1)[-2] \to q_*1_{\mathbb{P}E}$. Then the map induced by summing the maps c_1^i*

$$\oplus_{i=0}^{r-1} 1_S(-i)[-2i] \to q_*1_{\mathbb{P}E}$$

in $\mathscr{M}(S)$ is an isomorphism.

Proof. Notice that we have $q_\sharp \simeq q_!(r-1)[2(r-1)] \simeq q_*(r-1)[2(r-1)]$, so the desired isomorphism can also be written as an isomorphism $M_S(\mathbb{P}E) \simeq \oplus_{i=0}^{r-1} 1_S(i)[2i]$. This is shown in [7, Theorem 3.2]. □

Consider the algebra $A^{*,*}_{\mathscr{M}}(X)[U_k]_{k=1,\ldots,r}$ (with the U_k of bidegree $(1,1)$) and the ideal J_D generated by elements $c_t(E) - \sigma_t$, where σ_t is the t-th symmetric function in the U_k. We then have a map

$$\alpha_D : A^{*,*}_{\mathscr{M}}(X)[U_k]/J_D \to A^{*,*}_{\mathscr{M}}(D), \quad U_k \mapsto c_1(L_k),$$

where L_k is the k-th universal quotient on D. Factoring $f : D \to X$ as a sequence of projective bundles one sees that the map α_D is an isomorphism. Now let

$$\theta : A^{*,*}_{\mathscr{M}}(X)[T_{i,j_i}]/J \to A^{*,*}_{\mathscr{M}}(X)[U_k]/J_D$$

be the map induced by the map sending T_{i,j_i} to the j_i-th elementary symmetric polynomial in the variables

$$(U_{r_1+\cdots+r_{i-1}+s})_{1\leq s\leq r_i}.$$

We then have a commutative diagram

$$\begin{array}{ccc} A^{*,*}_{\mathscr{M}}(\mathcal{F}) & \longrightarrow & A^{*,*}_{\mathscr{M}}(D) \\ \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \alpha_D} \\ A^{*,*}_{\mathscr{M}}(X)[T_{i,j_i}]/J & \xrightarrow{\theta} & A^{*,*}_{\mathscr{M}}(X)[U_k]/J_D. \end{array}$$

Analyzing this as in [13, p. 310] one gets that α is an isomorphism as well. □

Remark 3.9. A theory of Chern classes with c_1 equal to a given orientation is unique if it exists. This follows from the usual argument, as discussed for example in [11, Remark 3.2.1], using the splitting principle and 3.8,

Corollary 3.10. *Let $X \in \mathscr{S}$, let $E_1, \dots, E_s$ be vector bundles on X, and let $v_1, \dots, v_s$ be integers ≥ 0. Let G_i denote the Grassmanian of v_i-planes in E_i, and let P_i denote the universal v_i-sub-bundle of $E_i|_{G_i}$. Then the $A^{*,*}_{\mathscr{M}}(X)$-algebra $A^{*,*}_{\mathscr{M}}(\prod_i G_i)$ is generated by the homogeneous components of the elements $\mathrm{pr}_i^* c.(P^i)$.*

Proof. This follows from the above description and factoring $\prod_i G_i \to X$ through a sequence of Grassman bundles. □

4. Local Chern classes

We continue with the notation of the preceding section.

4.1. For a closed imbedding $i : X \hookrightarrow M$ in $\mathscr{S}$, define

$$A^{n,m}_{\mathscr{M}}(M \text{ on } X) := \mathrm{Ext}^{2n}_{\mathscr{M}(X)}(1_X, i^! 1_M(m)),$$

and set

$$A^{*,*}_{\mathscr{M}}(M \text{ on } X) := \oplus_{n,m \in \mathbb{Z}} A^{n,m}_{\mathscr{M}}(M \text{ on } X).$$

This is a bigraded module over $A^{*,*}_{\mathscr{M}}(X)$. A *theory of local Chern classes* consists of an assignment to every bounded complex $K^\cdot$ of locally free sheaves on M with support in X cohomology classes

$$c_i^{M \text{ on } X}(K^\cdot) \in A^{i,i}_{\mathscr{M}}(M \text{ on } X)$$

satisfying the following properties:

(i) (*Pullback*) If $f : M' \to M$ is a morphism and $i' : X' \hookrightarrow M'$ denotes $f^{-1}(X)$ then $f^* c_i^{M \text{ on } X}(K^\cdot) \in A^{i,i}_{\mathscr{M}}(M' \text{ on } X')$ is equal to $c_i^{M' \text{ on } X'}(f^* K^\cdot)$.

(ii) Applying $\mathrm{Hom}_{\mathscr{M}(M)}(1_M, -)$ to the adjunction maps $i_* i^! 1_M(m)[2n] \to 1_M(m)[2n]$ we get a morphism $r : A^{*,*}_{\mathscr{M}}(M \text{ on } X) \to A^{*,*}_{\mathscr{M}}(M)$. If $c_\cdot^{M \text{ on } X}(K^\cdot) \in \prod_{s \geq 1} A^{s,s}_{\mathscr{M}}(M \text{ on } X)$ denotes the vector of the $c_s^{M \text{ on } X}(K^\cdot)$ then we require

$$r(c_\cdot^{M \text{ on } X}(K^\cdot)) + 1 = \prod_s c.(K^{2s}) c.(K^{2s-1})^{-1}.$$

Using 3.7 and the argument of Iversen [15] one obtains:

Proposition 4.2. *Suppose given a theory of Chern classes for $\mathscr{M}$. Then a theory of local Chern classes for $\mathscr{M}$ exists and is unique.*

□

4.3. In the case when R is a $\mathbb{Q}$-algebra one can introduce as in [15, §1] the localized Chern character

$$\mathrm{ch}^{M \text{ on } X}(K^\cdot) \in \prod_s A^{s,s}_{\mathscr{M}}(M \text{ on } X).$$

By the argument of [15] this satisfies the following properties:

(i) (*Functoriality*) If $f : M' \to M$ is a morphism and $i' : X' \hookrightarrow M'$ denotes $f^{-1}(X)$ then $f^*\mathrm{ch}^{M \text{ on } X}(K^\cdot) = \mathrm{ch}^{M' \text{ on } X'}(f^*K^\cdot)$.

(ii) $r(\mathrm{ch}^{M \text{ on } X}(K^\cdot)) = \mathrm{ch}(K^\cdot)$.

(iii) (*Decalage*) $\mathrm{ch}^{M \text{ on } X}(K^\cdot[1]) = -\mathrm{ch}^{M \text{ on } X}(K^\cdot)$.

(iv) For complexes $K^\cdot$ and $L^\cdot$ on M supported on X we have

$$\mathrm{ch}^{M \text{ on } X}(K^\cdot \oplus L^\cdot) = \mathrm{ch}^{M \text{ on } X}(K^\cdot) + \mathrm{ch}^{M \text{ on } X}(L^\cdot).$$

(v) (*Multiplicativity*) Let $K^\cdot$ (resp. $L^\cdot$) be a complex on M supported on Z (resp. V). Then

$$\mathrm{ch}^{M \text{ on } (Z\cap V)}(K^\cdot \otimes L^\cdot) = \mathrm{ch}^{M \text{ on } Z}(K^\cdot) \cdot \mathrm{ch}^{M \text{ on } V}(L^\cdot).$$

4.4. More generally, for a morphism $f : X \to Y$ in $\mathscr{S}$ we define $A^{*,*}_{\mathscr{M}}(f : X \to Y)$ by the formula

$$A^{n,m}_{\mathscr{M}}(f : X \to Y) := \mathrm{Ext}^{2n}_{\mathscr{M}(X)}(1_X, f^! 1_Y(m)).$$

Note that for a factorization of f

$$X \overset{i}{\hookrightarrow} M \overset{g}{\longrightarrow} Y, \quad f = g \circ i \tag{4.4.1}$$

with i an imbedding and g smooth of relative dimension d we have $f^! 1_Y(m) \simeq i^! g^! 1_Y(m) \simeq i^! 1_Y(m+d)[2d]$, whence a canonical isomorphism

$$A^{n,m}_{\mathscr{M}}(f : X \to Y) \simeq A^{n+d,m+d}_{\mathscr{M}}(i : X \hookrightarrow M).$$

4.5. For a quasi-projective morphism $f : X \to Y$ in $\mathscr{S}$, one has the Grothendieck group of f-perfect complexes defined as in [18, 3.10]. Moreover, the same argument is in loc. cit. shows that there is a transformation

$$\tau^X_Y : K(f\text{-perfect complexes on } X) \to \oplus_i A^{i,i}_{\mathscr{M}}(f : X \to Y).$$

This transformation is defined by choosing a factorization of f as in (4.4.1) and sending a complex $K^\cdot$ to

$$\mathrm{td}(i^*T_{M/Y}) \cdot \mathrm{ch}^X_M(K^\cdot) \in A^{*,*}_{\mathscr{M}}(X \hookrightarrow M) \simeq A^{*-d,*-d}_{\mathscr{M}}(X \to Y).$$

4.6. In particular, for $X \in \mathscr{S}$, quasi-projective over B, we can consider $A^{n,m}_{\mathscr{M}}(X \to B)$, with $X \to B$ the structure morphism. In this case we define

$$H^{\mathscr{M}}_{i,BM}(X) := A^{-i,-i}_{\mathscr{M}}(X \to B),$$

called the i-th *$\mathscr{M}$-valued Borel-Moore homology of* X (or just *i-th Borel-Moore homology* if the reference to $\mathscr{M}$ is clear). In this case the Grothendieck group of f-perfect complexes is simply the Grothendieck group of coherent sheaves on X (since B is regular) so we get a map

$$\tau_X : K(\mathrm{Coh}(X)) \to \oplus_i H^{\mathscr{M}}_{i,BM}(X). \tag{4.6.1}$$

Remark 4.7. In the case when B is the spectrum of a field, the map (4.6.1) can also be viewed as a cycle class map, using the identification of $K(\mathrm{Coh}(X))$ with Chow groups (tensor $\mathbb{Q}$).

Remark 4.8. Our construction of the map τ_Y^X above for a morphism $f : X \to Y$ in $\mathscr{S}$ uses the existence of a factorization through a smooth morphism, and therefore necessitates imposing quasi-projectivity hypotheses. It seems likely that another construction exists which generalizes to more general morphisms.

5. Local terms for motivic actions

Let k be a field and let $\mathscr{S}$ denote the category of finite type separated k-schemes. Fix a ring R and let $\mathscr{M}$ be an R-linear triangulated motivic category.

5.1. Let $X, Y \in \mathscr{S}$ be two objects. For $F \in \mathscr{M}(X)$ and $G \in \mathscr{M}(Y)$ let $F \boxtimes G \in \mathscr{M}(X \times Y)$ denote $\mathrm{pr}_1^*F \otimes_{X\times Y} \mathrm{pr}_2^*G$. There is a map

$$\epsilon_{X\times Y} : \Omega_X \boxtimes \Omega_Y \to \Omega_{X\times Y}$$

defined as follows. We have isomorphisms

$$\begin{aligned} \mathrm{Hom}_{X\times Y}(\Omega_X \boxtimes \Omega_Y, \Omega_{X\times Y}) &\simeq \mathrm{Hom}_{X\times Y}(\mathrm{pr}_1^*\Omega_X, \mathrm{pr}_2^!1_Y) \\ &\simeq \mathrm{Hom}_Y(\mathrm{pr}_{2!}\mathrm{pr}_1^*\Omega_X, 1_Y) \\ &\simeq \mathrm{Hom}_Y(g^*f_!f^!1_{\mathrm{Spec}(k)}, g^*1_{\mathrm{Spec}(k)}), \end{aligned} \tag{5.1.1}$$

where $f : X \to \mathrm{Spec}(k)$ (resp. $g : Y \to \mathrm{Spec}(k)$) is the structure morphism, the first isomorphism is induced by the isomorphism $\mathscr{H}om_{X\times Y}(\mathrm{pr}_2^*\Omega_Y, \Omega_{X\times Y}) \simeq \mathrm{pr}_2^!1_Y$ (coming from 2.9 (Duality) (c) which identifies $\mathrm{pr}_2^!(-)$ with $\mathscr{H}om_{X\times Y}(\mathrm{pr}_2^* \circ D_Y(-), \Omega_{X\times Y})$) and the adjunction isomorphism

$$\mathrm{Hom}_{X\times Y}(\Omega_X \boxtimes \Omega_Y, \Omega_{X\times Y}) \simeq \mathrm{Hom}_{X\times Y}(\mathrm{pr}_1^*\Omega_X, \mathscr{H}om_{X\times Y}(\mathrm{pr}_2^*\Omega_Y, \Omega_{X\times Y})),$$

the second isomorphism is by adjunction, and the third isomorphism is by base change. The map $\epsilon_{X\times Y}$ is the map corresponding under these isomorphisms to the adjunction map $f_!f^!1_{\mathrm{Spec}(k)} \to 1_{\mathrm{Spec}(k)}$.

Assumption 5.2. *Assume that the map $\epsilon_{X\times Y}$ is an isomorphism for all $X, Y \in \mathscr{S}$.*

Remark 5.3. For any $F \in \mathscr{M}(X)$ the adjunctions used in (5.1.1) define an isomorphism

$$\mathrm{Hom}_{X\times Y}(F \boxtimes \Omega_Y, \Omega_{X\times Y}) \simeq \mathrm{Hom}_Y(g^*f_!F, g^*1_{\mathrm{Spec}(k)})$$

functorial in F.

5.4. Fix $X, Y \in \mathscr{S}$. Note that there is a natural map

$$1_X \boxtimes \Omega_Y \to \mathscr{H}om_{X\times Y}(\Omega_X \boxtimes 1_Y, \Omega_X \boxtimes \Omega_Y)$$

which, with the above identification of $\Omega_X \boxtimes \Omega_Y$ with $\Omega_{X\times Y}$, gives a map

$$\rho_{X\times Y} : 1_X \boxtimes \Omega_Y \to D(\Omega_X \boxtimes 1_Y).$$

5.5. Consider a closed imbedding $i : Z \hookrightarrow Y$ with complement $j : U \hookrightarrow Y$, and let $\tilde{i} : X \times Z \hookrightarrow X \times Y$ and $\tilde{j} : X \times U \hookrightarrow X \times Y$ be the inclusions defined by base change. We then have a distinguished triangle (using assumption 5.2)

$$\tilde{i}_*(\Omega_X \boxtimes \Omega_Z) \to \Omega_X \boxtimes \Omega_Y \to \tilde{j}_*(\Omega_X \boxtimes \Omega_U) \to \tilde{i}_*(\Omega_X \boxtimes \Omega_Z)[1].$$

Applying $\mathscr{H}om_{X\times Y}(\Omega_X \boxtimes 1_Y, -)$ to this triangle we get a distinguished triangle

$$\tilde{i}_* D(\Omega_X \boxtimes 1_Z) \to D(\Omega_X \boxtimes 1_Y) \to \tilde{j}_* D(\Omega_X \boxtimes 1_U) \to \tilde{i}_* D(\Omega_X \boxtimes 1_Z)[1],$$

and a diagram

$$\begin{array}{ccccccc} \tilde{i}_*(1_X \boxtimes \Omega_Z) & \longrightarrow & 1_X \boxtimes \Omega_Y & \longrightarrow & \tilde{j}_*(1_X \boxtimes \Omega_U) & \longrightarrow & \tilde{i}_*(1_X \boxtimes \Omega_Z)[1] \\ \downarrow{\scriptstyle \rho_{X\times Z}} & & \downarrow{\scriptstyle \rho_{X\times Y}} & & \downarrow{\scriptstyle \rho_{X\times U}} & & \downarrow{\scriptstyle \rho_{X\times Z}} \\ \tilde{i}_* D(\Omega_X \boxtimes 1_Z) & \longrightarrow & D(\Omega_X \boxtimes 1_Y) & \longrightarrow & \tilde{j}_* D(\Omega_X \boxtimes 1_U) & \longrightarrow & \tilde{i}_* D(\Omega_X \boxtimes 1_Z)[1], \end{array} \tag{5.5.1}$$

where the horizontal rows are distinguished triangles.

Lemma 5.6. *The diagram (5.5.1) is commutative.*

Proof. Let $\epsilon'_{X\times Y}$ (resp. $\epsilon''_{X\times Y}$) denote the element of

$$\mathrm{Hom}_{X\times Y}(\mathrm{pr}_1^*\Omega_X, \mathrm{pr}_2^! 1_Y) \ \ (\text{resp. } \mathrm{Hom}_Y(\mathrm{pr}_{2!}\mathrm{pr}_1^*\Omega_X, 1_Y))$$

corresponding to $\epsilon_{X\times Y}$ under the isomorphisms in (5.1.1), and let $\alpha : f_! f^! 1_k \to 1_k$ denote the adjunction morphism, where to ease notation we write 1_k for $1_{\mathrm{Spec}(k)}$. Similarly define $\epsilon'_{X\times Z}$, $\epsilon''_{X\times Z}$, $\epsilon'_{X\times U}$, and $\epsilon''_{X\times U}$. Then $\rho_{X\times Y}$ (resp. $\rho_{X\times Z}$, $\rho_{X\times U}$) is Verdier dual to $\epsilon'_{X\times Y}$ (resp. $\epsilon'_{X\times Z}$, $\epsilon'_{X\times U}$). Let $g_Z : Z \to \mathrm{Spec}(k)$ and $g_U : U \to \mathrm{Spec}(k)$ denote the restrictions of $g : Y \to \mathrm{Spec}(k)$. For any $F \in \mathscr{M}(X \times Y)$ we have have by 2.9 (Loc) a distinguished triangle on Y

$$j_! j^* F \to F \to i_* i^* F \to j_! j^* F[1]$$

functorial in F. In particular, we get a morphism of distinguished triangles

$$\begin{array}{ccccccc} j_! g_U^* f_! f^! 1_k & \longrightarrow & g^* f_! f^! 1_k & \longrightarrow & i_* g_Z^* f_! f^! 1_k & \longrightarrow & j_! g_U^* f_! f^! 1_k[1] \\ \downarrow{\scriptstyle j_! g_U^*\alpha} & & \downarrow{\scriptstyle g^*\alpha} & & \downarrow{\scriptstyle i_* g_Z^*\alpha} & & \downarrow{\scriptstyle j_! g_U^*\alpha} \\ j_! g_U^* 1_k & \longrightarrow & g^* 1_k & \longrightarrow & i_* g_Z^* 1_k & \longrightarrow & j_! g_U^* 1_k[1]. \end{array} \tag{5.6.1}$$

Now observe that the natural morphism of distinguished triangles

$$\begin{array}{ccccccc} \tilde{j}_! \mathrm{pr}^!_{X\times U,2} 1_U & \longrightarrow & \mathrm{pr}^!_{X\times Y,2} 1_Y & \longrightarrow & \tilde{i}_* \mathrm{pr}^!_{X\times Z,2} 1_Z & \longrightarrow & \tilde{j}_! \mathrm{pr}^!_{X\times U,2} 1_U[1] \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow \\ \mathrm{pr}^!_{X\times Y,2} j_! 1_U & \longrightarrow & \mathrm{pr}^!_{X\times Y,2} 1_Y & \longrightarrow & \mathrm{pr}^!_{X\times Y,2} i_* 1_Z & \longrightarrow & \mathrm{pr}^!_{X\times Y,2} j_! 1_U[1] \end{array}$$

is an isomorphism, as the two middle vertical arrows are isomorphisms (using 2.9 (4)). By adjunction the commutativity of (5.6.1) then implies that the diagram

$$\begin{array}{ccccccc}
\tilde{j}_!\mathrm{pr}^*_{X\times U,1}\Omega_X & \longrightarrow & \mathrm{pr}^*_{X\times Y,1}\Omega_X & \longrightarrow & \tilde{i}_*\mathrm{pr}^*_{X\times Z,1}\Omega_X & \longrightarrow & \tilde{j}_!\mathrm{pr}^*_{X\times U,1}\Omega_X[1] \\
\downarrow{\scriptstyle \tilde{j}_!\epsilon'_{X\times U}} & & \downarrow{\scriptstyle \epsilon'_{X\times Y}} & & \downarrow{\scriptstyle i_*\epsilon'_{X\times Z}} & & \downarrow{\scriptstyle \tilde{j}_!\epsilon'_{X\times U}} \\
\tilde{j}_!\mathrm{pr}^!_{X\times U,2}1_U & \longrightarrow & \mathrm{pr}^!_{X\times Y,2}1_Y & \longrightarrow & \tilde{i}_*\mathrm{pr}^!_{X\times Z,2}1_Z & \longrightarrow & \tilde{j}_*\mathrm{pr}^!_{X\times U,2}1_U[1]
\end{array}$$

commutes, and finally dualizing this diagram we obtain that (5.5.1) commutes. □

Lemma 5.7. *For any $X, Y \in \mathscr{S}$ the map $\rho_{X\times Y}$ is an isomorphism.*

Proof. Consider a closed imbedding $i : Z \hookrightarrow Y$ with complement $j : U \hookrightarrow Y$ equidimensional of some dimension d and admitting a factorization of its structure morphism as in 2.14. Let $\tilde{i} : X \times Z \hookrightarrow X \times Y$ and $\tilde{j} : X \times U \hookrightarrow X \times Y$ be the inclusions defined by base change. Considering the morphism of distinguished triangles (5.5.1) it then suffices to show the result for the pair (X, Z) and (X, U), which by induction on the dimension of Y reduces the proof to the case when Y admits a factorization as in 2.14. In this case Ω_Y (rest. $\Omega_{X\times Y}$) is equal, up to a shift and Tate twist, to 1_Y (rest. $\Omega_X \boxtimes 1_Y$) and the result is immediate. □

5.8. Fix a correspondence $c : C \to X \times X$ with $C, X \in \mathscr{S}$ and quasi-projective, and let $\mathrm{Fix}(c) := C \times_{c, X\times X, \Delta_X} X$ denote the fixed point scheme. For an action (a morphism in $\mathscr{M}(C)$)

$$u : c_1^*1_X \simeq 1_C \to c_2^!1_X,$$

we define $\mathrm{Tr}_c(u) \in H^{\mathscr{M}}_{0,BM}(\mathrm{Fix}(c))$ as follows. Note that we have

$$c^!(\Omega_X \boxtimes 1_X) \simeq D_C c^*(1_X \boxtimes \Omega_X) \simeq D_C c_2^*\Omega_X \simeq c_2^!1_X,$$

where the first isomorphism uses 5.7. Therefore we can also view u as an element of $H^0_{\mathscr{M}}(C, c^!(\Omega_X \boxtimes 1_X))$. Let $\Delta : X \to X \times X$ be the diagonal morphism and note that there is a natural map

$$\Omega_X \boxtimes 1_X \to \Delta_*\Omega_X.$$

Applying this map and using the base change isomorphism for the cartesian diagram

$$\begin{array}{ccc}
C & \xleftarrow{\ \delta\ } & \mathrm{Fix}(c) \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c'} \\
X \times X & \xleftarrow{\ \Delta\ } & X
\end{array}$$

we get a morphism

$$c^!(\Omega_X \boxtimes 1_X) \to c^!\Delta_*\Omega_X \simeq \delta_*c'^!\Omega_X \simeq \delta_*\Omega_{\mathrm{Fix}(c)}.$$

Applying $\mathrm{Hom}_{\mathscr{M}(C)}(1_C, -)$ to this map we get a map

$$\mathrm{Tr}^{\mathscr{M}} : \mathrm{Hom}_{\mathscr{M}(C)}(c_1^*1_X, c_2^!1_X) \to H^{\mathscr{M}}_{0,BM}(\mathrm{Fix}(c)).$$

If no confusion seems likely we write simply Tr for $\mathrm{Tr}^{\mathscr{M}}$.

5.9. Let $\mathscr{N}$ be a second motivic category satisfying the assumption 5.2, and let

$$T : \mathscr{M} \to \mathscr{N}$$

be a morphism of fibered categories such that for every $X \in \mathscr{S}$ the morphism on fibers

$$T_X : \mathscr{M}(X) \to \mathscr{N}(X)$$

is a triangulated monoidal functor, and assume further that T is compatible with the operations $f_*, f^*, f_!, f^!$ for morphisms $f : X \to Y$ in $\mathscr{S}$, Tate twists, and internal Hom. Note that then T induces for every quasi-projective $X \in \mathscr{S}$ a map (which we abusively also denote simply by T)

$$T : H^{\mathscr{M}}_{*,BM}(X) \to H^{\mathscr{N}}_{*,BM}(X).$$

Let $c : C \to X \times X$ be a correspondence as in 5.8, and let $u : c_1^* 1_{X,\mathscr{M}} \to c_2^! 1_{X,\mathscr{M}}$ be an action in $\mathscr{M}(C)$, where we write $1_{X,\mathscr{M}}$ for the unit object in $\mathscr{M}(X)$. Then it follows from the construction and the fact that T is compatible with the six operations that

$$T(\mathrm{Tr}^{\mathscr{M}}(u)) = \mathrm{Tr}^{\mathscr{N}}(T(u))$$

in $H^{\mathscr{N}}_{0,BM}(\mathrm{Fix}(c))$.

6. Beilinson motives

In this section B is a regular excellent scheme of finite Krull dimension δ, and $\mathscr{S}$ denotes the category of quasi-projective B-schemes.

6.1. For $R = \mathbb{Q}$, there are several equivalent constructions of triangulated motivic categories. The one most convenient for us in this paper is the category of constructible Beilinson motives defined in [4, §14] which we will denote by $\mathscr{M}_B$.

The main properties of this category that we will need are the following 6.2 and 6.5.

Proposition 6.2. *For $X \in \mathscr{S}$ the map (4.6.1) induces for every i an isomorphism*

$$A_{\delta+i}(X)_{\mathbb{Q}} \simeq H^{\mathscr{M}_B}_{i,BM}(X),$$

where the left side refers to Chow homology groups, tensor $\mathbb{Q}$, as defined in [12, §1.8] *(or in the case when B is the spectrum of a field* [11, §1.3]*).*

Proof. To properly define $\mathscr{M}_B$ requires a lot of preparatory material, for which we refer to [4, §14]. One definition of the category of Beilinson motives (see [4, 14.2.9]) is as the homotopy category $\mathrm{Ho}(H_B - \mathrm{mod})$, where H_B is a cofibrant cartesian commutative monoid in the symmetric monoidal fibered model category of Tate spectra over the category of schemes; see [4, 7.2.10, 5.3.18]. The category $\mathscr{M}_B$ is defined as the subcategory of $\mathrm{Ho}(H_B - \mathrm{mod})$ of constructible objects [4, 15.1.1].

Let $KGL_{\mathbb{Q}}$ denote the absolute ring spectrum defined in [4, 14.1.1]. Then as in [4, 13.3.3] (which is an integral version) $\mathrm{Ho}(KLG_{\mathbb{Q}} - \mathrm{mod})$ forms a motivic category over $\mathscr{S}$ in the sense of loc. cit. (which is slightly weaker than the definition used in this paper).

If $X \in \mathscr{S}$ is a regular scheme, then by [4, 13.3.2.1] we have a canonical functorial isomorphism

$$\mathrm{Ext}^n_{\mathrm{Ho(KGL_{\mathbb{Q}}-mod}(X))}(KGL_{\mathbb{Q}}(X), KGL_{\mathbb{Q}}(X)(m)) \simeq K_{2n-m}(X)_{\mathbb{Q}},$$

where $K_{2n-m}(X)_{\mathbb{Q}}$ denotes algebraic K-theory.

By [4, 14.2.17] there is a map of ring spectra

$$H_B \to KGL_{\mathbb{Q}}, \tag{6.2.1}$$

which induces a morphism of motivic categories

$$\varphi^* : \mathrm{Ho}(H_B - \mathrm{mod}) \to \mathrm{Ho}(KGL_{\mathbb{Q}} - \mathrm{mod}).$$

The functor φ^* is the extension of scalars and the functor φ_* (the right adjoint to φ^*, which exists by 2.7 (i)) is the forgetful functor. By [4, 14.2.17 (3)] there is also a morphism

$$\pi_0 : \varphi_* KGL_{\mathbb{Q}} \to H_B \tag{6.2.2}$$

in $\mathrm{Ho}(H_B - \mathrm{mod})$ such that the composition $H_B \to \varphi_* KLG_{\mathbb{Q}} \to H_B$ is the identity map.

Fix a closed imbedding $i : X \hookrightarrow M$, with M smooth of constant dimension d over B, and let $j : U \hookrightarrow M$ be the complement of X. Taking cohomology of the distinguished triangle

$$i_*i^!1_M(d+a)[2d] \to 1_M(d+a)[2d] \to j_*1_U(d+a)[2d]$$

we get a long exact sequence

$$\cdots \to H^s_{\mathscr{M}_B}(X, \Omega_X(a)) \to H^{s+2d,d+a}_{\mathscr{M}_B}(M) \to H^{s+2d,d+a}_{\mathscr{M}_B}(U) \to \cdots . \tag{6.2.3}$$

To compare this with K-theory we following the argument of [21, Proof of Théorème 8]. For $X \in \mathscr{S}$ let $K'_m(X)_{\mathbb{Q}}$ denote the K-theory of the category of coherent sheaves on X tensor $\mathbb{Q}$. Recall from [21, 7.2 and Théoréme 8 (v)] that for any $X \in \mathscr{S}$ the filtration $F_{\cdot}$ on $K'_0(X)_{\mathbb{Q}}$ defined by the Riemann-Roch isomorphism $K'_0(X)_{\mathbb{Q}} \simeq A_*(X)_{\mathbb{Q}}$ (so $F_j/F_{j+1} \simeq A_j(X)_{\mathbb{Q}}$) is induced by operations ϕ^k on $K'_0(X)_{\mathbb{Q}}$. Following the proof of [21, Théorème 8 (ii)] we get the long exact sequence

$$\cdots \to K'_m(X) \to K'_m(M) \to K'_m(U) \to K'_{m-1}(X) \to \cdots ,$$

and as in loc. cit. this sequence is compatible with the Adams operations and induces an exact sequence on the associated graded pieces

$$\cdots \to \mathrm{gr}_j K'_m(X) \to \mathrm{gr}_j K'_m(M) \to \mathrm{gr}_j K'_m(U) \to \cdots . \tag{6.2.4}$$

The fact that the map from K-theory to the cohomology of Beilinson motives is defined by the map π_0 (6.2.2) and the description of the Adam's operations as coming from the decomposition in [4, 14.1.1 (K5)] implies that we get an induced map from the long exact sequence (6.2.4) to the long exact sequence (6.2.3) (using also the compatibility [4, 13.4.1 (K6)]. We therefore obtain a commutative diagram with exact rows

(6.2.5)

$$\begin{array}{ccccccccc}
\mathrm{gr}_{\delta+i}K'_1(M) & \longrightarrow & \mathrm{gr}_{\delta+i}K'_1(U) & \longrightarrow & \mathrm{gr}_{\delta+i}K'_0(X) & \longrightarrow & \mathrm{gr}_{\delta+i}K'_0(M) & \longrightarrow & \mathrm{gr}_{\delta+i}K'_0(U) \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\
H^{2(d-i)-1,d-i}_{\mathscr{M}_B}(M) & \longrightarrow & H^{2(d-i)-1,d-i}_{\mathscr{M}_B}(U) & \longrightarrow & H^{\mathscr{M}_B}_{i,BM}(X) & \longrightarrow & H^{2(d-i),d-i}_{\mathscr{M}_B}(M) & \longrightarrow & H^{2(d-i),d-i}_{\mathscr{M}_B}(U).
\end{array}$$

In general, if $M \to B$ is smooth of relative dimension d then for any integer $s \geq 0$ the map

$$\mathrm{gr}_{\delta+i}K'_s(M) \to H^{2(d-i)-s,d-i}_{\mathscr{M}_B}(M)$$

is an isomorphism. This follows from [4, 14.2.14], which identifies $H^{2(d-i)-s,d-i}_{\mathscr{M}_B}(M)$ with $\mathrm{gr}^{d-i}_\gamma K_s(M)$ (associated graded of the γ-filtration), and [21, 7.2 (vi)] which shows that $\mathrm{gr}^{d-i}_\gamma K_s(M) = \mathrm{gr}_{i+\delta} K'_s(M)$.

Therefore the maps a, b, d, and e in (6.2.5) are isomorphisms, which implies that the map c is also an isomorphism.

We therefore get an induced isomorphism $\alpha_X : A_{\delta+i}(X)_\mathbb{Q} \simeq H^{\mathscr{M}_B}_{i,BM}(X)$. Let $\beta_X : A_{\delta+i}(X)_\mathbb{Q} \to H^{\mathscr{M}_B}_{i,BM}(X)$ be the map defined by (4.6.1). We expect that $\alpha_X = \beta_X$ but this does not follow immediately from the construction. Nonetheless we get that β_X is an isomorphism as follows.

Observe that if $z : Z \hookrightarrow X$ is a closed subscheme, then the diagram

$$\begin{array}{ccc} A_{\delta+i}(Z)_\mathbb{Q} & \xrightarrow{z_*} & A_{\delta+i}(X)_\mathbb{Q} \\ \downarrow \alpha_Z & & \downarrow \alpha_X \\ H^{\mathscr{M}_B}_{i,BM}(Z) & \xrightarrow{z_*} & H^{\mathscr{M}_B}_{i,BM}(X) \end{array}$$

commutes by construction, and similarly for β_X. Furthermore, using [4, 13.6.4] we know that α_X and β_X agree in the case when X is regular.

Fix a sequence of closed subschemes

$$\emptyset = X_n \subset X_{n-1} \subset X_{n-2} \subset \cdots \subset X_1 \subset X_0 = X$$

such that for every t the complement $V_t := X_t - X_{t+1}$ is regular. Let $F_t \subset A_{\delta+i}(X)_\mathbb{Q}$ (resp. $G_t \subset H^{\mathscr{M}_B}_{i,BM}(X)$) denote the image of $A_{\delta+i}(X_t)_\mathbb{Q}$ (resp. $H^{\mathscr{M}_B}_{i,BM}(X_t)$). Then α_X and β_X respect these filtrations.

We claim that α_X and β_X induced the same maps on the graded vector spaces associated to these filtrations. Since α_X is an isomorphism this implies that β_X induces an isomorphism on the associated graded, which in turn implies that β_X is an isomorphism.

To see that α_X and β_X agree on the associated graded vector spaces, we proceed by induction on the length n of the sequence of closed subschemes. The base case $n = 1$ follows from the case when X itself is regular.

For the inductive step we assume that the result holds for each X_t, $t > 0$, with the sequence of closed subschemes

$$\emptyset = X_n \subset X_{n-1} \subset \cdots X_t.$$

Since F_t (resp. G_t) for $t > 0$ is the image of the corresponding step of the filtration on $A_{\delta+i}(X_{n-1})_\mathbb{Q}$ (resp. $H^{\mathscr{M}_B}_{i,BM}(X_{n-1})$) it follows from the inductive hypothesis that α_X and β_X induce the same map

$$F_t/F_{t+1} \to G_t/G_{t+1}$$

for $t > 0$. Thus it suffices to show that α_X and β_X induce the same map

$$A_{\delta+i}(X)_\mathbb{Q} \to \mathrm{Coker}(H^{\mathscr{M}_B}_{i,BM}(X_1) \to H^{\mathscr{M}_B}_{i,BM}(X)).$$

This follows from noting that the composition of this map (for either α_X or β_X) with the inclusion

$$\mathrm{Coker}(H^{\mathscr{M}_B}_{i,BM}(X_1) \to H^{\mathscr{M}_B}_{i,BM}(X)) \hookrightarrow H^{\mathscr{M}_B}_{i,BM}(V_0)$$

is equal to the restriction map $A_{\delta+i}(X)_{\mathbb{Q}} \to A_{\delta+i}(V_0)$ followed by α_{V_0} (which agrees with β_{V_0}). □

Remark 6.3. If $X \in \mathscr{S}$ is regular we have, by [4, 14.2.14], an isomorphism

$$H^{q,p}_{\mathscr{M}_B}(X) \simeq \mathrm{gr}^p_\gamma K_{2p-q}(X)_{\mathbb{Q}}.$$

This isomorphism implies various vanishing results for motivic cohomology. We will need two cases in what follows:

(i) ($p < 0$) If d is the dimension of X then by [21, Théorème 7.2 (vi)] we have

$$\mathrm{gr}^p_\gamma K_{2p-q}(X)_{\mathbb{Q}} = \mathrm{gr}^{d-p} K_{2p-q}(X)_{\mathbb{Q}},$$

which in the notation of [21, 7.4] is equal to $H_{2d-q}(X, d-p)$. If $p < 0$ then $d - p > d$ in which case this group is 0 by [21, Théorème 8 (i)].

(ii) (X affine $p = 0$ and $q < 0$) In this case we have

$$H^{q,p}_{\mathscr{M}_B}(X) \simeq \mathrm{gr}^0_\gamma K_{-q}(X).$$

The vanishing of this group is a known special case of Beilinson-Soulé vanishing [21, 2.9].

The second property of the category of Beilinson motives that we will need is the following result, which will enable us to use de Jong's results on equivariant alterations [8].

6.4. Let Y be a quasi-projective B-scheme and G a finite group acting on Y. Let X' denote the coarse moduli space of the quotient stack $[Y/G]$ and let $\pi : X' \to X$ be a finite surjective radicial morphism. Let $p : Y \to X$ be the projection and define $(p_*\Omega_Y)^G$ as in [4, 3.3.21], so $(-)^G$ denotes derived G-invariants. There is a natural morphism $p_*\Omega_Y \to \Omega_X$ (dual to the adjunction morphism $1_X \to p_*1_Y$) which induces a morphism

$$h : (p_*\Omega_Y)^G \to \Omega_X. \tag{6.4.1}$$

Proposition 6.5. *The map (6.4.1) is an isomorphism.*

Proof. Let $i : Z \hookrightarrow X$ be a closed imbedding with complement $j : U \hookrightarrow X$ such that the following hold:

(1) U is everywhere dense in X.
(2) If Y_U denotes $U \times_X Y$ then U_{red} and $Y_{U,\mathrm{red}}$ are regular, and the map $Y_{U,\mathrm{red}} \to U_{\mathrm{red}}$ is flat.

Let Y_Z denote $Y \times_X Z$ and let X'_Z denote the coarse moduli space of $[Y_Z/G]$. The formation of the coarse moduli space does not in general commute with base change. It is still true, however, that $Y_Z \to X'_Z \to Z$ satisfies the assumptions in 6.4. Consider the induced map of distinguished triangles

$$\begin{array}{ccccccc}
(i_*p_{Z*}\Omega_{Y_Z})^G & \longrightarrow & (p_*\Omega_Y)^G & \longrightarrow & (j_*p_{U*}\Omega_{Y_U})^G & \longrightarrow & (i_*p_{Z*}\Omega_{Y_Z})^G[1] \\
\downarrow h_Z & & \downarrow h & & \downarrow h_U & & \downarrow h_Z \\
i_*\Omega_Z & \longrightarrow & \Omega_X & \longrightarrow & j_*\Omega_U & \longrightarrow & i_*\Omega_Z[1].
\end{array}$$

By induction we may assume that the map $(p_{Z*}\Omega_{Y_Z})^G \to \Omega_Z$ is an isomorphism, and since the formation of derived G-invariants commutes with pushforward we may therefore assume that h_Z is an isomorphism. It therefore suffices to consider the case when X and Y are regular of the same dimension, where it follows from [4, 3.3.35 and 14.3.3] and the following lemma. □

Lemma 6.6. *Let $X \in \mathscr{S}$ be regular of dimension d. Then $\Omega_X \simeq 1_X(d-\delta)[2(d-\delta)]$.*

Proof. Since X is quasi-projective over B we can find a locally free sheaf E of finite rank $r+1$ on B and an imbedding $i : X \hookrightarrow \mathbb{P}E$ over B. Then i is a regular imbedding of codimension $r+\delta-d$. We have $\Omega_X \simeq i^! 1_{\mathbb{P}E}(r)[2r]$ so it suffices to show that $i^! 1_{\mathbb{P}E} \simeq 1_X(d-\delta-r)[2(d-\delta-r)]$. This follows from absolute purity for Beilinson motives [4, 14.4.1] (see also [5, A.2.8]). □

An immediate corollary is the following:

Corollary 6.7. *Let $Y \in \mathscr{S}$ be regular of dimension d, and let G be a finite group acting on Y. Let $X := Y/G$ be the coarse moduli space of the corresponding Deligne-Mumford stack $[Y/G]$. Then $\Omega_X \simeq 1_X(d-\delta)[2(d-\delta)]$.*

In particular, if $f : Z \to X$ is a morphism in $\mathscr{S}$ then we have $A^{i,i}_{\mathscr{M}_B}(Z \to X) \simeq A_{d-i}(Z)_{\mathbb{Q}}$.

Proof. The first statement follows immediately from 6.5 and 6.6. For the second statement observe that we have

$$\begin{aligned} A^{i,i}_{\mathscr{M}_B}(Z \to X) &= H^{2i}(Z, f^! 1_X(i)) \\ &= H^{2i}(Z, f^!\Omega_X(\delta - d + i)[2(\delta - d)]) \quad \text{using } \Omega_X \simeq 1_X(d-\delta)[2(d-\delta)] \\ &= H^{2(i+\delta-d)}(Z, \Omega_Z(i+\delta-d)) \\ &= H^{\mathscr{M}_B}_{-i-\delta+d,BM}(Z) \\ &\simeq A_{d-i}(Z)_{\mathbb{Q}} \quad \text{by 6.2.} \end{aligned}$$

□

Remark 6.8. Our proofs above use imbeddings into smooth schemes and therefore require imposing quasi-projectivity assumptions. It seems plausible that they could be generalized to statements without the quasi-projectivity assumption.

7. Assumption 5.2 over an algebraically closed field

7.1. In this section we restrict attention to the setting when B is the spectrum of an algebraically closed field. In this case, proposition 6.5 also implies that assumption 5.2 holds for $\mathscr{M}_B$. The proof of this is a bit more intricate and it is useful to consider the following variant statements and intermediate results.

7.2. Fix $X, Y \in \mathscr{S}$ and a closed imbedding $i : Z \hookrightarrow X$. Let $j : U \hookrightarrow X$ be the complement of Z and let

$$Z \times Y \xrightarrow{\tilde{i}} X \times Y \xleftarrow{\tilde{j}} U \times Y$$

denote the base changes to Y. For $F \in \mathscr{M}_B(X)$ let F_U denote the restriction to U, and consider the following conditions:

($A^Z_X(F)$) The natural map $(j_*F_U) \boxtimes \Omega_Y \to \tilde{j}_*(F_U \boxtimes \Omega_Y)$ is an isomorphism.

($B^Z_X(F)$) The natural map $(i^!F) \boxtimes \Omega_Y \to \tilde{i}^!(F \boxtimes \Omega_Y)$ is an isomorphism.

(C_X) The map $\epsilon_{X\times Y}:\Omega_X\boxtimes\Omega_Y\to\Omega_{X\times Y}$ is an isomorphism.

Lemma 7.3. *Properties $A^Z_X(F)$ and $B^Z_X(F)$ are equivalent.*

Proof. Indeed the maps in question fit into a morphism of distinguished triangles

$$\begin{array}{ccccccc}
\tilde{i}_*(i^!F\boxtimes\Omega_Y) & \longrightarrow & F\boxtimes\Omega_Y & \longrightarrow & (j_*F_U)\boxtimes\Omega_Y & \longrightarrow & \tilde{i}_*(i^!F\boxtimes\Omega_Y)[1] \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\tilde{i}_*\tilde{i}^!(F\boxtimes\Omega_Y) & \longrightarrow & F\boxtimes\Omega_Y & \longrightarrow & \tilde{j}_*(F_U\boxtimes\Omega_Y) & \longrightarrow & \tilde{i}_*\tilde{i}^!(F\boxtimes\Omega_Y)[1].
\end{array}$$

□

Lemma 7.4. *Let $p:X'\to X$ be a proper morphism and let $\tilde{p}:X'\times Y\to X\times Y$ denote the base change of p to Y. Then for any $F\in\mathscr{M}(X')$ and $G\in\mathscr{M}(Y)$ the natural map*

$$(p_*F)\boxtimes G\to\tilde{p}_*(F\boxtimes G) \tag{7.4.1}$$

is an isomorphism.

Proof. Indeed since p is proper we have

$$\begin{array}{rcll}
(p_*F')\boxtimes G & = & (\mathrm{pr}_1^*p_*F')\otimes\mathrm{pr}_2^*G & \\
& \simeq & (\tilde{p}_*\mathrm{pr}_1'^*F')\otimes\mathrm{pr}_2^*G & \text{by 2.9 (4)} \\
& \simeq & \tilde{p}_*(\mathrm{pr}_1'^*F'\otimes\tilde{p}^*\mathrm{pr}_2^*G) & \text{by 2.9 (5)} \\
& = & \tilde{p}_*(F'\boxtimes G), &
\end{array}$$

where we write

$$\mathrm{pr}_1':X'\times Y\to X',\quad \mathrm{pr}_1:X\times Y\to X,\quad \mathrm{pr}_2:X\times Y\to Y,$$

for the projections. □

Lemma 7.5. *Let $p:X'\to X$ be a proper morphism and let $\tilde{p}:X'\times Y\to X\times Y$ denote the base change of p to Y. Then the diagram*

$$\begin{array}{ccccc}
p_*\Omega_{X'}\boxtimes\Omega_Y & \xrightarrow{a} & \tilde{p}_*(\Omega_{X'}\boxtimes\Omega_Y) & \xrightarrow{\tilde{p}_*\epsilon_{X'\times Y}} & \tilde{p}_*\Omega_{X'\times Y} \\
\downarrow{\scriptstyle t_p\otimes\mathrm{id}} & & & & \downarrow{\scriptstyle t_{\tilde{p}}} \\
\Omega_X\boxtimes\Omega_Y & & \xrightarrow{\epsilon_{X\times Y}} & & \Omega_{X\times Y}
\end{array}$$

commutes, where the map t_p (resp. $t_{\tilde{p}}$) denotes the adjunction map $p_*\Omega_{X'}\to\Omega_X$ (resp. $\tilde{p}_*\Omega_{X'\times Y}\to\Omega_{X\times Y}$) and a is as in (7.4.1).

Proof. It follows from the definitions that under the isomorphism

$$\mathrm{Hom}_{X\times Y}(p_*\Omega_{X'}\boxtimes\Omega_Y,\Omega_{X\times Y})\simeq\mathrm{Hom}_Y(g^*f_!p_*\Omega_{X'},g^*1_{\mathrm{Spec}(k)})$$

defined in 5.3, the composition $t_{\tilde{p}}\circ(\tilde{p}_*\epsilon_{X'\times Y})\circ a$ corresponds to the map

$$g^*f_!p_*\Omega_{X'}\simeq g^*f_!p_*p^!f^!1_{\mathrm{Spec}(k)}\xrightarrow{\simeq}g^*(f\circ p)_!(f\circ p)^!1_{\mathrm{Spec}(k)}\longrightarrow g^*1_{\mathrm{Spec}(k)},$$

where the last map is induced by the adjunction $(f\circ p)_!(f\circ p)^!\to\mathrm{id}$.

The map $\epsilon_{X\times Y} \circ t_p \circ \mathrm{id}$ corresponds to the map

$$g^* f_! p_* \Omega_{X'} \simeq g^* f_! p_* p^! f^! 1_{\mathrm{Spec}(k)} \xrightarrow{p_* p^! \to \mathrm{id}} g^* f_! f^! 1_{\mathrm{Spec}(k)} \xrightarrow{f_! f^! \to \mathrm{id}} g^* 1_{\mathrm{Spec}(k)},$$

so the lemma follows from the fact that the adjunction maps are associative in the sense that the following diagram of functors and adjunction maps

$$\begin{array}{ccc} f_! p_* p^! f^! & \xrightarrow{p_* p^! \to \mathrm{id}} & f_! f^! \\ \downarrow \simeq & & \downarrow \\ (f\circ p)_! (f \circ p)^! & \longrightarrow & \mathrm{id} \end{array}$$

commutes. □

Lemma 7.6. *The diagram*

$$\begin{array}{ccccc} i^! \Omega_X \boxtimes \Omega_Y & \xrightarrow{\alpha} & \tilde{i}^! (\Omega_X \boxtimes \Omega_Y) & \xrightarrow{\tilde{i}^! \epsilon_{X\times Y}} & \tilde{i}^! \Omega_{X\times Y} \\ \downarrow \simeq & & & & \downarrow \simeq \\ \Omega_Z \boxtimes \Omega_Y & & \xrightarrow{\epsilon_{Z\times Y}} & & \Omega_{Z\times Y} \end{array}$$

commutes, where α denotes the map occurring in $B^Z_X(\Omega_X)$ and the vertical arrows are induced by the natural isomorphism $i^!\Omega_X \simeq \Omega_Z$ and $\tilde{i}^! \Omega_{X\times Y} \simeq \Omega_{Z\times Y}$.

Proof. Let $f_Z : Z \to \mathrm{Spec}(k)$ denote the structure morphism of Z.

The composition

$$i^! \Omega_X \boxtimes \Omega_Y \xrightarrow{\alpha} \tilde{i}^! (\Omega_X \boxtimes \Omega_Y) \xrightarrow{\tilde{i}^! \epsilon_{X\times Y}} \tilde{i}^! \Omega_{X\times Y} \tag{7.6.1}$$

is adjoint to the map

$$\tilde{i}_* (i^! \Omega_X \boxtimes \Omega_Y) \simeq (i_* i^! \Omega_X) \boxtimes \Omega_Y \xrightarrow{i_* i^! \to \mathrm{id}} \Omega_X \boxtimes \Omega_Y \xrightarrow{\epsilon_{X\times Y}} \Omega_{X\times Y},$$

from which it follows that under the isomorphism of 5.3

$$\mathrm{Hom}_{Z\times Y}(i^! \Omega_X \boxtimes \Omega_Y, \Omega_{Z\times Y}) \simeq \mathrm{Hom}_Y (g^* f_{Z!} i^! \Omega_X, g^* 1_{\mathrm{Spec}(k)})$$

the composition (7.6.1) corresponds to the map $g^* f_{Z!} i^! \Omega_X \to g^* 1_{\mathrm{Spec}(k)}$ induced by the composition

$$f_{Z!} i^! \Omega_X \simeq f_! i_* i^! \Omega_X \xrightarrow{i_* i^! \to \mathrm{id}} f_! f^! 1_{\mathrm{Spec}(k)} \xrightarrow{f_! f^! \to \mathrm{id}} 1_{\mathrm{Spec}(k)}.$$

By associativity of the adjunction maps as in the proof of 7.5 this composition is equal to composition

$$f_{Z!} i^! \Omega_X \simeq f_{Z!} \Omega_Z \simeq f_{Z!} f_Z^! 1_{\mathrm{Spec}(k)} \xrightarrow{f_{Z!} f_Z^! \to \mathrm{id}} 1_{\mathrm{Spec}(k)}.$$

which implies the lemma. □

Lemma 7.7. *Let $p : X' \to X$ be a proper morphism, let $i' : Z' \hookrightarrow X'$ be the preimage of Z, and let $F' \in \mathscr{M}_B(X')$ be an object. Then $B^{Z'}_{X'}(F')$ implies $B^Z_X(p_* F')$.*

Proof. Let $\tilde{p} : X' \times Y \to X \times Y$ (resp. $p_Z : Z' \to Z$, $\tilde{p}_Z : Z' \times Y \to Z \times Y$) denote the morphism induced by p so we have isomorphisms

(7.7.1) $$i^! p_* \simeq p_{Z*} i'^!, \quad \tilde{i}^! \tilde{p}_* \simeq \tilde{p}_{Z*} \tilde{i}'^!,$$

and let

$$\alpha' : i'^! F' \boxtimes \Omega_Y \to \tilde{i}'^! (F' \boxtimes \Omega_Y) \ \ (\text{resp. } \alpha : (i^! p_* F') \boxtimes \Omega_Y \to \tilde{i}^! ((p_* F') \boxtimes \Omega_Y))$$

denote the map in $B^{Z'}_{X'}(F')$ (resp. $B^Z_X(p_* F)$). It then suffices to show that the diagram

$$\begin{array}{ccc} \tilde{p}_{Z*}(i'^! F' \boxtimes \Omega_Y) & \xrightarrow{\tilde{p}_{Z*}\alpha'} & \tilde{p}_{Z*}\tilde{i}'^!(F' \boxtimes \Omega_Y) \\ \uparrow{\scriptstyle (7.4.1)} & & \downarrow{\scriptstyle (7.7.1)} \\ (p_{Z*} i'^! F') \boxtimes \Omega_Y & & \tilde{i}^! \tilde{p}_*(F' \boxtimes \Omega_Y) \\ \uparrow{\scriptstyle (7.7.1)} & & \uparrow{\scriptstyle (7.4.1)} \\ (i^! p_* F') \boxtimes \Omega_Y & \xrightarrow{\alpha} & \tilde{i}^!((p_* F') \boxtimes \Omega_Y) \end{array}$$

commutes, since all the vertical morphisms are isomorphisms. The adjoint of the composition

$$(p_{Z*} i'^! F') \boxtimes \Omega_Y \xrightarrow{(7.4.1)} \tilde{p}_{Z*}(i'^! F' \boxtimes \Omega_Y) \xrightarrow{\tilde{p}_{Z*}\alpha'} \tilde{p}_{Z*}\tilde{i}'^!(F' \boxtimes \Omega_Y) \xrightarrow{(7.7.1)} \tilde{i}^! \tilde{p}_*(F' \boxtimes \Omega_Y)$$

is the composition

$$\begin{array}{l} \tilde{i}_*((p_{Z*} i'^! F') \boxtimes \Omega_Y) \xrightarrow{(7.4.1)} (i_* p_{Z*} i'^! F') \boxtimes \Omega_Y \xrightarrow{i_* p_{Z*} \simeq p_* i'_*} (p_* i'_* i'^! F') \boxtimes \Omega_Y \\ \qquad \xrightarrow{i'_* i'^! \to \mathrm{id}} (p_* F') \boxtimes \Omega_Y \xrightarrow{(7.4.1)} \tilde{p}_*(F' \boxtimes \Omega_Y). \end{array}$$

The adjoint of the composition

$$(p_{Z*} i'^! F') \boxtimes \Omega_Y \xrightarrow{(7.7.1)^{-1}} (i^! p_* F') \boxtimes \Omega_Y \xrightarrow{\alpha} \tilde{i}^!((p_* F') \boxtimes \Omega_Y) \xrightarrow{(7.4.1)} \tilde{i}^! \tilde{p}_*(F' \boxtimes \Omega_Y),$$

on the other hand, is equal to the composition

$$\begin{array}{l} \tilde{i}_*((p_{Z*} i'^! F') \boxtimes \Omega_Y) \xrightarrow{(7.4.1)} (i_* p_{Z*} i'^! F') \boxtimes \Omega_Y \xrightarrow{p_{Z*} i'^! \simeq i^! p_*} (i_* i^! p_* F') \boxtimes \Omega_Y \\ \qquad \xrightarrow{i_* i^! \to \mathrm{id}} (p_* F') \boxtimes \Omega_Y \xrightarrow{(7.4.1)} \tilde{p}_*(F' \boxtimes \Omega_Y). \end{array}$$

It therefore suffices to show that the diagram of functors

$$\begin{array}{ccc} i_* p_{Z*} i'^! & \xrightarrow{i_* p_{Z*} \simeq p_* i'_*} & p_* i'_* i'^! \\ \downarrow{\scriptstyle p_{Z*} i'^! \simeq i^! p_*} & & \downarrow{\scriptstyle i'_* i'^! \to \mathrm{id}} \\ i_* i^! p_* & \xrightarrow{i_* i^! \to \mathrm{id}} & p_* \end{array}$$

commutes. Since i, p, and p_Z are proper this diagram is identified using the isomorphism α in 2.9 (2) with the diagram

$$\begin{array}{ccc} i_! p_{Z!} i'^! & \xrightarrow{i_! p_{Z!} \simeq i'_! p_!} & p_! i'_! i'^! \\ {\scriptstyle p_{Z!} i'^! \simeq i^! p_!} \downarrow & & \downarrow {\scriptstyle i'_! i'^! \to \mathrm{id}} \\ i_! i^! p_! & \xrightarrow{i_! i^! \to \mathrm{id}} & p_!, \end{array}$$

where the isomorphism $p_{Z!} i'^! \simeq i^! p_!$ is induced by adjunction by our assumptions in 2.9 (4), and this diagram commutes as all the maps are induced by the natural adjunctions. □

Lemma 7.8. *Let*

$$F_1 \to F_2 \to F_3 \to F_1[1]$$

be a distinguished triangle in $\mathscr{M}_B(X)$. If $B^Z_X(F_1)$ and $B^Z_X(F_2)$ hold then so does $B^Z_X(F_3)$.

Proof. This follows from noting that the morphisms in the properties $B^Z_X(F_i)$ fit into a morphism of distinguished triangles

$$\begin{array}{ccccccc} i^! F_1 \boxtimes \Omega_Y & \longrightarrow & i^! F_2 \boxtimes \Omega_Y & \longrightarrow & i^! F_3 \boxtimes \Omega_Y & \longrightarrow & i^! F_1 \boxtimes \Omega_Y[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{i}^!(F_1 \boxtimes \Omega_Y) & \longrightarrow & \tilde{i}^!(F_2 \boxtimes \Omega_Y) & \longrightarrow & \tilde{i}^!(F_3 \boxtimes \Omega_Y) & \longrightarrow & \tilde{i}^!(F_1 \boxtimes \Omega_Y)[1]. \end{array}$$

□

Lemma 7.9. *If $X \in \mathscr{S}$ be smooth over k.*

(i) Property C_X holds.

(ii) If $i : Z \hookrightarrow X$ is a closed subscheme and C_Z holds then $B^Z_X(\Omega_X)$ holds.

Proof. Let d be the dimension of X (a locally constant function on X). Then by 2.9 (3) applied to the morphisms $X \to \mathrm{Spec}(k)$ and $X \times Y \to Y$, we have $\Omega_X \simeq 1_X(d)[2d]$ and $\Omega_{X\times Y} \simeq 1_X \boxtimes \Omega_Y(d)[2d]$, and from its definition the map $\epsilon_{X\times Y}$ is equal to the isomorphism obtained from these identifications implying (i).

For (ii) observe that since $\mathrm{pr}_2 : X \times Y \to Y$ is smooth we have

$$\tilde{i}^!(\Omega_X \boxtimes \Omega_Y) \simeq \tilde{i}^! \mathrm{pr}_2^! \Omega_Y \simeq \Omega_{Z\times Y},$$

and it follows from 7.6 that the map

$$\Omega_Z \boxtimes \Omega_Y \simeq (i^! \Omega_X) \boxtimes \Omega_Y \to \tilde{i}^!(\Omega_X \boxtimes \Omega) \simeq \Omega_{Z\times Y}$$

arising in $B^Z_X(\Omega_X)$ is equal to $\epsilon_{Z\times Y}$. This implies (ii). □

Lemma 7.10. *Let $X \in \mathscr{S}$ be a scheme, and suppose that for every nowhere dense closed subscheme $i : Z \hookrightarrow X$ the properties C_Z and $B^Z_X(\Omega_X)$ hold. Then property C_X also holds.*

Proof. Let $j : U \hookrightarrow X$ be an everywhere dense open subscheme with U_{red} smooth over k, and let $i : Z \hookrightarrow X$ be the complementary closed subscheme (with the reduced structure). From the distinguished triangle

$$i_*\Omega_Z \to \Omega_X \to j_*\Omega_U \to i_*\Omega_Z[1]$$

and its variant for $X \times Y$ we get a commutative diagram

$$\begin{array}{cccccc} & i_*\Omega_Z \boxtimes \Omega_Y & \longrightarrow & \Omega_X \boxtimes \Omega_Y & \longrightarrow & j_*\Omega_U \boxtimes \Omega_Y \\ & \downarrow a & & \| & & \downarrow b \\ c\Big(& \tilde{i}_*(\tilde{i}^!(\Omega_X \boxtimes \Omega_Y)) & \longrightarrow & \Omega_X \boxtimes \Omega_Y & \longrightarrow & \tilde{j}_*(\Omega_U \boxtimes \Omega_Y) \\ & \downarrow d & & \downarrow f & & \downarrow e \\ & \tilde{i}_*\Omega_{Z\times Y} & \longrightarrow & \Omega_{X\times Y} & \longrightarrow & \tilde{j}_*\Omega_{U\times Y} \end{array}$$

By property $B_X^Z(\Omega_X)$ the map labelled a is an isomorphism, and by 7.9 (ii) and 7.3 the map e is also an isomorphism. Now by property C_Z the map c is an isomorphism whence the map d is also an isomorphism. From this it follows that the map f is an isomorphism as well. □

Lemma 7.11. *Let $i : Z \hookrightarrow X$ be a closed imbedding. The properties C_Z and C_X imply $B_X^Z(\Omega_X)$.*

Proof. Indeed we have

$$(i^!\Omega_X) \boxtimes \Omega_Y \simeq \Omega_Z \boxtimes \Omega_Y \simeq \Omega_{Z\times Y},$$

where the second isomorphism is by property C_Z. Similarly we have

$$\tilde{i}^!(\Omega_X \boxtimes \Omega_Y) \simeq \tilde{i}^!(\Omega_{X\times Y}) \simeq \Omega_{Z\times Y},$$

where the first isomorphism is by property C_X. Under these identifications the map occurring in property $B_X^Z(\Omega_X)$ is identified with the identity map on $\Omega_{Z\times Y}$. □

7.12. Let $p : P \to X$ be a proper morphism, and fix a distinguished triangle in $\mathscr{M}_B(X)$

$$1_X \to p_*1_P \to \mathcal{F} \to 1_X[1].$$

Assume there exists a closed imbedding $i : Z \hookrightarrow X$ with everywhere dense complement $j : U \to X$ such that the restriction $p_U : P_U \to U$ of p to U is finite radicial and surjective. Let $p_Z : P_Z \to Z$ be the restriction of p to Z. Let $\mathcal{F}_Z$ denote a cone of the morphism $1_Z \to p_{Z*}1_{P_Z}$ so we can find a morphism of distinguished triangles in $\mathscr{M}_B(X)$

$$\begin{array}{ccccccc} 1_X & \longrightarrow & p_*1_P & \longrightarrow & \mathcal{F} & \longrightarrow & 1_X[1] \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow a \\ i_*1_Z & \longrightarrow & i_*p_{Z*}1_{P_Z} & \longrightarrow & i_*\mathcal{F}_Z & \longrightarrow & i_*1_Z[1], \end{array}$$

where the maps labelled a and b are the adjunction maps.

Lemma 7.13. *The map $c : \mathcal{F} \to i_*\mathcal{F}_Z$ is an isomorphism.*

Proof. Considering the distinguished triangle

$$j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to j_!j^*\mathcal{F}[1]$$

it suffices to show that $j^*\mathcal{F} = 0$ and that the map $i^*\mathcal{F} \to \mathcal{F}_Z$ is an isomorphism. The first statement follows from the fact that the morphism $1_X \to p_*1_P$ is an isomorphism over U, since p is finite radicial and surjective over U (see 2.13), and the second statement follows from the fact that the base change map $i^*p_*1_P \to p_{Z*}1_{P_Z}$ is an isomorphism. □

Remark 7.14. Lemma 7.13 holds over a general base B, with the same proof.

Theorem 7.15. *Let $i : Z \hookrightarrow X$ be a closed imbedding in $\mathscr{S}$. Then properties $A^Z_X(1_X)$, $B^Z_X(1_X)$, $A^Z_X(\Omega_X)$, $B^Z_X(\Omega_X)$, and C_X hold.*

Proof. By induction on the dimension d of X we may assume that we have $X \in \mathscr{S}$ of dimension d and that the theorem is true for every object of $\mathscr{S}$ of dimension $< d$. To verify the theorem for X it then suffices by 7.3 and 7.10 to show that for every $i : Z \hookrightarrow X$ the properties $B^Z_X(1_X)$ and $B^Z_X(\Omega_X)$ hold.

By [8, 7.3] we can find a proper morphism $\tilde{p} : \widetilde{P} \to X$ with $\widetilde{P}$ smooth, quasi-projective, and equipped with an action of finite group G over X, such that if $p : P \to X$ is the coarse moduli space of the stack $[\widetilde{P}/G]$ then p is generically on X finite surjective and radicial. By 6.7 this implies that $\Omega_P \simeq 1_P(d)[2d]$, where d is the dimension of $\widetilde{P}$ (a locally constant function). This also implies that $\Omega_{P\times Y} \simeq 1_P(d)[2d] \boxtimes \Omega_Y$ since by 6.5 we have an isomorphism

$$\Omega_{P\times Y} \simeq (\tilde{p}_*\Omega_{\widetilde{P}\times Y})^G \simeq (\tilde{p}_*(1_{\widetilde{P}}(d)[2d] \boxtimes \Omega_Y))^G \simeq (\tilde{p}_*\tilde{p}^*(1_P(d)[2d] \boxtimes \Omega_Y))^G \simeq 1_P(d)[2d] \boxtimes \Omega_Y,$$

where the last isomorphism is by [4, 3.3.35] and the second isomorphism uses 7.9.

This also gives the following description of the map

$$\epsilon_{P\times Y} : \Omega_P \boxtimes \Omega_Y \to \Omega_{P\times Y}.$$

Let $\pi : \widetilde{P} \to P$ be the projection, and let $\tilde{\pi} : \widetilde{P} \times Y \to P \times Y$ be the base change to Y. Then by 7.5 the diagram

$$\begin{array}{ccc} \tilde{\pi}_*(\Omega_{\widetilde{P}} \boxtimes \Omega_Y) & \xrightarrow{\epsilon_{\widetilde{P}\times Y}} & \tilde{\pi}_*\Omega_{\widetilde{P}\times Y} \\ \downarrow & & \downarrow \\ \Omega_P \boxtimes \Omega_Y & \xrightarrow{\epsilon_{P\times Y}} & \Omega_{P\times Y} \end{array}$$

commutes, where the vertical maps are as in 7.5, and by 6.5 this identifies the map $\epsilon_{P\times Y}$ with the map on G-invariants

$$(\tilde{\pi}_*(\Omega_{\widetilde{P}} \boxtimes \Omega_Y))^G \xrightarrow{\epsilon_{\widetilde{P}\times Y}} (\tilde{\pi}_*\Omega_{\widetilde{P}\times Y})^G.$$

In particular since $\epsilon_{\widetilde{P}\times Y}$ is an isomorphism by 7.9 (i) property C_P holds.

From this we also get that for every closed $t : T \hookrightarrow P$ properties $B^T_P(\Omega_P)$ and $B^T_P(1_P)$ hold. Indeed since P is obtained as the coarse moduli space of a smooth Deligne-Mumford stack, for any connected component P_i of P the intersection $T \cap P_i$ is either all of P_i or of dimension $< d$. We can therefore apply 7.11 to get $B^T_P(\Omega_P)$ and since $1_P \simeq \Omega_P(-d)[-2d]$ (by 6.7) this also implies $B^T_P(1_P)$.

Let Q be a cone of the morphism $1_X \to p_*1_P$. By 7.7 we then have properties $B^Z_X(p_*1_P)$ and $B^Z_X(p_*\Omega_P)$. Lemma 7.8 then implies that to verify property $B^Z_X(1_X)$ it suffices to verify the property $B^Z_X(Q)$. Dualizing we also have a distinguished triangle

$$D_X(Q) \to p_*\Omega_P \to \Omega_X \to D_X(Q)[1],$$

and to verify property $B^Z_X(\Omega_X)$ it suffices to verify property $B^Z_X(D_X(Q))$.

Let $\alpha : T \hookrightarrow X$ be a nowhere dense closed subscheme such that the restriction of p to the complement of T is finite and radicial. Let Z_T (resp. P_T, Z'_T) denote $Z \cap T$ (resp. $T \times_X P$,

$Z_T \times_X P)$. Then by 7.13 we have $Q = \alpha_* Q_T$ for some $Q_T \in \mathscr{M}_B(T)$ fitting into a distinguished triangle

$$1_T \to p_{T*}1_{P_T} \to Q_T \to 1_T[1].$$

Dualizing we also get a distinguished triangle

$$D_T(Q_T) \to p_{T*}\Omega_{P_T} \to \Omega_T \to D_T(Q_T)[1].$$

By the induction hypothesis and applying 7.8 we conclude that $B_T^{Z_T}(Q_T)$ and $B_T^{Z_T}(D_T(Q_T))$ hold, and therefore by 7.7 properties $B_X^Z(Q)$ and $B_X^Z(D_X(Q))$ also hold. $\square$

8. Application: local terms for actions given by localized Chern classes

Let k be an algebraically closed field, and let $\mathscr{M}_B$ denote the motivic category of Beilinson motives over k.

8.1. For a prime ℓ invertible in k there is constructed in [5, 7.2.24] an étale realization functor

$$T_\ell : \mathscr{M}_B \to DM_{c,\ell}$$

where for $X \in \mathscr{S}$ the fiber $DM_{c,\ell}(X)$ is isomorphic to the idempotent completion $D_c^b(X, \mathbb{Q}_\ell)^\sharp$ of the triangulated category $D_c^b(X, \mathbb{Q}_\ell)$. Here the idempotent completion is defined as in [2]. This realization functor is compatible with the six operations and Chern classes. Note also that by [2, 1.4] the functor

$$D_c^b(X, \mathbb{Q}_\ell) \to D_c^b(X, \mathbb{Q}_\ell)^\sharp$$

is fully faithful.

8.2. Let $f : X \to Y$ be a morphism of quasi-projective schemes in $\mathscr{S}$. Let $A^{n,m}_{\mathrm{et},\ell}(X \to Y)$ denote $H^{2n}(X, f^!\mathbb{Q}_\ell(m))$, and let $H_{i,BM,\ell}(X)$ denote the i-th ℓ-adic Borel-Moore homology of X. These groups were considered in [18, 3.1 and 2.2] (with different notation). In [18, 4.2 and 2.10] there is constructed maps, which are special cases of the constructions in 4.5 and 4.6,

$$\tau^X_{Y,\ell} : K(f\text{-perfect complexes on } X) \to \oplus_i A^{i,i}_{\mathrm{et},\ell}(X \to Y), \quad \mathrm{cl}_\ell : A_i(X)_\mathbb{Q} \to H_{i,BM,\ell}(X),$$

where $A_i(X)_\mathbb{Q}$ denotes the Chow groups of X tensor $\mathbb{Q}$. By 4.5 and 4.6 we also have maps

$$\tau^X_Y : K(f\text{-perfect complexes on } X) \to \oplus_i A^{i,i}_{\mathscr{M}_B}(X \to Y), \quad \mathrm{cl}_\ell : A_i(X)_\mathbb{Q} \to H^{\mathscr{M}_B}_{i,BM}(X),$$

The realization functor also defines maps, which we somewhat abusively also denote by T_ℓ,

$$T_\ell : A^{i,i}_{\mathscr{M}_B}(X \to Y) \to A^{i,i}_{\mathrm{et},\ell}(X \to Y), \quad T_\ell : H^{\mathscr{M}_B}_{i,BM}(X) \to H_{i,BM,\ell}(X).$$

Lemma 8.3. *The diagrams*

(8.3.1)
$$\begin{array}{ccc} & K(f\text{-}perfect\ complexes\ on\ X) & \\ \swarrow^{\tau^X_Y} & & \searrow^{\tau^X_{Y,\ell}} \\ \oplus_i A^{i,i}_{\mathscr{M}_B}(X \to Y) & \xrightarrow{\quad T_\ell \quad} & \oplus_i A^{i,i}_{\mathrm{et},\ell}(X \to Y) \end{array}$$

and

(8.3.2)
$$\begin{array}{ccc} & A_i(X)_{\mathbb{Q}} & \\ {}^{\mathrm{cl}}\swarrow & & \searrow^{\mathrm{cl}_\ell} \\ H^{\mathscr{M}_B}_{i,BM}(X) & \xrightarrow{T_\ell} & H_{i,BM,\ell}(X) \end{array}$$

commute.

Proof. The commutativity of (8.3.2) is a special case of the commutativity of (8.3.1) taking $Y = \mathrm{Spec}(k)$. To see the commutativity of (8.3.1) observe that for f a closed imbedding the composition of τ^X_Y with the realization functor T_ℓ defines a theory of local Chern classes taking values in $\oplus_i A^{i,i}_{\mathrm{et},\ell}(X \to Y)$, which is compatible with the standard étale Chern classes (since T_ℓ is compatible with Chern classes). By the uniqueness part of 4.2 this implies that it agrees with the local Chern classes defined in [18, 3.11]. From this it follows that (8.3.1) commutes. □

Remark 8.4. Lemma 8.3 is stated and proven for the ℓ-adic realization functor, but can be generalized to a statement for morphisms of motivic categories as in 5.9.

Corollary 8.5. *Let $c : C \to X \times X$ be a correspondence with C and X quasi-projective schemes, and let $u : c_1^* 1_X \to c_2^! 1_X$ be an action in $\mathscr{M}_B(C)$. Then there exists an algebraic cycle $\Sigma \in A_0(\mathrm{Fix}(c))_{\mathbb{Q}}$ such that for any prime ℓ invertible in k we have $\mathrm{Tr}(u_\ell) = \mathrm{cl}_\ell(\Sigma)$, where $u_\ell : c_1^* \mathbb{Q}_\ell \to c_2^! \mathbb{Q}_\ell$ is the ℓ-adic realization of u.*

Proof. Let $\Sigma \in A_0(\mathrm{Fix}(c))_{\mathbb{Q}}$ be the class corresponding to $\mathrm{Tr}^{\mathscr{M}_B}(u)$ under the isomorphism $H^{\mathscr{M}_B}_{0,BM}(\mathrm{Fix}(c)) \simeq A_0(\mathrm{Fix}(c))_{\mathbb{Q}}$ given by 6.2. By 7.15 the assumption 5.2 is satisfied so by 5.9 we have $\mathrm{Tr}(u_\ell) = T_\ell(\mathrm{cl}(\Sigma))$, which by the commutativity of (8.3.2) is equal to $\mathrm{cl}_\ell(\Sigma)$. □

8.6. Let $c : C \to X \times X$ be a correspondence with C and X quasi-projective schemes, and let E be a c_2-perfect complex on C. We then get an action $u_\ell : c_1^* \mathbb{Q}_\ell \to c_2^! \mathbb{Q}_\ell$ from the class $\tau^C_{X,\ell}(E) \in H^0(C, c_2^! \mathbb{Q}_\ell)$ defined in [18, 4.2].

Theorem 8.7. *There exists a cycle $\Sigma \in A_0(\mathrm{Fix}(c))_{\mathbb{Q}}$, independent of ℓ, such that $\mathrm{Tr}_c(u_\ell) \in H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)})$ is equal to $\mathrm{cl}_\ell(\Sigma)$.*

Proof. Let $u : c_1^* 1_X \to c_2^! 1_X$ be the morphism in $\mathscr{M}_B(C)$ defines by $\tau^C_X(E)^0$. Then by the commutativity of (8.3.2) the action u_ℓ is the ℓ-adic realization of u. From this and 8.5 the result follows. □

9. Application: quasi-finite morphisms and correspondences

In this section B denotes a regular excellent scheme of dimension ≤ 2, and $\mathscr{S}$ is the category of finite type separated B-schemes.

9.1. Let ℓ be a prime invertible on B, and let $f : Y \to X$ be a quasi-finite morphism between quasi-projective B-schemes. Let $u_\ell \in H^0(Y, f^! \mathbb{Q}_\ell)$ be a section. We say that u_ℓ is *motivic* if there exists a morphism $u : 1_Y \to f^! 1_X$ in $\mathscr{M}_B(Y)$ such that u_ℓ is the ℓ-adic realization of u.

9.2. The condition that u_ℓ be motivic has the following more concrete characterization. Since f is quasi-finite, $f_!\mathbb{Q}_\ell$ is a sheaf. For any dense open subscheme $j : U \hookrightarrow X$ the adjunction map $\mathbb{Q}_{\ell,X} \to R^0 j_*\mathbb{Q}_{\ell,U}$ is injective so the restriction map

$$\mathrm{Hom}_X(f_!\mathbb{Q}_\ell, \mathbb{Q}_{\ell,X}) \to \mathrm{Hom}_X(f_!\mathbb{Q}_\ell, R^0 j_*\mathbb{Q}_{\ell,U}) \simeq \mathrm{Hom}_U(f_!\mathbb{Q}_\ell|_U, \mathbb{Q}_{\ell,U})$$

is injective. By adjunction it follows that the restriction map

$$\mathrm{Hom}_Y(\mathbb{Q}_{\ell,Y}, f^!\mathbb{Q}_{\ell,X}) \to \mathrm{Hom}_{f^{-1}(U)}(\mathbb{Q}_{\ell,f^{-1}(U)}, f^!\mathbb{Q}_{\ell,X}|_{f^{-1}(U)})$$

is injective, so the map u_ℓ is determined by its restriction to $f^{-1}(U)$. In particular, let $\{Y_i\}_{i\in I}$ be the irreducible components of Y which dominate an irreducible component of X via f, and choose a dense open subscheme $U \subset X$ such that U_{red} is regular and

$$f^{-1}(U) = \coprod_{i\in I} V_i,$$

where $V_i \subset Y_i$ is a dense open and $V_{i,\mathrm{red}}$ is regular of the same dimension of its image in U. We then have $f^!\mathbb{Q}_{\ell,U} \simeq \mathbb{Q}_{\ell,f^{-1}(U)}$ and a canonical isomorphism

$$H^0(f^{-1}(U), f^!\mathbb{Q}_{\ell,U}) \simeq \mathbb{Q}_\ell^I.$$

From this we obtain an inclusion

$$H^0(Y, f^!\mathbb{Q}_{\ell,X}) \hookrightarrow \mathbb{Q}_\ell^I. \tag{9.2.1}$$

It follows immediately from the construction that this is independent of the choice of U. The image of u_ℓ in $\mathbb{Q}_\ell^I$ will be called the *weight vector of* u_ℓ, and will be denoted $w(u_\ell)$.

Remark 9.3. In the above we do not assume that f is necessarily dominant. The argument shows that a morphism u_ℓ is determined by its restriction to those components of Y which dominate X.

Theorem 9.4. *(i) The section u_ℓ is motivic if and only if the weight vector $w(u_\ell)$ lies in $\mathbb{Q}^I \subset \mathbb{Q}_\ell^I$.*

(ii) If u_ℓ is the ℓ-adic realization of $u : 1_Y \to f^!1_X$, then for any other prime ℓ' invertible in k the ℓ'-adic realization $u_{\ell'}$ of u has $w(u_{\ell'}) = w(u_\ell)$ in $\mathbb{Q}^I$.

Remark 9.5. If the weight vector $w(u_\ell)$ lies in $\mathbb{Q}^I$ we say that u_ℓ has *rational weight vector.*

The proof of 9.4 occupies the following (9.6)–(9.19).

9.6. Fix a prime ℓ and an element $u_\ell \in H^0(Y, f^!\mathbb{Q}_\ell)$ with weight vector $w \in \mathbb{Q}^I$. We show that u_ℓ is motivic as follows.

9.7. By [9, 5.15] (in the case when B is the spectrum of a field one can also use [8, 7.3]) we can find a proper morphism $\tilde{p} : \widetilde{P} \to X$ with $\widetilde{P}$ regular and equipped with an action of finite group G over X, such that if $p : P \to X$ is the coarse moduli space of the stack $[\widetilde{P}/G]$ then p is generically on X finite surjective and radicial. Next choose a proper surjective generically finite morphism $\kappa : F \to Y \times_{f,X} P$, with F regular, which fits into a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{g} & P \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X. \end{array}$$

Let d_F (resp. d_P) be the dimension of F (resp. P), a locally constant function on F (resp. P). Note that we can also view d_P as a locally constant function on F via g. By 6.7 we have $\Omega_P \simeq \mathbb{Q}_\ell(d-\delta)[2(d-\delta)]$, and therefore $g^!\mathbb{Q}_\ell \simeq \Omega_F(\delta-d)[2(\delta-d)]$, which by 6.6 applied to F is isomorphic to $\mathbb{Q}_\ell$. From this it follows that if $\pi_0(F)$ denotes the set of connected components of F then $H^0(F, g^!\mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^{\pi_0(F)}$. Thus giving a map $\mathbb{Q}_\ell \to g^!\mathbb{Q}_\ell$ is equivalent to specifying a function $\pi_0(F) \to \mathbb{Q}_\ell$.

9.8. Under these identifications the map $\mathbb{Q}_\ell \to g^!\mathbb{Q}_\ell$ corresponding to the function sending all elements of $\pi_0(F)$ to 1 restricts on a connected component F_i of F to the map

$$[F_i] : \mathbb{Q}_\ell \to g^!\mathbb{Q}_\ell \simeq \Omega_{F_i}(\delta-d)[2(\delta-d)]$$

given by the fundamental class of F_i in $H^0(F_i, \Omega_{F_i}(\delta-d)[2(\delta-d)])$ defined in [20, Exposé XVI] (see also [17, 3.1] in the case when B is the spectrum of an algebraically closed field).

9.9. For an irreducible component Y_i let N_i be the number of irreducible components of F which dominate Y_i, and let $\nu_\ell : \mathbb{Q}_\ell \to g^!\mathbb{Q}_\ell$ be the map corresponding to the function assigning to a connected component F_j of F dominating Y_i the number

$$w(u)_i/(N_i \cdot \deg(F_j/P)),$$

where $\deg(F_j/P)$ denotes the degree of F_j over its image in P, and taking value 0 on connected components of F not dominating an irreducible component of Y.

9.10. The description of ν_ℓ in terms of maps defined by algebraic cycles implies that the map $\nu_\ell : \mathbb{Q}_{\ell,F} \to g^!\mathbb{Q}_{\ell,P}$ is motivic. Indeed observe that the étale realization functor is compatible with the purity isomorphisms [4, 14.4.1] (this follows from [7, 4.3] and the compatibility with Chern classes), and therefore for each connected component F_i of F the map

$$[F_i] : \mathbb{Q}_{\ell,F_i} \to g_i^!\mathbb{Q}_{\ell,P}$$

defined by the fundamental class of F_i is the realization of the corresponding map in $\mathscr{M}_B(F_i)$. Since ν_ℓ is obtained by summing these maps multiplied by rational numbers it follows that there exists a morphism $\nu : 1_F \to g^!1_P$ in $\mathscr{M}_B(F)$ whose realization is equal to ν_ℓ.

Note that for u_ℓ we have no such description in terms of cycles which is why the proof of 9.4 is more complicated.

Lemma 9.11. *The diagram*

(9.11.1)
$$\begin{array}{ccc} \mathbb{Q}_{\ell,Y} & \longrightarrow & q_*\mathbb{Q}_{\ell,F} \\ \downarrow{\scriptstyle u_\ell} & & \downarrow{\scriptstyle q_*\nu_\ell} \\ f^!\mathbb{Q}_{\ell,X} & \longrightarrow & f^!p_*\mathbb{Q}_{\ell,P} \end{array}$$

commutes.

Proof. A morphism $\mathbb{Q}_{\ell,Y} \to f^!p_*\mathbb{Q}_{\ell,P}$ is equivalent by adjunction to a morphism $f_!\mathbb{Q}_{\ell,Y} \to p_*\mathbb{Q}_{\ell,P}$. Since f is quasi-finite $f_!\mathbb{Q}_{\ell,Y}$ is a sheaf and therefore such a morphism is in turn equivalent to a morphism of sheaves $f_!\mathbb{Q}_{\ell,Y} \to R^0p_*\mathbb{Q}_{\ell,P}$. From this it follows it follows that a morphism $\mathbb{Q}_{\ell,Y} \to f^!p_*\mathbb{Q}_{\ell,P}$ is determined by its restriction to the inverse of any dense open subset of X (using an argument as in 9.2).

To prove the lemma it therefore suffices, by shrinking on X, to consider the case when X_{red} is regular, $p : P \to X$ is finite surjective and radicial, g is finite, and Y_{red} regular. Restricting to a connected component of Y we may further assume that Y is connected. In this case $f^! p_* \mathbb{Q}_{\ell,P}$ is isomorphic to $\mathbb{Q}_{\ell,Y}$ and the map

$$\mathbb{Q}_{\ell,Y} \xrightarrow{u_\ell} f^! \mathbb{Q}_{\ell,X} \longrightarrow f^! p_* \mathbb{Q}_{\ell,P} \simeq \mathbb{Q}_{\ell,Y}$$

is given by multiplication by the weight vector $w(u)$ (which after our reductions is simply an element of $\mathbb{Q}_\ell$).

On the other hand, for a connected component F_i of F let $q_i : F_i \to Y$ be the restriction of q. Then by compatibility of the cycle class map with proper pushforward [17, 6.1] (the statement there is under the assumption that B is the spectrum of an algebraically closed field but the proof works in general) the composite map

$$\mathbb{Q}_{\ell,Y} \longrightarrow q_{i*} \mathbb{Q}_{\ell,F_i} \xrightarrow{[F_i]} q_{i*} g^! \mathbb{Q}_{\ell,P} \longrightarrow f^! p_* \mathbb{Q}_\ell \simeq \mathbb{Q}_{\ell,Y}$$

is equal to multiplication by the degree of F_i over P, where $[F_i]$ is defined as in 9.8. Therefore the composite map

$$\mathbb{Q}_{\ell,Y} \longrightarrow q_* \mathbb{Q}_{\ell,F} \xrightarrow{q_* \nu_\ell} f^! p_* \mathbb{Q}_\ell \simeq \mathbb{Q}_{\ell,Y}$$

is equal to multiplication by

$$\sum_{F_j} \frac{w(u)}{(N \cdot \deg(F_j/P))} \cdot \deg(F_j/P) = w(u)$$

proving the lemma. □

9.12. Fix a distinguished triangle in $\mathscr{M}_B(X)$

$$1_X \to p_* 1_P \to \mathcal{F} \to 1_X[1],$$

and let $\mathcal{F}_\ell$ denote the ℓ-adic realization of $\mathcal{F}$, so we have a distinguished triangle in $D^b_c(X, \mathbb{Q}_\ell)$

$$\mathbb{Q}_{\ell,X} \to p_* \mathbb{Q}_{\ell,P} \to \mathcal{F}_\ell \to \mathbb{Q}_{\ell,X}[1].$$

Let $i : Z \hookrightarrow X$ be a closed imbedding with everywhere dense complement $j : U \to X$ such that the restriction $p_U : P_U \to U$ of p to U is finite radicial and surjective. Let $p_Z : P_Z \to Z$ be the restriction of p to Z. Let $\mathcal{F}_Z$ denote a cone of the morphism $1_Z \to p_{Z*} 1_{P_Z}$ so we can find a morphism of distinguished triangles in $\mathscr{M}_B(X)$

$$\begin{array}{ccccccc}
1_X & \longrightarrow & p_* 1_P & \longrightarrow & \mathcal{F} & \longrightarrow & 1_X[1] \\
\downarrow a & & \downarrow b & & \downarrow c & & \downarrow a \\
i_* 1_Z & \longrightarrow & i_* p_{Z*} 1_{P_Z} & \longrightarrow & i_* \mathcal{F}_Z & \longrightarrow & i_* 1_Z[1],
\end{array}$$

where the maps labelled a and b are the adjunction maps. By 7.13 (and 7.14) the map $c : \mathcal{F} \to i_* \mathcal{F}_Z$ is an isomorphism.

Notation 9.13. If W is a quasi-projective B-scheme and $F \in \mathscr{M}_B(W)$ we write $H^i_{\mathscr{M}_B}(W, F)$ for $\mathrm{Ext}^i_{\mathscr{M}_B(W)}(1_W, F)$.

Lemma 9.14. *Let $f : Y \to X$ be a quasi-finite morphism of quasi-projective B-schemes. Then the ℓ-adic realization map*

$$H^i_{\mathscr{M}_B}(Y, f^!1_X) \to H^i(Y, f^!\mathbb{Q}_\ell) \tag{9.14.1}$$

is injective for $i \leq 0$, and $H^i_{\mathscr{M}_B}(Y, f^!1_X) = 0$ for $i < 0$.

Proof. The second statement follows from the first and the fact that the functor $f^! : D^b_c(X, \mathbb{Q}_\ell) \to D^b_c(Y, \mathbb{Q}_\ell)$ takes $D^{\geq 0}_c(X, \mathbb{Q}_\ell)$ to $D^{\geq 0}_c(Y, \mathbb{Q}_\ell)$ by [1, XVIII, 3.1.7].

Consider first the case when X is the coarse moduli space of a stack of the form $[M/G]$ with M regular of some dimension d and G a finite group acting on M. In this case we have by 6.7

$$H^i_{\mathscr{M}_B}(Y, f^!1_X) \simeq H^i_{\mathscr{M}_B}(Y, \Omega_Y(\delta - d)[2(\delta - d)]),$$

and in particular by 6.2

$$H^0_{\mathscr{M}_B}(Y, f^!1_X) \simeq H^{-2(d-\delta)}_{\mathscr{M}_B}(Y, \Omega_Y(-(d - \delta))) = H^{\mathscr{M}_B}_{d-\delta,BM}(Y) \simeq A_d(Y)_{\mathbb{Q}}.$$

Since f is quasi-finite this is canonically isomorphic to the $\mathbb{Q}$-vector space with basis the irreducible components of Y of dimension d. This implies the injectivity of (9.14.1) for $i = 0$, and also shows that if $j : V \subset Y$ is the preimage of a dense open subset in X then the restriction map

$$H^0_{\mathscr{M}_B}(Y, f^!1_X) \to H^0_{\mathscr{M}_B}(V, f^!_V 1_X)$$

is injective, where $f_V : V \to X$ is the restriction of f. Let $r : Z \hookrightarrow Y$ be the complement of V and let $f_Z : Z \to X$ be the restriction of f. Choose V such that V_{red} is affine and regular of some dimension $e \leq d$. In this case we have $f^!_V 1_X \simeq 1_V(e - d)[2(e - d)]$ so

$$H^i_{\mathscr{M}_B}(V, f^!_V 1_X) \simeq H^{i+2(e-d)}_{\mathscr{M}_B}(V, 1_V(e - d)).$$

By 6.3 these groups are zero if $i < 0$. Now from the distinguished triangle

$$r_*f^!_Z 1_X \to f^!1_X \to j_*f^!_V 1_X \to r_*f^!_Z 1_X[1]$$

we get a long exact sequence

$$\cdots \to H^i_{\mathscr{M}_B}(Z, f^!_Z 1_X) \to H^i_{\mathscr{M}_B}(Y, f^!1_X) \to H^i_{\mathscr{M}_B}(V, f^!_V 1_X) \to \cdots .$$

By induction on the dimension of Y (with base case handled by the case when Y is regular) we have $H^i(Z, f^!_Z 1_X) = 0$ for $i < 0$, and as discussed above we also have $H^i_{\mathscr{M}_B}(V, f^!_V 1_X) = 0$ for $i < 0$. This therefore completes the proof in the case when X is the coarse space of a stack $[M/G]$ as above.

For the general case we proceed by induction on the dimension of X. Let $p : P \to X$ be as in 9.7, and consider the resulting distinguished triangle

$$1_X \to p_*1_P \to \mathcal{F} \to 1_X[1].$$

Applying $f^!$ we get a distinguished triangle

$$f^!1_X \to f^!p_*1_P \to f^!\mathcal{F} \to f^!1_X[1].$$

Let P_Y denote the fiber product $Y \times_X P$ so we have a cartesian square

$$\begin{array}{ccc} P_Y & \xrightarrow{g} & P \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X. \end{array}$$

By base change, we have $f^! p_* 1_P \simeq q_* g^! 1_P$ and therefore

$$H^i_{\mathscr{M}_B}(Y, f^! p_* 1_P) \simeq H^i_{\mathscr{M}_B}(P_Y, g^! 1_P).$$

By the case considered at the beginning of the proof, it follows that the ℓ-adic realization map

$$H^i_{\mathscr{M}_B}(Y, f^! p_* 1_P) \to H^i(Y, f^! p_* \mathbb{Q}_\ell)$$

is injective for $i \leq 0$. To prove the lemma it therefore suffices to show that $H^i(Y, f^! \mathcal{F}) = 0$ for $i < 0$. Let $i : Z \hookrightarrow X$, Y_Z and $\mathcal{F}_Z$ be as in 9.12 so we have

$$H^i_{\mathscr{M}_B}(Y, f^! \mathcal{F}) \simeq H^i_{\mathscr{M}_B}(Y_Z, f_Z^! \mathcal{F}_Z).$$

Now consider the distinguished triangle on Z

$$1_Z \to p_{Z*} 1_{P_Z} \to \mathcal{F}_Z \to 1_Z[1],$$

and the resulting distinguished triangle

$$f_Z^! 1_Z \to f_Z^! p_{Z*} 1_{P_Z} \to f_Z^! \mathcal{F}_Z \to f_Z^! 1_Z[1]$$

on Y_Z. By induction the lemma holds for the quasi-finite morphisms $f_Z : Y_Z \to Z$ and $g_Z : P_{Y_Z} \to P_Z$. To prove that $H^i_{\mathscr{M}_B}(Y_Z, f_Z^! \mathcal{F}_Z) = 0$ for $i < 0$ it therefore suffices to show that the map on étale cohomology

$$H^0(Y_Z, f_Z^! \mathbb{Q}_\ell) \to H^0(P_{Y_Z}, g_Z^! \mathbb{Q}_\ell) \tag{9.14.2}$$

is injective. For this, observe that since f_Z is quasi-finite it suffices to show injectivity after restricting to a dense open of Z (using the argument of 9.2). Thus it suffices to prove injectivity when Z_{red} and $Y_{Z,\text{red}}$ are regular, connected, and of the same dimension, in which case $H^0(Y_Z, f_Z^! \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$. Similarly using 9.2 the group $H^0(P_{Y_Z}, g_Z^! \mathbb{Q}_\ell)$ injects into a vector space $\mathbb{Q}_\ell^I$, where I is the set of irreducible components of P_{Y_Z} which dominate an irreducible component of P_Z. From this the result follows as there exists an irreducible component of P_{Y_Z} which dominates an irreducible component of P and also Y_Z, and the map $\mathbb{Q}_\ell \to \mathbb{Q}_\ell^I$ induced by (9.14.2) is nonzero on the factor corresponding to such a component. □

9.15. Returning to the setting of 9.7 and 9.12, fix also a distinguished triangle in $\mathscr{M}_B(Y)$

$$1_Y \to q_* 1_F \to \mathcal{G} \to 1_Y[1],$$

and let $\mathcal{G}_\ell \in D^b_c(Y, \mathbb{Q}_\ell)$ be the ℓ-adic realization of $\mathcal{G}$. Let $\tilde{i} : Y_Z \hookrightarrow Y$ denote $f^{-1}(Z)$, let $q_Z : F_Z \to Y_Z$ denote the pullback of q, and let $\mathcal{G}_Z$ denote a cone of $1_{Y_Z} \to q_{Z*} 1_{F_Z}$. Then we have $\mathcal{G} \simeq \tilde{i}_* \mathcal{G}_Z$ by 7.13 (and 7.14).

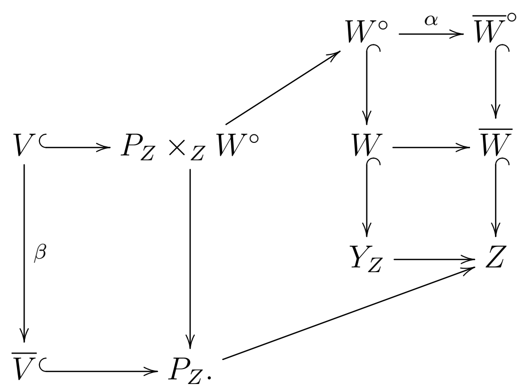
9.16. Applying i^* to the diagram (9.11.1) and using base change morphisms we obtain a commutative diagram

(9.16.1)
$$\begin{array}{ccc} \mathbb{Q}_{\ell,Y_Z} & \longrightarrow & q_{Z*}\mathbb{Q}_{\ell,F_Z} \\ \downarrow{\scriptstyle u_{Z,\ell}} & & \downarrow{\scriptstyle q_{Z*}\nu_{Z,\ell}} \\ f_Z^!\mathbb{Q}_{\ell,Z} & \longrightarrow & f_Z^!p_{Z*}\mathbb{Q}_{\ell,P_Z}. \end{array}$$

As noted in 9.10 the map ν_ℓ is motivic. The pullback $\nu_{Z\ell}$ is therefore also motivic, hence the map $p_{Z*}\nu_{Z,\ell}$ is motivic as well.

Lemma 9.17. *The weight vector of $u_{Z,\ell}$ is rational.*

Proof. Let $W \subset Y_Z$ be an irreducible component which dominates an irreducible component $\overline{W} \subset Z$ via f_Z, and let $W^\circ \subset W$ be a nonempty regular open subset mapping to a regular open subset $\overline{W}^\circ \subset \overline{W}$. Let $V \subset P_Z \times_Z W^\circ$ be a nonempty connected regular open subset mapping to a regular open subset $\overline{V} \subset P_Z$. Note that since W° is quasi-finite over Z, V is also quasi-finite over P_Z, and therefore V and $\overline{V}$ have the same dimension. Let $\alpha : W^\circ \to \overline{W}^\circ$ and $\beta : V \to \overline{V}$ denote the projections, so we have a commutative diagram



Then $\alpha^!\mathbb{Q}_\ell \simeq \alpha^*\mathbb{Q}_\ell$ and $\beta^!\mathbb{Q}_\ell \simeq \beta^*\mathbb{Q}_\ell$. By the commutativity of (9.16.1), we then have a commutative diagram

(9.17.1)
$$\begin{array}{ccccc} H^0(W^\circ, \mathbb{Q}_\ell) & \xrightarrow{u_{Z,\ell}} & H^0(W^\circ, \alpha^!\mathbb{Q}_\ell) & \xrightarrow{\simeq} & H^0(W^\circ, \alpha^*\mathbb{Q}_\ell) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(V, \mathbb{Q}_\ell) & \xrightarrow{\epsilon} & H^0(V, \beta^!\mathbb{Q}_\ell) & \xrightarrow{\simeq} & H^0(V, \beta^*\mathbb{Q}_\ell), \end{array}$$

where the map ϵ is induced by the composite map

$$\mathbb{Q}_{\ell,Y_Z} \longrightarrow q_{Z*}\mathbb{Q}_{\ell,F_Z} \xrightarrow{p_{Z*}\nu_{Z,\ell}} f_Z^!p_{Z*}\mathbb{Q}_{\ell,P_Z} \simeq \mathrm{pr}_{P_Z\times_Z Y_Z,2*}\mathrm{pr}^!_{P_Z\times_Z Y_Z,1}\mathbb{Q}_{\ell,P_Z}.$$

The composition of the top horizontal line in (9.17.1) is given by multiplication by the W-component of the weight vector of $u_{Z,\ell}$, so to prove this is a rational number it suffices to show that the composition of the bottom horizontal line in (9.17.1) is given by multiplication by a rational number. Since the map $p_{Z*}\nu_{Z,\ell}$ is motivic, the bottom horizontal line is, by induction, multiplication by an element of $H^0(V, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$ in the image of $H^0_{\mathscr{M}_B}(V, 1_V) \simeq \mathbb{Q}$, which implies the result. □

9.18. By induction we can find $u_Z : 1_{Y_Z} \to f_Z^! 1_Z$ in $\mathscr{M}_B(Y_Z)$ with ℓ-adic realization $u_{Z,\ell}$. Let $\nu_Z : 1_{F_Z} \to g_Z^! 1_{P_Z}$ denote a morphism in $\mathscr{M}_B(F_Z)$ inducing $\nu_{Z,\ell}$. By 9.14 the induced diagram

$$\begin{array}{ccc} 1_{Y_Z} & \longrightarrow & q_{Z*}1_{F_Z} \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle q_*\nu_Z} \\ f_Z^! 1_Z & \longrightarrow & f_Z^! p_{Z*} 1_{P_Z} \end{array}$$

commutes since this holds for the ℓ-adic realizations. Let $\rho_Z : \mathcal{G}_Z \to f_Z^! \mathcal{F}_Z$ be a morphism filling in the diagram

$$\begin{array}{ccccccc} 1_{Y_Z} & \longrightarrow & q_{Z*}1_{F_Z} & \longrightarrow & \mathcal{G}_Z & \longrightarrow & 1_{Y_Z}[1] \\ \downarrow{\scriptstyle u_Z} & & \downarrow{\scriptstyle q_*\nu_Z} & & \downarrow{\scriptstyle \rho_Z} & & \downarrow{\scriptstyle u_Z} \\ f_Z^! 1_Z & \longrightarrow & f_Z^! p_{Z*} 1_{P_Z} & \longrightarrow & f_Z^! \mathcal{F}_Z & \longrightarrow & f_Z^! 1_Z[1], \end{array}$$

and let $\rho : \mathcal{G} \to f^! \mathcal{F}$ be the morphism obtained by applying i_* to ρ_Z and using the isomorphisms 7.13 and 9.15. Then the diagram

$$\begin{array}{ccc} q_*1_F & \longrightarrow & \mathcal{G} \\ \downarrow{\scriptstyle q_*\nu} & & \downarrow{\scriptstyle \rho} \\ f^! p_* 1_P & \longrightarrow & f^! \mathcal{F} \end{array}$$

commutes. Indeed this can be verified after applying i^* where the result follows from the construction. We can therefore find a morphism $\lambda : 1_Y \to f^! 1_X$ so that we have a morphism of distinguished triangles

$$\begin{array}{ccccccc} 1_Y & \longrightarrow & q_*1_F & \longrightarrow & \mathcal{G} & \longrightarrow & 1_Y[1] \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \lambda} \\ f^! 1_X & \longrightarrow & f^! p_* 1_P & \longrightarrow & f^! \mathcal{F} & \longrightarrow & f^! 1_X[1]. \end{array}$$

The ℓ-adic realization of λ is then equal to u_ℓ, as this can be verified over a regular dense open of X where it follows from 9.11. This completes the proof of the "if" part of statement (i).

9.19. To see the "only if" part of statement (i) as well as statement (ii) in 9.4 it suffices to define the weight vector of u without passing to realizations. For this choose $U \subset X$ as in 9.2 so that $f^{-1}(U) = \coprod_i V_i$ with each $V_{i,\mathrm{red}}$ regular. By 6.6 the restriction of $f^!$ to V_i is isomorphic to 1_{V_i}, and in particular

$$H^0_{\mathscr{M}_B}(V_i, f^! 1_X) \simeq \mathbb{Q}.$$

We then get a map

$$H^0_{\mathscr{M}_B}(Y, f^! 1_X) \to \prod_i H^0_{\mathscr{M}_B}(V_i, f^! 1_X) \simeq \mathbb{Q}^I,$$

where the last isomorphism uses purity. By compatibility of the ℓ-adic realization functor with the purity isomorphisms, as discussed in 9.16, the image of u under this map is equal to the weight vector of the ℓ-adic realization u_ℓ. This completes the proof of 9.4. $\square$

Remark 9.20. The proof (in particular 9.14) shows that if u_ℓ is motivic, then the morphism $u : 1_Y \to f^! 1_X$ in $\mathscr{M}_B(X)$ inducing u_ℓ is unique.

9.21. We apply this to correspondences as follows. Assume now that B is the spectrum of an algebraically closed field k, and let $c : C \to X \times X$ be a correspondence with X and C quasi-projective schemes, and c_2 quasi-finite.

Theorem 9.22. *(i) Let $w \in \mathbb{Q}^I$ be a vector and assume that for some prime ℓ_0 invertible in k there exists an action $u_{\ell_0} : c_1^*\mathbb{Q}_{\ell_0} \to c_2^!\mathbb{Q}_{\ell_0}$ with $w(u_{\ell_0}) = w$. Then there exists an algebraic cycle $\Sigma \in A_0(\mathrm{Fix}(c))_{\mathbb{Q}}$ such that for any prime ℓ invertible in k and action $u_\ell : c_1^*\mathbb{Q}_\ell \to c_2^!\mathbb{Q}_\ell$ with weight vector w we have $\mathrm{Tr}_c(u_\ell) = \mathrm{cl}_\ell(\Sigma)$ in $H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)})$.*

(ii) Let ℓ be a prime invertible in k and let $u_\ell : c_1^*\mathbb{Q}_\ell \to c_2^!\mathbb{Q}_\ell$ be an action with rational weight vector. Then for every proper component $\Gamma \subset \mathrm{Fix}(c)$ the local term $\mathrm{lt}_\Gamma(\mathbb{Q}_{\ell,X}, u_\ell)$ is in $\mathbb{Q}$.

(iii) Let ℓ and ℓ' be two primes invertible in k (possibly equal), and let $u_\ell \in H^0(C, c_2^!\mathbb{Q}_{\ell,X})$ and $u_{\ell'} \in H^0(C, c_2^!\mathbb{Q}_{\ell',X})$ be sections with weight vectors $w(u)$ and $w(u')$ in $\mathbb{Q}^I$ and equal. Then for every proper component $\Gamma \subset \mathrm{Fix}(c)$, we have equality of rational numbers

$$\mathrm{lt}_\Gamma(\mathbb{Q}_\ell, u_\ell) = \mathrm{lt}_\Gamma(\mathbb{Q}_{\ell'}, u_{\ell'}).$$

Proof. Statements (ii) and (iii) follow from (i). Statement (i) follows from 9.4 which implies that there exists a morphism $u : c_1^* 1_X \to c_2^! 1_X$ in $\mathscr{M}_B(C)$ such that for any prime ℓ invertible in k the ℓ-adic realization $u_\ell : c_1^*\mathbb{Q}_\ell \to c_2^!\mathbb{Q}_\ell$ of u has weight vector w. □

9.23. **Global consequences.**

Theorem 9.24. *Let $c : C \to X \times X$ be a correspondence over an algebraically closed field with C and X Deligne-Mumford stacks, and assume c_2 is finite and c_1 is quasi-finite.*

(i) If $u : c_1^*\mathbb{Q}_\ell \to c_2^!\mathbb{Q}_\ell$ is an action with rational weight vector $w(u)$, then the trace $\mathrm{tr}(u|R\Gamma(X, \mathbb{Q}_\ell))$ of the induced action of u on $R\Gamma(X, \mathbb{Q}_\ell)$ is in $\mathbb{Q}$.

(ii) If ℓ and ℓ' are two primes and $u : c_1^*\mathbb{Q}_\ell \to c_2^!\mathbb{Q}_\ell$ and $u' : c_1^*\mathbb{Q}_{\ell'} \to c_2^!\mathbb{Q}_{\ell'}$ are actions with rational weight vectors and $w(u) = w(u')$, then $\mathrm{tr}(u|R\Gamma(X, \mathbb{Q}_\ell)) = \mathrm{tr}(u'|R\Gamma(X, \mathbb{Q}_{\ell'}))$.

Remark 9.25. The action of u on $R\Gamma(X, \mathbb{Q}_\ell)$ is defined as the composite map

$$R\Gamma(X, \mathbb{Q}_\ell) \xrightarrow{c_1^*} R\Gamma(C, \mathbb{Q}_\ell) \xrightarrow{u} R\Gamma(C, c_2^!\mathbb{Q}_\ell) \xrightarrow{\alpha} R\Gamma(X, c_{2!}c_2^!\mathbb{Q}_\ell) \xrightarrow{c_{2!}c_2^! \to \mathrm{id}} R\Gamma(X, \mathbb{Q}_\ell),$$

where the map labelled α is the isomorphism induced by the isomorphism $c_{2*} \simeq c_{2!}$, using that c_2 is proper (in the stack case this isomorphism is given by [19, 5.1]).

Proof of 9.24. By spreading out and using the generic base change theorem it suffices to consider the case when k is the algebraic closure of a finite field.

Fix a model $c : C_0 \to X_0 \times X_0$ for c over a finite field $\mathbb{F}_q \subset k$ such that all irreducible components of C are defined over $\mathbb{F}_q$. Then any map $c_1^*\mathbb{Q}_\ell \to c_2^!\mathbb{Q}_\ell$ is defined over $\mathbb{F}_q$, and in particular commutes with Frobenius. For $n \geq 0$ let

$$c^{(n)} : C \to X \times X$$

be the correspondence given by $(c_1, F_X^n \circ c_2)$, where $F_X : X \rightarrow X$ is the base change to k of the q-power Frobenius morphism on X_0.

If $u : c_1^*\mathbb{Q}_\ell \rightarrow c_2^!\mathbb{Q}_\ell$ is an action, let $u^{(n)} : c_1^{(n)*}\mathbb{Q}_\ell \rightarrow c_2^{(n)!}\mathbb{Q}_\ell$ be the action obtained by composing u with the n iterates of the canonical isomorphism $F_X^!\mathbb{Q}_\ell \rightarrow \mathbb{Q}_\ell$. Then as in [14, 3.5 (c)] to prove (i) it suffices to show that there exists n_0 such that for $n \geq n_0$ we have $\mathrm{tr}(u^{(n)}|R\Gamma(X, \mathbb{Q}_\ell)) \in \mathbb{Q}$ for all $n \geq n_0$, and to prove (ii) it suffices to show that there exists n_0 such that for all $n \geq n_0$ we have an equality of rational numbers

$$\mathrm{tr}(u^{(n)}|R\Gamma(X, \mathbb{Q}_\ell)) = \mathrm{tr}(u'^{(n)}|R\Gamma(X, \mathbb{Q}_{\ell'})).$$

Let $d : C \rightarrow X \times X$ be the transpose of c given by (c_2, c_1). For $n \geq 0$ let ${}^{(n)}d : C \rightarrow X \times X$ denote the correspondence $(F_X^n c_2, c_1)$, so ${}^{(n)}d$ is the transpose of $c^{(n)}$. Let $v : d_1^*\Omega_X \rightarrow d_2^!\Omega_X$ denote the transpose of u, and for $n \geq n_0$ let ${}^{(n)}v : {}^{(n)}d_1^*\Omega_X \rightarrow {}^{(n)}d_2^!\Omega_X$ denote the map obtained by n iterates of the isomorphism $F_X^*\Omega_X \rightarrow \Omega_X$.

By Fujiwara's theorem [10, 5.4.5] and its variant for Deligne-Mumford stacks (see [19, 1.26]; the necessary compactification results can be found in [6, 1.2.1]) there exists an integer n_0 such that for all $n \geq n_0$ the following hold:

(i) The fixed points $\mathrm{Fix}({}^{(n)}d) = \mathrm{Fix}(c^{(n)})$ consists of a finite set of isolated points.
(ii) We have

$$\mathrm{tr}({}^{(n)}v|R\Gamma_c(X, \Omega_X)) = \sum_{y \in \mathrm{Fix}({}^{(n)}d)} \mathrm{lt}_y(\Omega_X, {}^{(n)}v).$$

(iii) If $U \rightarrow X$ is an étale morphism and $d_U : C_U \rightarrow U \times U$ denotes the pullback of c along $U \times U \rightarrow X \times X$, and if $v_U : d_{U1}^*\Omega_U \rightarrow d_{U2}^!\Omega_U$ denotes the pullback of v, then for every $y' \in \mathrm{Fix}({}^{(n)}d_U) = U \times_X \mathrm{Fix}({}^{(n)}d)$ mapping to $y \in \mathrm{Fix}({}^{(n)}d)$ we have

$$\mathrm{lt}_{y'}(\Omega_U, {}^{(n)}d_U) = \mathrm{lt}_y(\Omega_X, {}^{(n)}d).$$

Now since ${}^{(n)}v$ is adjoint to $u^{(n)}$ we have

$$\mathrm{tr}({}^{(n)}v|R\Gamma_c(X, \Omega_X)) = \mathrm{tr}(u^{(n)}|R\Gamma(X, \mathbb{Q}_\ell)),$$

and by [13, III, 5.1.6] we have

$$\mathrm{lt}_y(\Omega_X, {}^{(n)}v) = \mathrm{lt}_y(\mathbb{Q}_\ell, u^{(n)}).$$

It follows that for $n \geq n_0$

$$\mathrm{tr}(u^{(n)}|R\Gamma(X, \mathbb{Q}_\ell)) = \sum_{y \in \mathrm{Fix}(c^{(n)})} \mathrm{lt}_y(\mathbb{Q}_\ell, u^{(n)}).$$

It suffices to show that each local term belongs to $\mathbb{Q}$ and is independent of ℓ. Now by (iii), the local term $\mathrm{lt}_y(\mathbb{Q}_\ell, u^{(n)})$ can be computed after replacing X by an étale covering, which reduces the computation to the case when X is quasi-projective. Combining this with 9.22 we get the theorem (note that if $w(u)_i$ denotes the component of the weight vector corresponding to an irreducible component $C_i \subset C$ then $q^{\dim(C_i)}w(u)_i = w(u^{(n)})_i$). □

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