COMPATIBLE SYSTEMS AND CORRESPONDENCES

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Contents

1. Introduction

This paper is part of a series of papers studying independence of ℓ properties of correspondences acting on complexes of sheaves. The purpose of the present paper is to develop the foundations for a general theory of compatible systems of complexes of sheaves with actions of correspondences. The idea to develop such a theory is due to Illusie [9, 3.8].

As in [13], one motivation for the work in this paper comes from questions of independence of ℓ for traces of correspondences acting on various cohomology groups. One of the key tools for proving such independence of ℓ results is the trace formula of Fujiwara [5]. To help motivate the work in this paper let us recall this result.

Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$, and let k be an algebraic closure of $\mathbb{F}_q$. Let $X_0/\mathbb{F}_q$ be a separated scheme of finite type over $\mathbb{F}_q$, and let

$$c = (c_1, c_2) : C_0 \to X_0 \times X_0$$

be a morphism of $\mathbb{F}_q$-schemes, with C_0 also separated and of finite type over $\mathbb{F}_q$. Assume that c_1 is proper and that c_2 is quasi-finite, and let $\ell \neq p$ be a prime. Let $K \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ be a complex (where X denotes $X_0 \otimes_{\mathbb{F}_q} k$), and let

$$u : c_1^* K \to c_2^! K$$

be a morphism in $D^b_c(C, \overline{\mathbb{Q}}_\ell)$ (which by adjunction corresponds to a morphism $c_{2!}c_1^*K \to K$ in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$). We refer to such a morphism u as a *c-structure on K*. We then get an endomorphism

$$R\Gamma_c(u) : R\Gamma_c(X, K) \to R\Gamma_c(X, K),$$

defined as the composite

$$R\Gamma_c(X,K) \longrightarrow R\Gamma_c(X, c_{1*}c_1^*K) \xrightarrow{c_{1*}\simeq c_{1!}} R\Gamma_c(C, c_1^*K) \xrightarrow{\simeq} R\Gamma_c(X, c_{2!}c_1^*K) \xrightarrow{u} R\Gamma_c(X,K).$$

Suppose further given an isomorphism

$$\epsilon : F_X^*K \to K,$$

where $F_X : X \to X$ is the relative Frobenius of X/k (which equals the base change to k of the q-power Frobenius on X_0). For an integer $n \geq 0$ define

$$c^{(n)} : C \to X \times X$$

to be the composite map

$$C \xrightarrow{c} X \times X \xrightarrow{F_X^n \times \mathrm{id}} X \times X.$$

Let

$$u^{(n)} : (c^{(n)})_1^*K \to (c^{(n)})_2^!K$$

be the composite map

$$(c^{(n)})_1^*K \xrightarrow{\simeq} c_1^*F_X^{n*}K \xrightarrow{\epsilon^n} c_1^*K \xrightarrow{u} c_2^!K = (c^{(n)})_2^!K.$$

Also define $\mathrm{Fix}(c^{(n)})$ to be the fiber product of the diagram

$$\begin{tikzcd} & C \arrow[d, "c^{(n)}"] \\ X \arrow[r, "\Delta"] & X \times X. \end{tikzcd}$$

Fujiwara's theorem is then the following:

Theorem 1.1 (Fujiwara [5, 5.4.5])**.** *There exists an integer n_0 such that for every $n \geq n_0$ the set $\mathrm{Fix}(c^{(n)})(k)$ is finite and*

$$\mathrm{tr}(R\Gamma(u^{(n)})|R\Gamma_c(X,K)) = \sum_{z \in \mathrm{Fix}(c^{(n)})(k)} \mathrm{lt}_z^{\mathrm{na}}(K, u^{(n)}),$$

where $\mathrm{lt}_z^{\mathrm{na}}(K, u^{(n)})$ denotes the naive local term at z (see section 3.1 for precise definitions).

Remark 1.2. Fujiwara's theorem also holds for algebraic spaces, and Deligne-Mumford stacks. The case of algebraic spaces follows from Varshavsky's argument [17] together with the Nagata compactification theorem for algebraic spaces [2, 1.2.1]. The case of Deligne-Mumford stacks can be deduced from this by passing to the coarse moduli spaces (see [15]).

Remark 1.3. Below we will also need to consider the composition of c with itself. If $c : C \to X \times X$ and $d : D \to X \times X$ are two correspondences, then their composition is defined to be the fiber product $C \times_{c_2, X, d_1} D$ with map e to $X \times X$ given by

$$e = (c_1 \circ \mathrm{pr}_1, d_2 \circ \mathrm{pr}_2) : C \times_{c_2, X, d_1} D \to X \times X.$$

If $K \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ is a complex and $u : c_1^*K \to c_2^!K$ (resp. $v : d_1^*K \to d_2^!K$) is a c-structure (resp. d-structure), then K also has an e-structure, the *composition of u and v*, given by the map

$$e_1^*K = \mathrm{pr}_1^*c_1^*K \xrightarrow{\mathrm{pr}_1^*u} \mathrm{pr}_1^*c_2^!K \xrightarrow{bc} \mathrm{pr}_2^!d_1^*K \xrightarrow{\mathrm{pr}_2^!v} \mathrm{pr}_2^!d_2^!K = e_2^!K,$$

where the map labelled 'bc' is the base change morphism. In particular, for a single $c : C \to X \times X$ and $m \geq 1$ we can consider its m-fold composition c_m, and for a c-structure u its m-fold composition u_m.

Fujiwara's theorem 1.1 can be used to deduce independence of ℓ results for compactly supported cohomology. For example, let X/k be a separated finite type k-scheme, and let $\alpha : X \to X$ be a proper endomorphism. Then by standard limit arguments there exists a finite subfield $\mathbb{F}_q \subset k$ and a finite type separated $\mathbb{F}_q$-scheme X_0 with an endomorphism $\alpha_0 : X_0 \to X_0$ inducing (X, α) by base change to k. Let $C_0 = X_0$, and let

$$c : C_0 \to X_0 \times X_0$$

be the map given by $\alpha_0 \times \mathrm{id}$. Since c_2 is the identity, the canonical isomorphism

$$\alpha^* \mathbb{Q}_\ell \to \mathbb{Q}_\ell$$

defines a map

$$u : c_1^* \mathbb{Q}_\ell \to c_2^! \mathbb{Q}_\ell.$$

In this case the induced action $R\Gamma_c(u)$ on $R\Gamma_c(X, \mathbb{Q}_\ell)$ is the natural action induced by functoriality of compactly supported cohomology. Moreover, it follows from the definitions that all the naive local terms are 1. If

$$f : R\Gamma_c(X, \mathbb{Q}_\ell) \to R\Gamma_c(X, \mathbb{Q}_\ell)$$

denotes the Frobenius automorphism, then f commutes with $R\Gamma_c(u)$ and Fujiwara's theorem gives that there exists an integer n_0 such that for all $n \geq n_0$ we have

$$\mathrm{tr}(R\Gamma_c(u) \circ f^n | R\Gamma_c(X, \mathbb{Q}_\ell)) = |X^{\alpha^{(n)}}(k)|,$$

where $X^{\alpha^{(n)}}$ denotes the fixed points of $F_X^n \circ \alpha$. In particular, for $n \geq n_0$ the trace

$$\mathrm{tr}(R\Gamma_c(u) \circ f^n | R\Gamma_c(X, \mathbb{Q}_\ell))$$

is in $\mathbb{Q}$ (even $\mathbb{Z}$) and independent of ℓ. By some linear algebra (see section 4.12), this implies that in fact

$$\mathrm{tr}(R\Gamma_c(u) | R\Gamma_c(X, \mathbb{Q}_\ell))$$

is in $\mathbb{Q}$ and independent of ℓ.

Following a suggestion of Illusie [9, 3.8] we take Fujiwara's theorem as a starting point for developing a theory of 'compatible systems'.

We describe this notion here in the case of field of definition $\mathbb{Q}$, though we consider a more general field of definition $E/\mathbb{Q}$ in section 4.12.

Let us first review Gabber's notion of a compatible system of Weil complexes [6]. Let $\mathbb{F}_q$ be a finite field and let $X_0/\mathbb{F}_q$ be a separated scheme of finite type. Recall (see for example [4, 1.1.14]) that an *ℓ-adic Weil complex* (or simply *Weil complex* if the reference to ℓ is clear) on X_0 is a pair (K, ϵ), where $K \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ and $\epsilon : F_X^* K \to K$ is an isomorphism. For such a pair (K, ϵ) we get for every closed point $x \in X_0$ and a geometric point $\bar{x}$ over x an automorphism $\epsilon_{\bar{x}} : K_{\bar{x}} \to K_{\bar{x}}$ defined by taking the stalk of the $[k(x) : \mathbb{F}_q]$-th power of ϵ. In particular, we can consider the trace $\mathrm{Tr}_{\bar{x}}(\epsilon) \in \overline{\mathbb{Q}}_\ell$.

Let $\{\ell_\alpha\}_{\alpha \in I}$ be a set of primes indexed by a set I, with each $\ell_\alpha \neq p$ (but we allow $\ell_\alpha = \ell_\beta$ for $\alpha \neq \beta$). A *compatible system of Weil complexes* is a collection of pairs $\{(K_\alpha, \epsilon_\alpha)\}_{\alpha \in I}$ indexed by I, where each $(K_\alpha, \epsilon_\alpha)$ is an ℓ_α-adic Weil complex on X_0 and such that the following condition holds:

(i) For every closed point $x \in X_0$ and $\alpha \in I$ we have $\mathrm{Tr}_{\bar{x}}(\epsilon_\alpha) \in \mathbb{Q} \subset \overline{\mathbb{Q}}_\ell$.
(ii) For every closed point $x \in X_0$ and two elements $\alpha, \beta \in I$ we have an equality of rational numbers $\mathrm{Tr}_{\bar{x}}(\epsilon_\alpha) = \mathrm{Tr}_{\bar{x}}(\epsilon_\beta)$.

1.4. **Weakly compatible systems.** Suppose given in addition a correspondence over $\mathbb{F}_q$

$$c : C_0 \to X_0 \times X_0,$$

with c_2 quasi-finite. A *weakly compatible system of Weil complexes with c-structure on X* (or *I-compatible system* to make clear the reference to I) is a collection of data $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}_{\alpha \in I}$ as follows (see 4.17 for more details):

(i) $\{(K_\alpha, \epsilon_\alpha)\}_{\alpha \in I}$ is a compatible system of Weil complexes.
(ii) $u_\alpha : c_1^* K_\alpha \to c_2^! K_\alpha$ is a morphism which commutes with ϵ_α in the sense that the diagram

$$\begin{tikzcd}
F_C^* c_1^* K_\alpha \arrow[r, "F_C^* u_\alpha"] \arrow[d, "\simeq"] & F_C^* c_2^! K_\alpha \arrow[d, "\simeq"] \\
c_1^* F_C^* K_\alpha \arrow[d, "c_1^* \epsilon_\alpha"] & c_2^! F_X^* K_\alpha \arrow[d, "c_2^! \epsilon_\alpha"] \\
c_1^* K_\alpha \arrow[r, "u_\alpha"] & c_2^! K_\alpha
\end{tikzcd}$$

commutes, where the arrows labelled $\simeq$ are the canonical isomorphisms.

For any $n \geq 0$ and $r \geq 1$ we then get a map

$$u_{r,\alpha}^{(n)} : c_{r1}^{(n)*} K_\alpha \to c_{r2}^! K_\alpha,$$

as discussed in 3.11, where c_r denotes the r-fold composition of c with itself.
(iii) For every $\alpha \in I$, $n \geq 0$, $r \geq 1$, and $z \in \mathrm{Fix}(c_r^{(n)})(k)$ the local term $\mathrm{lt}_z^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(n)})$ is in $\mathbb{Q}$, and for every $\alpha, \beta \in I$ we have an equality of rational numbers

$$\mathrm{lt}_z^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(n)}) = \mathrm{lt}_z^{\mathrm{na}}(K_\beta, u_{r,\beta}^{(n)}).$$

Remark 1.5. We refer to a triple (K, ϵ, u) consisting of a Weil complex (K, ϵ) and a morphism $u : c_1^* K \to c_2^! K$ satisfying (i), and (ii) as a *Weil complex with c-structure.*

1.6. **Strongly compatible systems.** Instead of considering the naive local terms we can also consider the so-called true local terms. If

$$c : C \to X \times X$$

is a correspondence, with C and X separated Deligne-Mumford stacks of finite type over k, then we can form the fixed point stack

$$\mathrm{Fix}(c) := C \times_{c, X \times X, \Delta} X,$$

and for a complex $K \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ with a c-structure $u : c_1^* K \to c_2^! K$ the construction of [7, III, 4.2] (see also [17, 1.2.2]), defines a class in $H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)})$, where $\Omega_{\mathrm{Fix}(c)}$ denotes the dualizing complex on $\mathrm{Fix}(c)$. For each proper connected component $Z \subset \mathrm{Fix}(c)$ there is a natural map $H^0(Z, \Omega_Z) \to \overline{\mathbb{Q}}_\ell$, and so from (K, u) and Z we obtain a number

$$\mathrm{lt}_Z(K, u) \in \overline{\mathbb{Q}}_\ell,$$

called the *true local term at* Z.

We can then consider compatible systems using true local terms rather than naive local terms. As in 1.4, let k be an algebraic closure of a finite field $\mathbb{F}_q$, and let $\{\ell_\alpha\}_{\alpha \in I}$ be a set of primes not equal to the characteristic p of k. Let

$$c : C_0 \to X_0 \times X_0$$

be a correspondence with C_0 and X_0 separated Deligne-Mumford stacks of finite type over $\mathbb{F}_q$ (but we make no assumption on c_1 and c_2). In subsection 4.21, we then introduce the notion of a *strongly compatible system of Weil complexes with c-structure*, which is a collection of data $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}_{\alpha \in I}$ as follows (note that the only difference between this definition and the one in 1.4 is in (iii)):

(i) $\{(K_\alpha, \epsilon_\alpha)\}_{\alpha \in I}$ is a compatible system of Weil complexes.
(ii) $u_\alpha : c_1^* K_\alpha \to c_2^! K_\alpha$ is a morphism which commutes with ϵ_α.
(iii) For every $\alpha \in I$, $n \geq 0$, $r \geq 1$, and proper connected component $Z \subset \mathrm{Fix}(c_r^{(n)})$ the true local term $\mathrm{lt}_Z(K_\alpha, u_{r,\alpha}^{(n)})$ is in $\mathbb{Q}$, and for every $\alpha, \beta \in I$ we have an equality of rational numbers
$$\mathrm{lt}_Z(K_\alpha, u_{r,\alpha}^{(n)}) = \mathrm{lt}_Z(K_\beta, u_{r,\beta}^{(n)}).$$

Here $u_{r,\alpha}$ and c_r are as in 1.4.

1.7. **The main conjecture on compatible systems.** Our main conjecture on compatible systems is the following:

Conjecture 1.8. *Let $c : C_0 \to X_0 \times X_0$ be a correspondence with C_0 and X_0 separated $\mathbb{F}_q$-schemes of finite type, and assume that c_2 is quasi-finite. Let $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}_{\alpha \in I}$ be a weakly compatible system of Weil complexes with c-structure. Then $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}_{\alpha \in I}$ is a strongly compatible system.*

We will discuss this conjecture in several following papers. In this paper, however, we restrict ourselves to discussing consequences of the conjecture in addition to developing the general formalism of compatible systems.

1.9. **Applications.** If conjecture 1.8 holds one obtains a partial six functor formalism for compatible systems with actions of correspondences. The key result for this is stability under Verdier duality (here and in the following we label results conditional on 1.8 with a †):

Theorem† 1.10 (Statement 5.2 in the text)**.** *Let $c_0 : C_0 \to X_0 \times X_0$ be a correspondence as above, and assume that c_1 and c_2 are quasi-finite. Let $c^t : C_0 \to X_0 \times X_0$ be the transpose of c obtained by composing c with the automorphism of $X_0 \times X_0$ interchanging the factors. If $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ is a compatible system of Weil complexes with c-structure, then the Verdier duals DK_α are naturally part of a compatible system $\{(DK_\alpha, \epsilon_\alpha^t, u_\alpha^t)\}$ of Weil complexes with c^t-structure.*

This result also enables us to develop a six-operations formalism for weakly compatible systems and suitable morphisms of correspondences. It is not hard to show using Fujiwara's theorem, that for suitable morphisms of correspondences

$$\begin{tikzcd} & C_0 \arrow[dl, "c_1"'] \arrow[dr, "c_2"] \arrow[dd, "g"] & \\ X_0 \arrow[dd, "f"] & & X_0 \arrow[dd, "f"'] \\ & D_0 \arrow[dl, "d_1"] \arrow[dr, "d_2"'] & \\ Y_0 & & Y_0, \end{tikzcd}$$

the pushforwards $f_! K_\alpha$ have a natural structure of Weil complexes with d-structure, and that the resulting collection of data $\{(f_! K_\alpha, f_! \epsilon_\alpha, f_! u_\alpha)\}$ is a weakly compatible system.

To get a handle on the ordinary pushforwards $f_* K_\alpha$, we use Verdier duality to write this as

$$f_* K_\alpha = D_Y f_! D_X K_\alpha,$$

where $D_X : D^b_c(X, \overline{\mathbb{Q}}_{\ell_\alpha}) \to D^b_c(X, \overline{\mathbb{Q}}_{\ell_\alpha})$ is the Verdier duality functor and then apply 1.10.

Similarly it is not hard to show that suitable pullbacks f^* take weakly compatible systems to weakly compatible systems, and using duality again this implies that so does the functor $f^!$.

The paper is organized as follows.

In sections 2 and 3 we discuss definitions and basics on local terms (both true and naive). This is all fairly standard material, except for some of the generalizations to stacks.

In section 4 we discuss the basics on weakly and strongly compatible systems.

Section 5 develops the formalism of six operations for weakly compatible systems assuming 1.8. The key ingredient is Fujiwara's theorem which implies stability of weakly compatible systems under $f_!$.

Finally in section 6 we discuss some group theoretic aspects of the notion of compatible systems. The main ingredient here is proposition 6.14 which is a type of Chebotarev density theorem for fixed points of correspondences. This result is obtained using results of Hrushovski [8]. The main result from this section that will be used in subsequent work is proposition 6.23.

1.11. **Terminology and conventions.** For subsequent work it will be important to consider Deligne-Mumford stacks and not just schemes (essentially because we want to use equivariant alterations for various devissage arguments).

By a *coefficient ring* Λ, we mean either a Gorenstein local ring of dimension 0 and finite residue field of characteristic ℓ invertible in k, $\overline{\mathbb{Q}}_\ell$, $\mathbb{Z}_\ell$, or a finite extension of $\mathbb{Q}_\ell$, where ℓ is a prime invertible in k. For a separated Deligne-Mumford stack X of finite type over k we write $D^b_{ctf}(X, \Lambda)$ for the derived category of bounded complexes of Λ-modules which are of finite tor-dimension and have constructible cohomology sheaves. In the case when $\Lambda = \overline{\mathbb{Q}}_\ell$, $\mathbb{Z}_\ell$, or $\mathbb{Q}_\ell$ we have $D^b_{ctf}(X, \Lambda) = D^b_c(X, \Lambda)$, the usual derived category of constructible Λ-modules as defined in [4, 1.1.2]. If the coefficient ring Λ is understood we sometimes write $D^b_{ctf}(X)$ instead of $D^b_{ctf}(X, \Lambda)$. We write $\Omega_X \in D^b_{ctf}(X, \Lambda)$ for the dualizing complex of X normalized so that the dualizing complex of $\mathrm{Spec}(k)$ is Λ, and

$$D_X : D^b_{ctf}(X, \Lambda)^{\mathrm{op}} \to D^b_{ctf}(X, \Lambda)$$

for the Verdier dualizing functor. If no confusion is likely to arise we sometimes write simply D for D_X.

If X is a normal scheme and $\bar{x} \to X$ is a geometric point we get a fiber functor

$$\omega_{\bar{x}} : (\text{finite étale } X\text{-schemes}) \to \mathrm{Set}$$

sending Y/X to the fiber $Y_{\bar{x}}$. If $\bar{x}$ and $\bar{x}'$ are two geometric points then an *étale path between x and x'* is an isomorphism of fiber functors $\omega_{\bar{x}} \simeq \omega_{\bar{x}'}$.

Throughout we denote results conditional on 1.8 with a †.

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2. True local terms

Throughout this section we work over an algebraically closed field k and sheaves of Λ-modules, where Λ is a coefficient ring as in 1.11.

2.1. Definition of local terms and basic properties.

2.2. Let

$$c : C \to X \times X$$

be a correspondence, with C and X separated Deligne-Mumford stacks of finite type over k. Let F denote the fixed point stack $\mathrm{Fix}(c)$ of this correspondence, so we have a cartesian diagram

$$\begin{tikzcd} F \arrow[r, "\delta"] \arrow[d, "c'"'] & C \arrow[d, "c"] \\ X \arrow[r, "\Delta"] & X \times X. \end{tikzcd}$$

The same argument as in [11, 4.5.3] (in the case of schemes this is [1, XVIII.3.1.12.2]) shows that for any $K, L \in D^b_{ctf}(X, \Lambda)$ we have a canonical isomorphism

$$c^! \mathscr{R}Hom_{X\times X}(\mathrm{pr}_1^* K, \mathrm{pr}_2^! L) \simeq \mathscr{R}Hom_C(c_1^* K, c_2^! L)$$

and as in [7, III.3.1.1], we have

$$\mathscr{R}Hom_{X\times X}(\mathrm{pr}_1^* K, \mathrm{pr}_2^! L) \simeq D(K) \boxtimes L.$$

We therefore obtain a canonical isomorphism

$$c^!(D(K) \boxtimes L) \simeq \mathscr{R}Hom(c_1^* K, c_2^! L). \tag{2.2.1}$$

In particular, for $K = L$, we can view a c-structure $u : c_1^*K \to c_2^!K$ as an element of $H^0(C, c^!(D(K) \boxtimes K))$.

Now we have a map

$$D(K) \boxtimes K \longrightarrow \Delta_*(D(K) \otimes K) \xrightarrow{\text{eval}} \Delta_*\Omega_X.$$

Applying $c^!$ to this composite and using the base change theorem we get a morphism

$$c^!(D(K) \boxtimes K) \to \delta_*\Omega_F.$$

Taking global sections we get a map

$$\mathrm{Tr}_c : \mathrm{Hom}(c_1^*K, c_2^!K) \simeq H^0(C, c^!(D(K) \boxtimes K)) \to H^0(F, \Omega_F). \tag{2.2.2}$$

For any open and closed substack $Z \subset F$ proper over k, define the *local term of u at Z*, denoted $\mathrm{lt}_Z(K, u) \in \Lambda$, to be the image of $\mathrm{Tr}_c(u)$ under the map

$$H^0(F, \Omega_F) \xrightarrow{\text{restriction}} H^0(Z, \Omega_Z) \xrightarrow{\int} \Lambda.$$

Here $\int : H^0(Z, \Omega_Z) \to \Lambda$ is the proper pushforward map.

Remark 2.3. This definition of local terms differs from the one given in [7, III.4.2.7]. However, the equivalence of the two definitions is shown in [17, A.2].

2.4. **Proper pushforward and true local terms.**

2.5. Consider a commutative diagram separated Deligne-Mumford stacks of finite type over k

$$\begin{tikzcd}
& C \arrow[dl, "c_1"'] \arrow[dd, "g"] \arrow[dr, "c_2"] & \\
X \arrow[dd, "f"] & & X \arrow[dd, "f"] \\
& D \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\
Y & & Y,
\end{tikzcd}$$

with f and g proper. Assume either that f is representable or that Λ is a $\mathbb{Q}_\ell$-algebra. This assumption implies that $f_! \simeq f_*$ by [15, 5.17].

Let F_c (resp. F_d) denote the fixed point stack of c (resp. d), and let $q : F_c \to F_d$ be the projection. If $K \in D^b_{ctf}(X, \Lambda)$ is a complex with c-structure $u : c_1^*K \to c_2^!K$, then by [17, 1.1.6] there is an induced action $g_!u$ of D on $f_!K$.

This action is defined as follows. Let C' denote $D \times_{c_2,Y} X$, so we have a commutative diagram

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[rr, bend left, "c_2"] \arrow[dr, "g"'] & C' \arrow[r, "c_2'"] \arrow[d, "k"] & X \arrow[d, "f"] \\ & D \arrow[r, "d_2"] & Y, \end{tikzcd}$$

with the square cartesian. The map $g_!u$ is then defined as the composite map

$$d_1^* f_! K = d_1^* f_* K \longrightarrow g_* c_1^* K \xrightarrow{g_* u} g_* c_2^! K \xrightarrow{\simeq} k_* h_* h^! c_2'^! K \xrightarrow{h_* h^! \to \mathrm{id}} k_* c_2'^! K \xrightarrow{\simeq} d_2^! f_! K,$$

where the last isomorphism is by the base change theorem. The relative version of the Grothendieck-Lefschetz trace formula is the following:

Theorem 2.6 ([17, 1.2.5])**.** *The morphism $q : F_c \to F_d$ is proper, and the diagram*

$$\begin{tikzcd} \mathrm{Hom}(c_1^* K, c_2^! K) \arrow[r, "\mathrm{Tr}_c"] \arrow[d, "u \mapsto g_! u"] & H^0(F_c, \Omega_{F_c}) \arrow[d, "q_*"] \\ \mathrm{Hom}(d_1^* f_! K, c_2^! f_! K) \arrow[r, "\mathrm{Tr}_d"] & H^0(F_d, \Omega_{F_d}) \end{tikzcd}$$

commutes.

2.7. **True local terms and duality.**

2.8. Consider again a correspondence $c : C \to X \times X$ as in 2.2, and let

$$c^t = (c_2, c_1) : C \to X \times X$$

be its transpose. For $K \in D^b_{ctf}(X)$ and $u : c_1^* K \to c_2^! K$ an action of c on K define

$$u^t : c_1^{t*} D(K) \to c_2^{t!} D(K)$$

to be the action of c^t on $D(K)$ given by the composite

$$c_1^{t*} D(K) \xrightarrow{\simeq} c_2^* D(K) \xrightarrow{D(u)} c_1^! D(K) \xrightarrow{\simeq} c_2^{t!} D(K).$$

Since the diagonal of X is invariant under switching the two factors of $X \times X$, there is a canonical isomorphism

$$\sigma : \mathrm{Fix}(c) \to \mathrm{Fix}(c^t).$$

Proposition 2.9 (See also [7, III, 5.1.6])**.** *For $K \in D^b_{ctf}(X, \Lambda)$, the diagram*

$$\begin{tikzcd} \mathrm{Hom}(c_1^* K, c_2^! K) \arrow[r, "\mathrm{Tr}_c"] \arrow[d, "u \mapsto u^t"] & H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)}) \arrow[d, "\sigma"] \\ \mathrm{Hom}(c_1^{t*} DK, c_2^{t!} K) \arrow[r, "\mathrm{Tr}_{c^t}"] & H^0(\mathrm{Fix}(c^t), \Omega_{\mathrm{Fix}(c^t)}) \end{tikzcd}$$

commutes.

Proof. It suffices to note that the diagram

$$\begin{tikzcd}
\mathscr{R}Hom(c_1^*K, c_2^!K) \arrow[r, "(2.2.1)"] \arrow[d, "v\mapsto D(v)"'] & c^!(D(K)\boxtimes K) \arrow[r] \arrow[d, "\simeq"] & c^!\Delta_*(D(K)\otimes K) \arrow[r, "\mathrm{eval}"] \arrow[d, "\mathrm{flip}"] & c^!\Delta_*\Omega_X \arrow[r, "\simeq"] & \delta_*\Omega_{\mathrm{Fix}(c)} \arrow[d, "\simeq"] \\
\mathscr{R}Hom(c_1^{t*}DK, c_2^{t!}DK) \arrow[r, "(2.2.1)"] & c^{t!}(K\boxtimes DK) \arrow[r] & c^{t!}\Delta_*(K\otimes DK) \arrow[r, "\mathrm{eval}"] & c^{t!}\Delta_*\Omega_X \arrow[r, "\simeq"] & \delta_*\Omega_{\mathrm{Fix}_{\mathrm{Fix}(c^t)}}
\end{tikzcd}$$

commutes. □

2.10. **Compatibility with pushforward along closed embeddings.**

2.11. Let $i : Z \hookrightarrow X$ be a closed embedding, and let

$$c_Z : C_Z \to Z \times Z$$

be the correspondence on Z obtained by taking the fiber product of the diagram

$$\begin{tikzcd}
 & Z\times Z \arrow[d, "i\times i"] \\
C \arrow[r, "c"] & X\times X.
\end{tikzcd}$$

If C_j denotes the fiber product $C \times_{c_j,X,i} Z$, then we have a cartesian diagram

(2.11.1)
$$\begin{tikzcd}
C_Z \arrow[r, hook, "\alpha"] \arrow[d, hook, "\beta"'] & C_2 \arrow[d, hook, "\gamma"] \\
C_1 \arrow[r, hook, "\delta"] & C,
\end{tikzcd}$$

where all the morphisms are closed embeddings.

Let $K \in D^b_{ctf}(Z)$ be a complex, let $u : c_{Z,1}^*K \to c_{Z,2}^!K$ be a c_Z-structure on K, and let $i_*u : c_1^*(i_*K) \to c_2^!(i_*K)$ be the pushforward of u to C.

Observe that there is a natural cartesian diagram

(2.11.2)
$$\begin{tikzcd}
\mathrm{Fix}(c_Z) \arrow[r, hook] \arrow[d] & \mathrm{Fix}(c) \arrow[d] \\
Z \arrow[r, hook, "i"] & X.
\end{tikzcd}$$

Proposition 2.12. *For any open and closed $W \subset \mathrm{Fix}(c)$ we have*

$$\mathrm{lt}_W(i_*K, i_*u) = \mathrm{lt}_{W\cap \mathrm{Fix}(c_Z)}(K, u).$$

Proof. This follows immediately from 2.6. □

2.13. Any c-structure

$$v : c_{2!}c_1^*(i_*K) \to i_*K$$

is of the form i_*u for some c_Z-structure u on K.

To see this, let $\chi_s : C_s \to Z$ $(s = 1, 2)$ be the map such that $i \circ \chi_1 = c_1 \circ \delta$ (resp. $i \circ \chi_2 = c_2 \circ \gamma$). By the base change theorem we have

$$c_1^*i_*K \simeq \delta_*\chi_1^*K, \quad c_2^!i_*K \simeq \gamma_*\chi_2^!K.$$

Also since (2.11.1) is cartesian we have

$$\gamma^*\delta_* \simeq \alpha_*\beta^*.$$

Therefore

$$\begin{aligned}
\mathrm{Hom}(c_1^*i_*K, c_2^!i_*K) &\simeq \mathrm{Hom}(\delta_*\chi_1^*K, \gamma_*\chi_2^!K)\\
&\simeq \mathrm{Hom}(\gamma^*\delta_*\chi_1^*K, \chi_2^!K)\\
&\simeq \mathrm{Hom}(\alpha_*\beta^*\chi_1^*K, \chi_2^!K)\\
&\simeq \mathrm{Hom}(\beta^*\chi_1^*K, \alpha^!\chi_2^!K)\\
&\simeq \mathrm{Hom}(c_{Z,1}^*K, c_{Z,2}^!K).
\end{aligned}$$

Proposition 2.14. *Under this isomorphism*

$$\mathrm{Hom}(c_{Z,1}^*K, c_{Z,2}^!K) \simeq \mathrm{Hom}(c_1^*i_*K, c_2^!i_*K) \tag{2.14.1}$$

a map $u : c_{Z1}^*K \to c_{Z2}^!K$ is sent to i_*u.

Proof. Note first of all that for any $G \in D^b_{ctf}(C)$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}(G, c_2^!i_*K) \arrow[r, "c_2^!i_*\simeq\gamma_*\chi_2^!"] \arrow[d, "\text{adj.}"] & \mathrm{Hom}(G, \gamma_*\chi_2^!K) \arrow[d, "\text{adj.}"] \\
\mathrm{Hom}(c_{2!}G, i_*K) \arrow[d, "\text{adj.}"] & \mathrm{Hom}(\gamma^*G, \chi_2^!K) \arrow[d, "\text{adj.}"] \\
\mathrm{Hom}(i^*c_{2!}G, K) \arrow[r, "i^*c_{2!}\simeq\chi_{2!}\gamma^*"] & \mathrm{Hom}(\chi_{2!}\gamma^*G, K)
\end{tikzcd}$$

commutes, as explained in [1, XVIII, 3.1.11] (in particular the commutativity of the last diagram in this paragraph). It follows that the isomorphism

$$\mathrm{Hom}(i^*c_{2!}c_1^*i_*K, K) \simeq \mathrm{Hom}(c_{Z,2!}c_{Z,1}^*K, K)$$

obtained from (2.14.1) by adjunction, is induced by the map

$$i^*c_{2!}c_1^*i_*K \to c_{Z,2!}c_{Z,1}^*K$$

obtained by going around the top and right sides of the following diagram:

$$\begin{tikzcd}
i^*c_{2!}c_1^*i_*K \arrow[r, "i^*c_{2!}\simeq\chi_{2!}\gamma^*"] \arrow[dr, "c_1^*i_*\simeq\delta_*\chi_1^*"] \arrow[ddd, "c_1^*i_*\to\epsilon_*c_{Z1}^*"] & \chi_{2!}\gamma^*c_1^*i_*K \arrow[r, "c_1^*i_*\simeq\delta_*\chi_1^*"] & \chi_{2!}\gamma^*\delta_*\chi_1^*K \arrow[dl, "\mathrm{id}\to\beta_*\beta^*"] \arrow[dd, "\gamma^*\delta_*\simeq\alpha_*\beta^*"] \\
 & i^*c_{2!}\delta_*\chi_1^*K \arrow[ur, "i^*c_{2!}\simeq\chi_{2!}\gamma^*"] \arrow[ddl, "\mathrm{id}\to\beta_*\beta^*"] & \\
 & \chi_{2!}\gamma^*\delta_*\beta_*\beta^*\chi_1^*K \arrow[r, "\gamma^*\delta_*\beta_*\simeq\alpha_*"] & \chi_{2!}\alpha_*\beta^*\chi_1^*K \arrow[d, "\simeq"] \\
i^*c_{2!}\delta_*\beta_*\beta^*\chi_1^*K \arrow[ur, "i^*c_{2!}\simeq\chi_{2!}\gamma^*"] \arrow[rr, "\simeq"] & & c_{Z2!}c_{Z1}^*K.
\end{tikzcd} \tag{2.14.2}$$

On the other hand, by definition for $u : c_{Z2!}c_{Z1}^*K \to K$ the map i_*u is obtained by composing u with the bottom and left side of this diagram. To prove the lemma it therefore suffices to show that (2.14.2) commutes.

For this note that all the small inside diagrams clearly commute, except possibly the bottom right diagram. The commutativity of this last square amounts to the commutativity of the following diagram of functors

$$\begin{tikzcd}
i^*c_{2!}\delta_*\beta_* \arrow[r, "i^*c_{2!}\simeq\chi_{2!}\gamma^*"] \arrow[d, "\simeq"] & \chi_{2!}\gamma^*\delta_*\beta_* \arrow[d, "\simeq"] \\
i^*i_*c_{Z2!} \arrow[r, "\simeq"] & \chi_{2!}\alpha_*,
\end{tikzcd}$$

which follows from [1, XVII, 5.2.5] applied to the diagram

$$\begin{tikzcd}
C_1 \arrow[d, "\delta"] & C_Z \arrow[l, "\beta"'] \arrow[d, "\alpha"] \\
C \arrow[d, "c_2"] & C_2 \arrow[l, "\gamma"'] \arrow[d, "\chi_2"] \\
X & Z. \arrow[l, "i"']
\end{tikzcd}$$

□

2.15. **Comparison with coarse moduli spaces.**

2.16. Consider an algebraic space X/k, and a correspondence

$$d : \mathscr{C} \to X \times X$$

with $\mathscr{C}$ a Deligne-Mumford stack. Assume further that the orders of the stabilizer groups are all invertible in Λ. Let $\pi : \mathscr{C} \to C$ be the coarse moduli space, so that d factors through a correspondence

$$c : C \to X \times X.$$

Note that since π is proper, we have for any $F \in D^b_{ctf}(C)$ a natural map

$$F \to \pi_*\pi^*F \simeq \pi_!\pi^*F,$$

which is an isomorphism by [15, 5.11]. We therefore obtain an isomorphism

$$c_{2!}c_1^*K \simeq c_{2!}\pi_!\pi^*c_1^*K \simeq d_{2!}d_1^*K.$$

which defines a bijection

$$\tau : \{d\text{-structures on } K\} \to \{c\text{-structures on } K\}. \tag{2.16.1}$$

Lemma 2.17. *Let $K \in D^b_{ctf}(X)$, and let $v : d_1^*K \to d_2^!K$ be a d-structure. Let $u : c_1^*K \to c_2^!K$ be the c-structure $\tau(v)$. Then if $\pi' : \mathrm{Fix}(d) \to \mathrm{Fix}(c)$ denotes the projection we have $\mathrm{Tr}_c(v) = \pi'_*\mathrm{Tr}_d(u)$ in $H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)})$.*

Proof. This follows immediately from 2.6. □

2.18. **Variant local terms.**

2.19. One can define local terms in a more general setting than the one of 2.2 . However, as explained in [7, III, 5.2.10] and reviewed here, the calculation of these more general local terms reduce to the calculation of the local terms as defined in 2.2

2.20. Let X and Y be separated finite type Deligne-Mumford stacks over k. Let $F \in D^b_c(X, \Lambda)$ and $G \in D^b_c(Y, \Lambda)$ be complexes. Fix morphisms $c : C \to X \times Y$ and $d : D \to Y \times X$ and suppose given maps

$$u : c_1^* F \to c_2^! G, \quad v : d_1^* G \to d_2^! F.$$

Let $d^t : D \to X \times Y$ be the composition of d with the map $Y \times X \to X \times Y$ switching the factors, and let $\mathscr{F}$ denote the fiber product of the diagram

$$\begin{tikzcd} & D \arrow[d, "d^t"] \\ C \arrow[r, "c"] & X \times Y, \end{tikzcd}$$

so we have a cartesian square

$$\begin{tikzcd} \mathscr{F} \arrow[r, "c'"] \arrow[d, "\delta"] & D \arrow[d, "d^t"] \\ C \arrow[r, "c"] & X \times Y. \end{tikzcd}$$

As in 2.2.1 the map u corresponds to an element of $H^0(C, c^!(D(F) \boxtimes G))$, or equivalently by duality to a morphism

$$c^*(F \boxtimes D(G)) \to \Omega_C$$

which we again denote by u. Applying $\delta^!$ to this map we get a morphism

$$c'^* d^{t!}(F \boxtimes D(G)) \xrightarrow{bc} \delta^! c^*(F \boxtimes D(G)) \xrightarrow{u} \delta^! \Omega_C \simeq \Omega_{\mathscr{F}}.$$

Taking global sections we get a morphism

$$\langle u, - \rangle : \mathrm{Hom}(d_1^* G, d_2^! F) \simeq H^0(D, d^{t!}(F \boxtimes D(G))) \to H^0(\mathscr{F}, \Omega_{\mathscr{F}}).$$

In particular, evaluating on v we get an element

$$\langle u, v \rangle \in H^0(\mathscr{F}, \Omega_{\mathscr{F}}).$$

2.21. Let E denote $C \times_{c_2, Y, d_1} D$, so we have a commutative diagram

$$\begin{tikzcd} & & E \arrow[dl, "\tilde{d}_1"'] \arrow[dr, "\tilde{c}_2"] \arrow[dll, bend right, "e_1"'] \arrow[drr, bend left, "e_2"] & & \\ & C \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & & D \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\ X & & Y & & X. \end{tikzcd}$$

Let $vu : e_1^* F \to e_2^! F$ denote the composite

$$e_1^* F \simeq \tilde{d}_1^* c_1^* F \xrightarrow{u} \tilde{d}_1^* c_2^! G \xrightarrow{bc} \tilde{c}_2^! d_1^* G \xrightarrow{v} \tilde{c}_2^! d_2^! F \simeq e_2^! F.$$

Observe also that there is a canonical isomorphism $\mathrm{Fix}(e) \simeq \mathscr{F}$, so we also get an element

$$\mathrm{Tr}(F, vu) \in H^0(\mathscr{F}, \Omega_{\mathscr{F}}).$$

Proposition 2.22. *We have* $\mathrm{Tr}(F, vu) = \langle u, v\rangle$.

Proof. See [7, III, 5.2.10]. □

3. Naive local terms

Throughout this section we work over an algebraically closed field k, and with a coefficient ring Λ as in 1.11.

3.1. **Definition of naive local terms and basic properties.**

3.2. Let

$$c : C \to X \times X$$

be a correspondence, with C and X algebraic spaces separated and of finite type over k, and assume c_2 is quasi-finite. Let $K \in D^b_{ctf}(X)$ be a complex, and let

$$u : c_1^* K \to c_2^! K$$

be a c-structure on K. For any point $y \in \mathrm{Fix}(c)(k)$ with image $x \in X(k)$ (under either c_1 or c_2), we define the *naive local term*, denoted

$$\mathrm{lt}^{\mathrm{na}}_y(K, u) \in \Lambda,$$

as the trace of the composite map

$$K_x \xrightarrow{\simeq} (c_1^*K)_y \xrightarrow{u} (c_2^!K)_y \hookrightarrow \oplus_{z \in c_2^{-1}(x)} (c_2^!K)_z \xrightarrow{\simeq} (c_{2!}c_2^!K)_x \xrightarrow{c_{2!}c_2^! \to \mathrm{id}} K_x, \tag{3.2.1}$$

where the isomorphism

$$\oplus_{z \in c_2^{-1}(x)} (c_2^!K)_z \simeq (c_{2!}c_2^!K)_x$$

is given by the proper base change theorem.

3.3. **Compatibility with pushforward along closed embeddings.**

3.4. Let $i : Z \hookrightarrow X$ be a closed embedding, and let

$$c_Z : C_Z \to Z \times Z$$

be the correspondence on Z obtained by taking the fiber product of the diagram

$$\begin{tikzcd} & Z \times Z \arrow[d, "i \times i"] \\ C \arrow[r, "c"] & X \times X. \end{tikzcd}$$

Note that $c_{Z,2} : C_Z \to Z$ is again quasi-finite. Also note that if C_i denotes the fiber product $C \times_{c_i,X,i} Z$, then we have a cartesian diagram

$$\begin{tikzcd} C_Z \arrow[r, hook, "\alpha"] \arrow[d, hook, "\beta"'] & C_2 \arrow[d, hook, "\gamma"] \\ C_1 \arrow[r, hook, "\delta"] & C, \end{tikzcd} \tag{3.4.1}$$

where all the morphisms are closed embeddings.

Now let $K \in D^b_{ctf}(Z)$ be a complex, and let $u : c^*_{Z,1}K \to c^!_{Z,2}K$ be a c_Z-structure on K. We can then define a c-structure

$$i_*u : c_1^*(i_*K) \to c_2^!(i_*K)$$

as follows.

Note that if

$$\epsilon : C_Z \hookrightarrow C$$

is the inclusion (a closed embedding), then there is a natural map of functors

$$c_1^* i_* \to \epsilon_* c^*_{Z1} \tag{3.4.2}$$

obtained by adjunction from the isomorphism

$$\epsilon^* c_1^* i_* \simeq c^*_{Z1} i^* i_* \simeq c^*_{Z1}.$$

We therefore get a morphism of functors

$$c_{2!}c_1^* i_* \longrightarrow c_{2!}\epsilon_* c^*_{Z1} \xrightarrow{\epsilon_* \simeq \epsilon_!} (c_2\epsilon)_! c^*_{Z1} \xrightarrow{\simeq} i_* c_{Z2!} c^*_{Z1}. \tag{3.4.3}$$

We define

$$i_*u : c_{2!}c_1^* i_* K \to i_*K$$

to be the composite of this map $c_{2!}c_1^* i_* K \to i_* c_{Z2!} c^*_{Z1} K$ with

$$u : i_* c_{Z2!} c^*_{Z1} K \to i_* K.$$

Observe that there is a natural cartesian diagram

$$\begin{tikzcd} \mathrm{Fix}(c_Z) \arrow[r, hook] \arrow[d] & \mathrm{Fix}(c) \arrow[d] \\ Z \arrow[r, hook, "i"] & X. \end{tikzcd} \tag{3.4.4}$$

Proposition 3.5. *Let $y \in \mathrm{Fix}(c)(k)$ be a fixed point. Then*

$$\mathrm{lt}^{\mathrm{na}}_y(i_*K, i_*u) = \mathrm{lt}^{\mathrm{na}}_y(K, u)$$

if $y \in \mathrm{Fix}(c_Z)(k)$, and

$$\mathrm{lt}^{\mathrm{na}}_y(i_*K, i_*u) = 0$$

otherwise.

Proof. Let $x \in X(k)$ be the image of y.

Since the square (3.4.4) is cartesian, it is immediate that $\mathrm{lt}_y^{\mathrm{na}}(i_*K, i_*u) = 0$ for $y \notin \mathrm{Fix}(c_Z)$, for then $(i_*K)_x = 0$.

So assume $y \in \mathrm{Fix}(c_Z)$. Taking the stalk at x we obtain from (3.4.3) a map

$$\Psi : \oplus_{w \in c_2^{-1}(x)}(c_1^* i_* K)_w \to \oplus_{t \in c_{Z2}^{-1}(x)} K_t$$

defined by the commutativity of the diagram

$$\begin{tikzcd}
(c_{2!}c_1^*i_*K)_x \arrow[r] \arrow[d, "\simeq"] & (c_{2!}\epsilon_* c_{Z1}^*K)_x \arrow[r, "\epsilon_* \simeq \epsilon_!"] & ((c_2\epsilon)_! c_{Z1}^*K)_x \arrow[r, "\simeq"] & (i_* c_{Z2!} c_{Z1}^* K)_x \arrow[d, "\simeq"] \\
\oplus_{w \in c_2^{-1}(x)}(c_1^* i_* K)_w \arrow[rrr, "\Psi"] & & & \oplus_{t \in c_{Z2}^{-1}(x)} K_t.
\end{tikzcd}$$

It follows from the functoriality of the base change isomorphism, that the map Ψ is the projection induced by the natural inclusion

$$c_{Z2}^{-1}(x) \subset c_2^{-1}(x).$$

Since the local term $\mathrm{lt}_y^{\mathrm{na}}(i_*K, i_*u)$ is by definition the trace of the composite

$$\begin{tikzcd}
K_x \arrow[r, hook, "y"] & \oplus_{w \in c_2^{-1}(x)}(c_1^* i_* K)_w \arrow[r, "\Psi"] & \oplus_{t \in c_{Z2}^{-1}(x)} K_t \arrow[r, "u"] & K_x
\end{tikzcd}$$

this implies the equality

$$\mathrm{lt}_y^{\mathrm{na}}(i_*K, i_*u) = \mathrm{lt}_y^{\mathrm{na}}(K, u).$$

□

Remark 3.6. By 2.13 any c-structure

$$v : c_{2!}c_1^*(i_*K) \to i_*K$$

is of the form i_*u for some c_Z-structure u on K.

3.7. **Compatibility with modification of C.**

3.8. Consider a commutative diagram of algebraic spaces of finite type over k

$$\begin{tikzcd}
& \widetilde{C} \arrow[d, "q"] \arrow[ddl, "d_1"'] \arrow[ddr, "d_2"] & \\
& C \arrow[dl, "c_1"] \arrow[dr, "c_2"'] & \\
X & & X,
\end{tikzcd}$$

where q is proper, and c_2 and d_2 are quasi-finite. Let $K \in D^b_{ctf}(X)$ be a complex and let

$$u : d_{2!}d_1^*K \to K$$

be a d-structure. We then obtain a c-structure

$$q_*u : c_{2!}c_1^*K \to K$$

on K from the composite

$$\begin{array}{rcll} c_{2!}c_1^*K & \to & c_{2!}q_*q^*c_1^*K & (\mathrm{id} \to q_*q^*) \\ & \simeq & c_{2!}q_!q^*c_1^*K & (q_! \simeq q_*) \\ & \simeq & d_{2!}d_1^*K & \\ & \stackrel{u}{\to} & K. & \end{array}$$

Let

$$\gamma : \mathrm{Fix}(d) \to \mathrm{Fix}(c)$$

be the natural map. Note that since d_2 is quasi-finite, this map γ is also quasi-finite.

Proposition 3.9. *For any $y \in \mathrm{Fix}(c)(k)$ we have*

$$\mathrm{lt}^{\mathrm{na}}_y(K, q_*u) = \sum_{z \in \gamma^{-1}(y)} \mathrm{lt}^{\mathrm{na}}_z(K, u).$$

Proof. Let $x \in X(k)$ be the image of y. The result then follows from noting that the diagram

$$\begin{tikzcd} K_x \arrow[r, hook, "y"] & \oplus_{z \in c_2^{-1}(x)}(c_1^*K)_z \arrow[r, "q^*"] \arrow[d, "\simeq"] & \oplus_{w \in d_2^{-1}(x)}(d_1^*K)_w \arrow[d, "\simeq"] \\ & (c_{2!}c_1^*K)_x \arrow[r, "q^*"] \arrow[dr, "q_*u"'] & (d_{2!}d_1^*K)_x \arrow[d, "u"] \\ & & K_x \end{tikzcd}$$

commutes. □

3.10. Compatibility with composition.

3.11. If $c : C \to X \times X$ and $d : D \to X \times X$ are correspondences, with C, D, and X algebraic spaces of finite type over k, then their composition $e : E \to X \times X$ is defined by setting $E := C \times_{c_2,X,d_1} D$ and $e = (c_1 \circ \mathrm{pr}_1, d_2 \circ \mathrm{pr}_2)$. As mentioned in 1.3, given $K \in D^b_{ctf}(X)$, a c-structure $u : c_1^*K \to c_2^!K$, and a d-structure $v : d_1^*K \to d_2^!K$, we get an e-structure $v \circ u : e_1^*K \to e_2^!K$, called the *composition of u and v*. Note that if c_2 and d_2 are both quasi-finite, then so is e_2.

The composition E is a subspace of $C \times D$, and the composite

$$\mathrm{Fix}(c) \times_X \mathrm{Fix}(d) \hookrightarrow \mathrm{Fix}(c) \times \mathrm{Fix}(d) \hookrightarrow C \times D$$

factors through $\mathrm{Fix}(e) \subset E$ defining a morphism

$$\mathrm{Fix}(c) \times_X \mathrm{Fix}(d) \hookrightarrow \mathrm{Fix}(e).$$

Proposition 3.12. *Let $(z, w) \in \mathrm{Fix}(c) \times_X \mathrm{Fix}(d)$ be a k-point mapping to $x \in X(k)$, and let $u_z : K_x \to K_x$ and $v_w : K_x \to K_x$ be the endomorphisms obtained from u and v respectively. Then the naive local term $\mathrm{lt}^{\mathrm{na}}_{(z,w)}(K, v \circ u)$ is equal to the trace of the composition $v_w \circ u_z : K_x \to K_x$.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
 & & E \arrow[dl, "\tilde{d}_1"'] \arrow[dr, "\tilde{c}_2"] & & \\
 & C \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & & D \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\
X & & X & & X,
\end{tikzcd}$$

and to prove the proposition it suffices to note that the diagram

$$\begin{tikzcd}
K_x \arrow[r, "c_1^*"] \arrow[drr, bend right, "u_z"'] & (c_1^*K)_z \arrow[r, "\tilde{d}_1^*"] \arrow[d, "u"] & (\tilde{d}_1^* c_1^* K)_{(z,w)} \arrow[r, "v \circ u"] & (\tilde{c}_2^! d_2^! K)_{(z,w)} \arrow[dr, "\mathrm{tr}_{\tilde{c}_2}"'] \arrow[drr, "\mathrm{tr}_{e_2}"] & & \\
 & (c_2^! K)_z \arrow[r, "\mathrm{tr}_{c_2}"] & K_x \arrow[r, "d_1^*"] \arrow[rrr, bend right, "v_w"'] & (d_1^* K)_w \arrow[r, "v"] & (d_2^! K)_w \arrow[r, "\mathrm{tr}_{d_2}"'] & K_x
\end{tikzcd}$$

commutes. ☐

3.13. **Naive local terms for Deligne-Mumford stacks.**

3.14. One can also define naive local terms for Deligne-Mumford stacks. Correspondences involving Deligne-Mumford stacks arise naturally when using devissage arguments and alterations.

Let

$$c : C \to X \times X$$

be a correspondence with X and C separated Deligne-Mumford stacks of finite type over k, and as in the case of spaces assume that c_2 is quasi-finite. We can then consider the *fixed point stack*, denoted $\mathrm{Fix}(c)$, which is defined to be the fiber product of the diagram

$$\begin{tikzcd}
 & C \arrow[d, "c"] \\
X \arrow[r, "\Delta"] & X \times X.
\end{tikzcd}$$

Note that $\mathrm{Fix}(c)$ is again a Deligne-Mumford stack. The category $\mathrm{Fix}(c)(k)$ is the category of pairs

$$\lambda = (y \in C(k), \sigma : c_1(y) \to c_2(y)),$$

where $y \in C(k)$ is an object, and σ is an isomorphism in $X(k)$.

As before let $K \in D^b_{ctf}(X)$ be a complex of Λ-modules and let

$$u : c_1^* K \to c_2^! K$$

be a c-structure. For $\lambda = (y, \sigma) \in \mathrm{Fix}(c)(k)$ we will consider two different (though closely related) notions of naive local terms

$$\widetilde{\mathrm{lt}^{\mathrm{na}}}_{\lambda}(K, u) \quad \text{and} \quad \mathrm{lt}^{\mathrm{na}}_{\lambda}(K, u)$$

defined as follows. Let $x \in X(k)$ denote $c_1(y)$. As before we have a map

$$K_x \xrightarrow{\simeq} (c_1^* K)_y \xrightarrow{u} (c_2^! K)_y, \tag{3.14.1}$$

so we need to define a map

$$(c_2^! K)_y \to K_x.$$

For this note that again by the proper base change theorem (in the stack case this is [12, 0.1.1]) we have

$$(c_{2!} c_2^! K)_x \simeq R\Gamma_c(C_x^{(2)}, c_2^! K),$$

where

$$C_x^{(2)} := C \times_{c_2, X, x} \mathrm{Spec}(k).$$

Since c_2 is quasi-finite, the stack $C^{(2)}_{x,\mathrm{red}}$ is isomorphic to a finite disjoint union of stacks of the form BH, where H is a finite group. In particular, the point y defines an open and closed embedding

$$B\mathrm{Aut}(y/x) \hookrightarrow C^{(2)}_{x,\mathrm{red}},$$

where

$$\mathrm{Aut}(y/x) := \mathrm{Ker}(\mathrm{Aut}(y) \xrightarrow{c_{2*}} \mathrm{Aut}(x)),$$

where we have used σ to identify $c_2(y)$ with x.

Let

$$\delta : \mathrm{Spec}(k) \to B\mathrm{Aut}(y/x)$$

be the natural projection. Then δ is finite and étale, so we have an adjunction map

$$\delta_! \delta^* \xrightarrow{\delta^* \simeq \delta^!} \delta_! \delta^! \longrightarrow \mathrm{id}.$$

This map defines a morphism

$$(c_2^! K)_y \simeq R\Gamma_c(\mathrm{Spec}(k), \delta^*(c_2^! K)|_{B\mathrm{Aut}(y/x)}) \to R\Gamma_c(C_x^{(2)}, c_2^! K) \simeq (c_{2!} c_2^! K)_x,$$

and hence, by composing (3.14.1) with the composite

$$(c_2^! K)_y \to (c_{2!} c_2^! K)_x \to K_x,$$

we get the desired endomorphism

$$K_x \to K_x.$$

We define $\widetilde{\mathrm{lt}^{\mathrm{na}}}_{\lambda}(K, u) \in \Lambda$ to be the trace of this map.

It is convenient to also consider the group $\mathrm{Aut}(\lambda)$, which is the set of automorphisms $\alpha : y \to y$ in $C(k)$ for which the diagram in $X(k)$

$$\begin{tikzcd} c_1(y) \arrow[r, "c_1(\alpha)"] \arrow[d, "\sigma"] & c_1(y) \arrow[d, "\sigma"] \\ c_2(y) \arrow[r, "c_2(\alpha)"] & c_2(y) \end{tikzcd}$$

commutes. In the case of $\Lambda = \overline{\mathbb{Q}}_\ell$, we then define

$$\mathrm{lt}^{\mathrm{na}}_\lambda(K,u) := \frac{|\mathrm{Aut}(y)(k)|}{|\mathrm{Aut}(c_1(y))(k)|\cdot|\mathrm{Aut}(\lambda)|}\widetilde{\mathrm{lt}^{\mathrm{na}}}_\lambda(K,u).$$

Remark 3.15. Note that the preceding paragraph shows that for any quasi-finite morphism $f : X \to Y$ of Deligne-Mumford stacks over k, $x \in X(k)$, and $K \in D^b_{ctf}(Y)$, there is a canonical map

$$\tau_{K,x} : (f^!K)_x \to K_{f(x)}.$$

Example 3.16. To get a sense for the above definition of local terms for Deligne-Mumford stacks consider the following special case. Let G and H be two finite groups, and let

$$\rho_1, \rho_2 : H \to G$$

be two homomorphisms. This gives rise to a correspondence

$$\begin{tikzcd} & BH \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ BG & & BG. \end{tikzcd}$$

Let V be a finite-dimensional representation of G over $\overline{\mathbb{Q}}_\ell$, and let ρ_i^*V denote the pullback of V to an H-representation via ρ_i (so ρ_i^*V has the same underlying vector space as V with H-action through ρ_i). A c-structure on the sheaf on BG associated to V is then equivalent to a morphism of H-representations

$$u : \rho_1^*V \to \rho_2^*V.$$

Let V_G denote the coinvariants of V. We then get an endomorphism

$$\bar{u} : V_G \to V_G$$

from the composite

$$V_G \xrightarrow{\simeq} V^G \hookrightarrow \rho_1^*V \xrightarrow{u} \rho_2^*V \twoheadrightarrow (\rho_2^*V)_H \twoheadrightarrow V_G.$$

For $g \in G$ define

$$H_g := \{h \in H | \rho_1(h)g\rho_2(h)^{-1} = g\},$$

and set

$$L_g := \frac{1}{|H_g|}\mathrm{tr}(V \xrightarrow{g} V \xrightarrow{u} V).$$

Define an equivalence relation $\sim$ on G by declaring that $g \sim g'$ if there exists $h \in H$ such that $g' = \rho_1(h)g\rho_2(h)^{-1}$. Then L_g depends only on the equivalence class of g. Indeed if $g' = \rho_1(h)g\rho_2(h)^{-1}$, then we have a commutative diagram

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "\rho_2(h)"] & V \arrow[r, "u"] \arrow[d, "\rho_1(h)"] & V \arrow[d, "\rho_2(h)"] \\ V \arrow[r, "g'"] & V \arrow[r, "u"] & V, \end{tikzcd}$$

where the right square commutes since u is a map of H-representations. Therefore

$$\mathrm{tr}(V \xrightarrow{g} V \xrightarrow{u} V) = \mathrm{tr}(V \xrightarrow{g'} V \xrightarrow{u} V).$$

Also we have a bijection

$$H_g \to H_{g'}, \quad \tilde{h} \mapsto h\tilde{h}h^{-1}.$$

We can therefore unambiguously write $L_{[g]}$ for a class $[g] \in G/\sim$.

Let

$$T : V \to V$$

denote the map

$$v \mapsto \frac{1}{|G|}\sum_{g \in G} g \cdot v.$$

Then T factors as

$$V \twoheadrightarrow V_G \xrightarrow{\simeq} V^G \hookrightarrow V.$$

It follows that we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(T) \arrow[r] \arrow[d, "0"] & V \arrow[r] \arrow[d, "u\circ T"] & V_G \arrow[r] \arrow[d, "\bar{u}"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(T) \arrow[r] & V \arrow[r] & V_G \arrow[r] & 0,
\end{tikzcd}$$

and therefore

$$\mathrm{tr}(\bar{u}|V_G) = \frac{1}{|G|}\sum_{g \in G} \mathrm{tr}(V \xrightarrow{g} V \xrightarrow{u} V).$$

On the other hand, by the standard orbit decomposition formula for the action of H on G given by

$$h * g := \rho_1(h) g \rho_2(h)^{-1},$$

we have

$$|H| = |H_g| \cdot |[g]|.$$

We conclude that

$$\mathrm{tr}(\bar{u}|V_G) = \frac{|H|}{|G|}\sum_{[g] \in G/\sim} L_{[g]}.$$

In terms of the correspondence c, this is equivalent to the formula

$$\mathrm{tr}(u|R\Gamma_c(BG, \mathcal{V})) = \sum_{[\lambda] \in |\mathrm{Fix}(c)(k)|} \mathrm{lt}^{\mathrm{na}}_{\lambda}(\mathcal{V}, u),$$

where $\mathcal{V}$ denotes the sheaf on BG corresponding to V, and the sum on the right is over isomorphism classes of objects in $\mathrm{Fix}(c)(k)$.

3.17. **Change of coefficient rings.**

3.18. Let $\rho : \Lambda \to \Lambda'$ be a morphism of coefficient rings (the ones we will consider are $\mathbb{Z}_\ell \hookrightarrow \overline{\mathbb{Q}}_\ell$ or the reduction maps $\mathbb{Z}_\ell \to \mathbb{Z}/(l^n)$, $n \geq 1$).

Let

$$c : C \to X \times X$$

be a correspondence with C and X Deligne-Mumford stacks, and the maps c_1 and c_2 quasi-finite. Let $K \in D^b_{ctf}(X, \Lambda)$ be a complex with a c-structure

$$u : c_{2!}c_1^* K \to K.$$

We have a natural isomorphism [1, XVII.5.2.9]

$$c_{2!}c_1^*(K \otimes^{\mathbb{L}}_{\Lambda} \Lambda') \simeq (c_{2!}c_1^*K) \otimes^{\mathbb{L}}_{\Lambda} \Lambda',$$

so u induces a c-structure

$$u' : c_{2!}c_1^*K' \to K'$$

on $K' := K \otimes^{\mathbb{L}}_{\Lambda} \Lambda'$.

Proposition 3.19. *For any fixed point $\lambda \in \mathrm{Fix}(c)(k)$ we have an equality in Λ'*

$$\rho(\widetilde{\mathrm{lt}^{\mathrm{na}}}_{\lambda}(K, u)) = \widetilde{\mathrm{lt}^{\mathrm{na}}}_{\lambda}(K', u').$$

Proof. This is immediate from the definitions. □

3.20. **Compatibility with tensor product.**

3.21. Let

$$c : C \to X \times X$$

be a correspondence, with X and C Deligne-Mumford stacks of finite type over k, and the maps c_1 and c_2 quasi-finite. Let $K, L \in D^b_{ctf}(X, \Lambda)$ be two complexes and suppose given c-structures

$$u : c_{2!}c_1^*K \to K, \quad v : c_{2!}c_1^*L \to L.$$

Recall that there is a natural map (see for example [1, XVII, §5.4])

$$(c_{2!}c_1^*K) \otimes^{\mathbb{L}} (c_{2!}c_1^*L) \to c_{2!}((c_1^*K) \otimes^{\mathbb{L}} (c_1^*L)). \tag{3.21.1}$$

Suppose given a c-structure

$$w : c_{2!}c_1^*(K \otimes^{\mathbb{L}} L) \to K \otimes^{\mathbb{L}} L$$

such that the diagram

$$\begin{tikzcd}
c_{2!}c_1^*(K \otimes^{\mathbb{L}} L) \arrow[r, "w"] & K \otimes^{\mathbb{L}} L \\
c_{2!}((c_1^*K) \otimes^{\mathbb{L}} (c_1^*L)) \arrow[u, "\simeq"] & (c_{2!}c_1^*K) \otimes^{\mathbb{L}} (c_{2!}c_1^*L) \arrow[l, "(3.21.1)"'] \arrow[u, "u\otimes v"']
\end{tikzcd}$$

commutes.

Lemma 3.22. *For any $\lambda \in \mathrm{Fix}(c)(k)$, we have*

$$\widetilde{\mathrm{lt}^{\mathrm{na}}}(K \otimes^{\mathbb{L}} L, w) = \widetilde{\mathrm{lt}^{\mathrm{na}}}(K, u) \cdot \widetilde{\mathrm{lt}^{\mathrm{na}}}(L, v).$$

Proof. This is immediate from the definitions. □

3.23. **Comparison with coarse moduli spaces.** For the remainder of this section, we prove some results in the case when $\Lambda = \overline{\mathbb{Q}}_\ell$.

Consider again the setup of 2.15

Lemma 3.24. *Let $K \in D^b_c(X)$, and let $v : d_1^*K \to d_2^!K$ be a d-structure. Let $u : c_1^*K \to c_2^!K$ be the c-structure $\tau(v)$, and let $y \in \mathrm{Fix}(d)(k)$ be a fixed point with image $\bar{y} \in \mathrm{Fix}(c)(k)$. Then*

$$\mathrm{lt}^{\mathrm{na}}_{\bar{y}}(K, u) = \mathrm{lt}^{\mathrm{na}}_{y}(K, v).$$

Moreover,

$$\mathrm{lt}^{\mathrm{na}}_{\bar{y}}(K, u) = \widetilde{\mathrm{lt}^{\mathrm{na}}}_{\bar{y}}(K, u), \quad \mathrm{lt}^{\mathrm{na}}_{y}(K, v) = \widetilde{\mathrm{lt}^{\mathrm{na}}}_{y}(K, v).$$

Proof. Let $x \in X(k)$ be the image of y. By definition of the map τ we have a commutative diagram

(3.24.1)
$$\begin{tikzcd} d_{2!}d_1^*K \arrow[d, "v"] & c_{2!}\pi_*\pi^*c_1^*K \arrow[l, "\pi_*\simeq\pi_!"'] \\ K & c_{2!}c_1^*K. \arrow[l, "u"] \arrow[u, "\mathrm{id}\to\pi_*\pi^*"'] \end{tikzcd}$$

Let $C'^{(2)}_x$ denote $c_2^{-1}(x)$, and let $C^{(2)}_x$ denote the fiber product

$$C^{(2)}_x := C \times_{d_2, X, x} \mathrm{Spec}(k).$$

Consider the diagram

$$\begin{tikzcd} (d_1^*K)_y \arrow[r] & R\Gamma_c(C^{(2)}_x, d_1^*K) \arrow[dr, "v"] & \\ K_x \arrow[u, "d_1^*"] \arrow[d, "c_2^*"'] & R\Gamma_c(C'^{(2)}_x, \pi_*d_1^*K) \arrow[u, "\pi_*\simeq\pi_!"] & K_x \\ (c_1^*K)_{\bar{y}} \arrow[r, hook] & R\Gamma_c(C'^{(2)}_x, c_1^*K). \arrow[u, "\mathrm{id}\to\pi_*\pi^*"] \arrow[ur, "u"'] & \end{tikzcd}$$

The left pentagon commutes by the definition of the horizontal maps, and the right part of the diagram commutes by the commutativity of (3.24.1). From this and the definitions of $\mathrm{lt}^{\mathrm{na}}_{\bar{y}}(K, u)$ and $\mathrm{lt}^{\mathrm{na}}_{y}(K, v)$ the first part of lemma follows.

The second statement in the lemma is immediate from the definitions. $\square$

3.25. **Pulling back to closed substacks.**

3.26. Consider a correspondence

$$c : C \to X \times X,$$

with X and C separated finite type Deligne-Mumford stacks over k, and assume that c_1 and c_2 are quasi-finite. Let $i : Z \hookrightarrow X$ be a closed substack such that

$$c_1^{-1}(Z)_{\mathrm{red}} = c_2^{-1}(Z)_{\mathrm{red}},$$

and let $K \in D^b_c(X)$ be a complex with a c-structure

$$u : c_1^*K \to c_2^!K.$$

Let C_Z denote $c_1^{-1}(Z)_{\mathrm{red}}$ so we have a correspondence

$$c_Z : C_Z \to Z \times Z.$$

We then get a c_Z-structure i^*u on i^*K as follows. Let

$$\tilde{i} : C_Z \hookrightarrow C$$

be the inclusion, and note that for $s = 1, 2$ the map

$$C_Z \to C \times_{c_s, X, i} Z$$

is a closed embedding defined by a nilpotent ideal. We therefore get a map

$$c_{Z1}^* i^* K \simeq \tilde{i}^* c_1^* K \xrightarrow{u} \tilde{i}^* c_2^! K \xrightarrow{bc} c_{Z2}^! i^* K, \tag{3.26.1}$$

where the map $\tilde{i}^* c_2^! \to c_{Z2}^! i^*$ is adjoint to the composition of the base change isomorphism

$$c_{Z2!} \tilde{i}^* c_2^! K \simeq i^* c_{2!} c_2^! K$$

with the adjunction map

$$i^* c_{2!} c_2^! K \to i^* K.$$

We denote the map (3.26.1) by i^*u.

Note also that there is a natural closed embedding

$$\gamma : \mathrm{Fix}(c_Z) \hookrightarrow \mathrm{Fix}(c).$$

Lemma 3.27. *For every $\lambda \in \mathrm{Fix}(c_Z)$ we have*

$$\mathrm{lt}_\lambda^{\mathrm{na}}(i^*K, i^*u) = \mathrm{lt}_{\gamma(\lambda)}^{\mathrm{na}}(K, u).$$

Proof. We can without loss of generality replace X by an étale covering, and may therefore assume that X is a scheme. Furthermore, using 3.24 we can replace C by its coarse moduli space so we may also assume that C is a scheme (since an algebraic space quasi-finite over a scheme is a scheme). This reduces the proof to the case of schemes in which case the result is immediate from the definitions. □

Remark 3.28. In fact the diagram

$$\begin{array}{ccc} c_1^*K & \xrightarrow{\mathrm{id} \to i_*i^*} & c_1^* i_* i^* K \\ \downarrow u & & \downarrow i_*i^*u \\ c_2^!K & \xrightarrow{\mathrm{id} \to i_*i^*} & c_2^! i_* i^* K \end{array}$$

commutes. This follows from noting that the diagram

$$\begin{tikzcd}
c_1^*K \arrow[r, "\mathrm{id}\to i_*i^*"] \arrow[d, "u"] \arrow[ddr, "\mathrm{id}\to\tilde{i}_*\tilde{i}^*"] & c_1^*i_*i^*K \arrow[d, "bc"] \\
c_2^!K \arrow[d, "\mathrm{id}\to i_*i^*"] & \tilde{i}_*c_{Z1}^*i^*K \arrow[d, "\simeq"] \\
c_2^!i_*i^*K \arrow[d, "bc"] & \tilde{i}_*\tilde{i}^*c_1^*K \arrow[d, "u"] \\
\tilde{i}_*c_{Z2}^!i^*K & \tilde{i}_*\tilde{i}^*c_2^!K \arrow[l, "bc"']
\end{tikzcd}$$

commutes, which is immediate.

4. Compatible systems

Throughout this section, we work over a finite field $\mathbb{F}_q$ with q elements and characteristic p (so $q = p^a$ for some a). We fix an algebraic closure $\mathbb{F}_q \hookrightarrow k$. We usually denote a scheme or stack over $\mathbb{F}_q$ with a subscript "0", and its base change to k by the corresponding unadorned symbol (so X_0 denotes a scheme over $\mathbb{F}_q$ and X denotes the base change $X_0 \otimes_{\mathbb{F}_q} k$).

4.1. **Twisting by Frobenius.**

4.2. Let X_0 be a separated Deligne-Mumford stack of finite type over $\mathbb{F}_q$, and let

$$F_X : X \to X$$

denote the relative Frobenius of X/k.

Definition 4.3 (See for example [4, 1.1.10])**.** An *ℓ-adic Weil complex* on X (or just *Weil complex* if the reference to ℓ is clear) is a pair (K, ϵ), where $K \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ and

$$\epsilon : F_X^*K \to K$$

is an isomorphism.

Let

$$c : C_0 \to X_0 \times X_0$$

be a correspondence over $\mathbb{F}_q$, with C_0 and X_0 Deligne-Mumford stacks.

Definition 4.4. A *c-structure* on a Weil complex (K, ϵ) is a morphism

$$u : c_1^*K \to c_2^!K$$

such that the diagram

(4.4.1)
$$\begin{tikzcd} c_1^*F_X^*K \arrow[r, "\epsilon"] \arrow[d, "\simeq"] & c_1^*K \arrow[d, "u"] \\ F_C^*c_1^*K \arrow[d, "u"] & c_2^!K \\ F_C^*c_2^!K \arrow[r] & c_2^!F_X^*K \arrow[u, "\epsilon"] \end{tikzcd}$$

commutes, where the bottom horizontal arrow is obtained from the natural isomorphisms $F_X^* \simeq F_X^!$ and $F_C^* \simeq F_C^!$.

Example 4.5. If $K_0 \in D_c^b(X_0, \overline{\mathbb{Q}}_\ell)$ is a complex and $u : c_1^*K_0 \to c_2^!K_0$ is a morphism in $D_c^b(C_0, \overline{\mathbb{Q}}_\ell)$ is a morphism, then the pullback K of K_0 to X has a natural structure of a Weil complex. Indeed if $g : X \to X_0$ denotes the projection then the diagram

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[dr, "g"] & X \arrow[d, "g"] \\ & X_0 \end{tikzcd}$$

commutes and we get an isomorphism $F_X^*K = F_X^*g^*K_0 \simeq g^*K_0 = K$. The pullback of u to C is compatible with this structure in the above sense.

4.6. For natural numbers $n, m \in \mathbb{N}$ we write $c^{(n,m)}$ for the correspondence

$$\begin{tikzcd} & & C_0 \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & & \\ & X_0 \arrow[dl, "F_{X_0}^n"'] & & X_0 \arrow[dr, "F_{X_0}^m"] & \\ X_0 & & & & X_0. \end{tikzcd}$$

If (K, ϵ, u) is an ℓ-adic Weil complex with c-structure, then for every $n, m \in \mathbb{N}$ the complex K has a natural $c^{(n,m)}$-structure $u^{(n,m)}$ defined as the composite

$$c_1^{(n,m)*}K \xrightarrow{\simeq} c_1^*F_X^{n*}K \xrightarrow{\epsilon^n} c_1^*K \xrightarrow{u} c_2^!K \xrightarrow{\epsilon^{-m}} c_2^!F_X^{m*}K \xrightarrow{\simeq} c_2^{(n,m)!}K,$$

where the last isomorphism is obtained from the canonical isomorphism $F_X^! \simeq F_X^*$ (since F_X is a universal homeomorphism).

4.7. Let

$$f : X_0 \to Y_0$$

be a proper morphism between separated Deligne-Mumford stacks over $\mathbb{F}_q$. Let

$$c : C_0 \to X_0 \times X_0$$

be a correspondence, and let

$$d : C_0 \to Y_0 \times Y_0$$

be the composite

$$C_0 \xrightarrow{c} X_0 \times X_0 \xrightarrow{f \times f} Y_0 \times Y_0.$$

Let (K, ϵ, u) be an ℓ-adic Weil sheaf with c-structure on X. We give $f_! K \in D_c^b(Y, \overline{\mathbb{Q}}_\ell)$ the structure of a Weil sheaf with d-structure as follows.

Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[rr, bend left, "F_X"] \arrow[dr, "f"'] & X' \arrow[r, "\pi"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y \arrow[r, "F_Y"] & Y,
\end{tikzcd}$$

where the square is cartesian, and g is the relative Frobenius of X/Y. We then get an isomorphism

$$F_Y^* f_! K \xrightarrow{bc} f'_! \pi^* K \xrightarrow{\mathrm{id} \to g_* g^*} f'_! g_* g^* \pi^* K \xrightarrow{g_* \simeq g_!} f_! F_X^* K \xrightarrow{\epsilon} f_! K,$$

which we denote by $f_! \epsilon$ (here the adjunction map $\mathrm{id} \to g_* g^*$ is an isomorphism since g is a universal homeomorphism).

Define

$$f_! u : d_1^* f_! K \to d_2^! f_! K$$

to be the composite map

$$\begin{array}{rcll}
d_1^* f_! K & \simeq & c_1^* f^* f_* K & (f_! \simeq f_* \text{ since } f \text{ is proper}) \\
& \to & c_1^* K & (f^* f_* \to \mathrm{id}) \\
& \xrightarrow{u} & c_2^! K & \\
& \to & c_2^! f^! f_! K & (\mathrm{id} \to f^! f_!) \\
& \simeq & d_2^! f_! K. &
\end{array}$$

Lemma 4.8. *The data $(f_! K, f_! \epsilon, f_! u)$ is a Weil complex with d-structure.*

Proof. We need to prove that the diagram

$$\begin{tikzcd}
d_1^* F_Y^* f_! K \arrow[r, "f_! \epsilon"] \arrow[d, "\simeq"] & d_1^* f_! K \arrow[d, "f_! u"] \\
F_C^* d_1^* f_! K \arrow[d, "f_! u"] & d_2^! f_! K \\
F_C^* d_2^! f_! K \arrow[r] & d_2^! F_Y^* f_! K \arrow[u, "f_! \epsilon"']
\end{tikzcd}$$

commutes. Expanding out the definitions of $f_!\epsilon$ and $f_!u$, this amounts to showing that the big outside diagram in the following diagram commutes:

(4.8.1)
$$\begin{tikzcd}
c_1^*f^*F_Y^*f_!K \arrow[r, "bc"] \arrow[d, "\simeq"] \arrow[dr] & c_1^*f^*f_!'\pi^*K \arrow[r, "\mathrm{id}\to g_*g^*"] & c_1^*f^*f_!'g_*g^*\pi^*K \arrow[r, "\simeq"] & c_1^*f^*f_!F_X^*K \arrow[d, "\epsilon"] \\
F_C^*c_1^*f^*f_!K \arrow[d, "f_!\simeq f_*"] & c_1^*F_X^*f^*f_!K \arrow[d, "f_!\simeq f_*"] & & c_1^*f^*f_!K \arrow[d, "f_!\simeq f_*"] \\
F_C^*c_1^*f^*f_*K \arrow[d, "f^*f_*\to\mathrm{id}"] & c_1^*F_X^*f^*f_*K \arrow[d, "f^*f_*\to\mathrm{id}"] & & c_1^*f^*f_*K \arrow[d, "f^*f_*\to\mathrm{id}"] \\
F_C^*c_1^*K \arrow[r, "\simeq"] \arrow[d, "u"] & c_1^*F_X^*K \arrow[rr, "\epsilon"] & & c_1^*K \arrow[d, "u"] \\
F_C^*c_2^!K \arrow[d, "\simeq"] & & & c_2^!K \arrow[ddd, "\mathrm{id}\to f^!f_!"] \\
c_2^!F_X^*K \arrow[urrr, "\epsilon"] \arrow[d, "\mathrm{id}\to f^!f_!"] & & & \\
c_2^!F_X^*f^!f_!K \arrow[d, "\simeq"] & & & \\
c_2^!f^!F_Y^*f_!K \arrow[rrr, "f_!\epsilon"] & & & c_2^!f^!f_!K.
\end{tikzcd}$$

The top left inner diagram clearly commutes, and the center inner diagram commutes since (K, ϵ, u) is a Weil complex with c-structure. The commutativity of the top right diagram follows from noting that the diagram

$$\begin{tikzcd}
f^*F_Y^*f_!K \arrow[r, "bc"] \arrow[d, "\simeq"] & f^*f_!'\pi^*K \arrow[r, "\mathrm{id}\to g_*g^*"] \arrow[d, "f_!'\simeq f_*'"] & f^*f_!'g_*g^*\pi^*K \arrow[r, "\simeq"] \arrow[d, "f_!'\simeq f_*'"] & f^*f_!F_X^*K \arrow[d, "f_*\simeq f_!"] \\
F_X^*f^*f_!K \arrow[ur, "bc"] \arrow[d, "f_!\simeq f_*"] & f^*f_*'\pi^*K \arrow[r, "\mathrm{id}\to g_*g^*"] & f^*f_*'g_*g^*\pi^*K \arrow[r, "\simeq"] & f^*f_*F_X^*K \arrow[d, "\epsilon"] \arrow[dll, "f^*f_*\to\mathrm{id}"'] \\
F_X^*f^*f_*K \arrow[ur, "bc"] \arrow[r, "f^*f_*\to\mathrm{id}"] & F_X^*K \arrow[r, "\epsilon"] & K & f^*f_*K \arrow[l, "f^*f_*\to\mathrm{id}"']
\end{tikzcd}$$

commutes, and the commutativity of the bottom inner diagram follows from noting that

$$\begin{tikzcd}
F_X^*K \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}\to f^!f_!"'] \arrow[drrr, "\mathrm{id}\to f^!f_!"] & K \arrow[rr, "\mathrm{id}\to f^!f_!"] & & f^!f_!K \\
F_X^*f^!f_!K \arrow[d, "\simeq"] & & & \\
f^!F_Y^*f_!K \arrow[r, "bc"] & f^!f'_!\pi^*K \arrow[r, "\mathrm{id}\to g_*g^*"] & f^!f'_!g_*g^*\pi^*K \arrow[r, "\simeq"] & f^!f_!F_X^*K \arrow[uu, "\epsilon"]
\end{tikzcd}$$

commutes. ∎

4.9. Assume now that C_0 and X_0 are algebraic spaces. Let $y \in \mathrm{Fix}(c)(k)$ be a fixed point with image $x \in X(k)$ defined over $\mathbb{F}_q$ (so x is obtained from a point $x_0 \in X(\mathbb{F}_q)$). Let

$$f_x : K_x \to K_x$$

denote the automorphism defined by the Weil complex structure, and let

$$u_y : K_x \to K_x$$

denote the endomorphism (3.2.1).

Observe that since x is defined over $\mathbb{F}_q$, for any $n, m \in \mathbb{N}$ we also have $y \in \mathrm{Fix}(c^{(n,m)})$.

Lemma 4.10. *For any $n, m \in \mathbb{N}$ we have*

$$\mathrm{lt}_y^{\mathrm{na}}(K, u^{(n,m)}) = \mathrm{tr}(f_x^{-m}u_yf_x^n|K_x).$$

Proof. The local term $\mathrm{lt}_y^{\mathrm{na}}(K, u^{(n,m)})$ is by definition the trace of the composite

$$K_x \xrightarrow{c_1^{(n,m)*}} (c_1^{(n,m)*}K)_y \hookrightarrow \oplus_{z\in c_2^{(n,m)-1}(x)}(c_1^{(n,m)*}K)_z \xrightarrow{\simeq} (c_{2!}^{(n,m)}c_1^{(n,m)*}K)_x \xrightarrow{u^{(n,m)}} K_x.$$

This sequence of morphisms fits into the following larger commutative diagram

$$\begin{tikzcd}
K_x \arrow[r, "c_1^{(n,m)*}"] \arrow[dd, "f_x^n"] & (c_1^{(n,m)*}K)_y \arrow[r, hook] \arrow[d, "\simeq"] & \oplus_{z\in c_2^{(n,m)-1}(x)}(c_1^{(n,m)*}K)_z \arrow[r, "\simeq"] \arrow[d, "\simeq"] & (c_{2!}^{(n,m)}c_1^{(n,m)*}K)_x \arrow[r, "u^{(n,m)}"] \arrow[d, "\simeq"] & K_x \\
 & (c_1^*F_X^{n*}K)_y \arrow[r, hook] \arrow[d, "\epsilon^n"] & \oplus_{z\in(F_X^mc_2)^{-1}(x)}(c_1^*F_X^{n*}K)_z \arrow[r, "\simeq"] \arrow[d, "\epsilon^n"] & (F_{X!}^mc_{2!}c_1^*F_X^{n*}K)_x \arrow[d, "\epsilon^n"] & K_x \arrow[u, "(f_x^m)^{-1}"'] \\
K_x \arrow[r, "\simeq"] & (c_1^*K)_y \arrow[r, hook] & \oplus_{z\in(F_X^mc_2)^{-1}(x)}(c_1^*K)_z \arrow[r, "\simeq"] & (F_{X!}^mc_{2!}c_1^*K)_x \arrow[r, "u"] & (F_{X*}^mK)_x \arrow[u, "\simeq"']
\end{tikzcd}$$

which implies the lemma, as the composite map $K_x \to K_x$ obtained by going around the left, bottom, and right side of the diagram is $f_x^{-m}u_yf_x^n$. ∎

Lemma 4.11. *The endomorphisms u_y and f_x of K_x commute. In particular, for $n \geq m$ we have*

$$\mathrm{lt}_y^{\mathrm{na}}(K, u^{(n,m)}) = \mathrm{lt}_y^{\mathrm{na}}(K, u^{(n-m,0)}).$$

Proof. Taking the stalk at y of the diagram (4.4.1) we get that the diagram

$$\begin{tikzcd} K_x \arrow[r, "f_x"] \arrow[d, "\simeq"] & K_x \arrow[d, "\simeq"] \\ (c_1^*K)_{F_C(y)} \arrow[d, "u"] & (c_1^*K)_y \arrow[d, "u"] \\ (c_2^!K)_{F(y)} \arrow[r, "\epsilon"] \arrow[d] & (c_2^!K)_y \arrow[d] \\ K_x \arrow[r, "f_x"] & K_x \end{tikzcd}$$

commutes, which implies the lemma. □

4.12. ***I*-systems.**

4.13. Let E be a field of characteristic 0, and let I be a set. Assume given for every $\alpha \in I$ a pair $(\ell_\alpha, \iota_\alpha)$, where ℓ_α is a prime not equal to p, and $\iota_\alpha : E \hookrightarrow \overline{\mathbb{Q}}_{\ell_\alpha}$ is an embedding.

Definition 4.14 ([6, 1.2])**.** Let X_0 be a separated Deligne-Mumford stack of finite type over $\mathbb{F}_q$. An *E-compatible system of Weil complexes indexed by I* (or just E-compatible system if the reference to I is clear) is a collection of Weil complexes $\{(K_\alpha, \epsilon_\alpha)\}_{\alpha\in I}$, where $K_\alpha \in D^b_c(X, \overline{\mathbb{Q}}_{\ell_\alpha})$ and for every geometric point $\bar{x} \to X$ lying over a closed point $x \in X_0$ we have $\mathrm{Tr}(F_x|K_{\alpha,\bar{x}}) \in \iota_\alpha(E)$, and for any two indices $\alpha, \beta \in I$ we have an equality of elements of E

$$\iota_\alpha^{-1}(\mathrm{Tr}(F_x|K_{\alpha,\bar{x}})) = \iota_\beta^{-1}(\mathrm{Tr}(F_x|K_{\beta,\bar{x}})).$$

Remark 4.15. If the K_α are sheaves, then one can also consider the local L-function $L_x((K_\alpha, \epsilon_\alpha), t) := \det(1 - t^{\deg(x)}\epsilon_x, K_{\bar{x}})^{-1} \in \overline{\mathbb{Q}}_\ell[[t]]$. As remarked in [6, §1.2], a compatible system $\{(K_\alpha, \epsilon_\alpha)\}_{\alpha\in I}$ has the (a priori stronger) property that for every x the power series $L_x((K_\alpha, \epsilon_\alpha), t)$ lies in the image of $E[[t]]$ under ι_α and that for any two $\alpha, \beta \in I$ we have

$$\iota_\alpha^{-1}(L_x((K_\alpha, \epsilon_\alpha), t)) = \iota_\beta^{-1}(L_x((K_\beta, \epsilon_\beta), t))$$

in $E[[t]]$.

4.16. Let

$$c : C_0 \to X_0 \times X_0$$

be a correspondence over $\mathbb{F}_q$ with c_2 quasi-finite. For an integer $m \geq 1$ let $c_m : C_m \to X \times X$ denote the m-fold composition of c with itself.

Definition 4.17. An *I-system* consists of a ℓ_α-adic Weil complex with c-structure $(K_\alpha, \epsilon_\alpha, u_\alpha)$ for every $\alpha \in I$. A *compatible I-system* (or just *compatible system* if the reference to I is clear) is an I-system $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ such that the collection $\{(K_\alpha, \epsilon_\alpha)\}_{\alpha\in I}$ is an E-compatible system of Weil complexes indexed by I, and such that for every $n \geq 0$, $r \geq 1$, and $y \in \mathrm{Fix}(c_m^{(n,0)})$ we have

$$\mathrm{lt}_y^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(n,0)}) \in \iota_\alpha(E) \subset \overline{\mathbb{Q}}_{\ell_\alpha},$$

and

$$\mathrm{lt}_y^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(n,0)}) = \mathrm{lt}_y^{\mathrm{na}}(K_\beta, u_{r,\beta}^{(n,0)}),$$

for all $\alpha, \beta \in I$, where these local terms are viewed as elements of E via ι_α and ι_β respectively, and $u_{r,\alpha}$ denotes the c_m-structure on K_α defined by composition.

4.18. Recall the following facts of linear algebra (see [9, §8]). Let E be a field of characteristic 0, and let $V = V_0 \oplus V_1$ be a finite-dimensional $\mathbb{Z}/(2)$-graded E-vector space. Let

$$u, f : V \to V$$

be two commuting, graded, endomorphisms of V with f bijective. Define a function

$$S(t) := \sum_{n \geq 1} \mathrm{tr}(uf^n)t^n \in t \cdot E[[t]].$$

Lemma 4.19. *(i) $S(t) \in E(t)$.*

(ii) Let $s = 1/t$. Then $S(t)$ does not have a pole at $s = 0$ and $-\mathrm{tr}(u)$ is equal to the value of $S(t)$ at $s = 0$.

(iii) Suppose $K \subset E$ is a subfield, and that $S(t) \in t \cdot K[[t]] \subset t \cdot E[[t]]$. Then in fact $S(t) \in K(t) \subset E(t)$ and in particular $\mathrm{tr}(u) \in K$.

Proof. See [9, §8]. □

The main application for us of this linear algebra is the following:

Lemma 4.20. *Let $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}_{\alpha \in I}$ be an I-system.*

(i) Suppose that for every $\alpha, \beta \in I$ and $r \geq 1$ there exists an integer n_0 such that for every $n \geq n_0$ we have

$$\mathrm{lt}_y^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(n,0)}) \in \iota_\alpha(E), \quad \mathrm{lt}_y^{\mathrm{na}}(K_\beta, u_{r,\beta}^{(n,0)}) \in \iota_\beta(E),$$

and

$$\mathrm{lt}_y^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(n,0)}) = \mathrm{lt}_y^{\mathrm{na}}(K_\beta, u_{r,\beta}^{(n,0)})$$

for all $y \in \mathrm{Fix}(c_r^{(n,0)})$ (where again we view these local terms as elements of E). Then $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}_{\alpha \in I}$ is a compatible system.

(ii) If $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}_{\alpha \in I}$ is a compatible system, then for every $\alpha, \beta \in I$, $n, m \in \mathbb{N}$, $r \geq 1$, and $y \in \mathrm{Fix}(c_r^{(n,m)})$ we have

$$\mathrm{lt}_y^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(n,m)}) \in \iota_\alpha(E), \quad \mathrm{lt}_y^{\mathrm{na}}(K_\beta, u_{r,\beta}^{(n,m)}) \in \iota_\beta(E)$$

and

$$\mathrm{lt}_y^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(n,m)}) = \mathrm{lt}_y^{\mathrm{na}}(K_\beta, u_{r,\beta}^{(n,m)}).$$

Proof. For (i), let $n \geq 0$ and $r \geq 1$ be integers and let $y \in \mathrm{Fix}(c_r^{(n,0)})$ be a fixed point. Also choose $\alpha, \beta \in I$. We need to show that

$$\mathrm{lt}_y^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(n,0)}) \in \iota_\alpha(E), \quad \mathrm{lt}_y^{\mathrm{na}}(K_\beta, u_{r,\beta}^{(n,0)}) \in \iota_\beta(E), \tag{4.20.1}$$

and that

$$\mathrm{lt}_y^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(n,0)}) = \mathrm{lt}_y^{\mathrm{na}}(K_\beta, u_{r,\beta}^{(n,0)}).$$

Replacing c by $c_r^{(n,0)}$ and u by $u_r^{(n,0)}$, we may assume that $n=0$ and that $r=1$. Furthermore, after making a field extension $\mathbb{F}_q \to \mathbb{F}_{q^r}$, we may assume that $y \in C(\mathbb{F}_q)$, and that $n_0 = 1$. Let $x \in X(k)$ be the image of y, and set

$$V_\alpha = \oplus_{i \in \mathbb{Z}} H^i(K_{\alpha,x}),$$

a $\mathbb{Z}/(2)$-graded $\overline{\mathbb{Q}}_{\ell_\alpha}$-vector space. Let

$$u_\alpha : V_\alpha \to V_\alpha$$

be the endomorphism defined by the action of the correpondence on $K_{\alpha,x}$, and let

$$f_\alpha : V_\alpha \to V_\alpha$$

be the automorphism defined by the Weil complex structure. Then u_α and f_α are commuting endomorphisms of V_α, by 4.11. Set

$$S_\alpha(t) := \sum_{n \geq 1} \mathrm{tr}(u_\alpha f_\alpha^n | V_\alpha),$$

which by (4.19 (i)) is an element of $\overline{\mathbb{Q}}_{\ell_\alpha}(t)$. Similarly define $(V_\beta, u_\beta, f_\beta)$ and $S_\beta(t) \in \overline{\mathbb{Q}}_{\ell_\beta}(t)$. By (4.19 (iii)) we have

$$S_\alpha(t), S_\beta(t) \in E(t),$$

and

$$S_\alpha(t) = S_\beta(t),$$

so from (4.19 (ii)) we have (4.20.1) and

$$\mathrm{tr}(u_\alpha | V_\alpha) = \mathrm{tr}(u_\beta | V_\beta),$$

which is equivalent to the equality

$$\mathrm{lt}_y^{\mathrm{na}}(K_\alpha, u_\alpha) = \mathrm{lt}_y^{\mathrm{na}}(K_\beta, u_\beta).$$

For (ii), note that by 4.11 it suffices to show that for $m, r \geq 1$ we have

$$\mathrm{lt}_y^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(0,m)}) \in \iota_\alpha(E), \quad \mathrm{lt}_y^{\mathrm{na}}(K_\beta, u_{r,\beta}^{(0,m)}) \in \iota_\beta(E)$$

and

$$\mathrm{lt}_y^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(0,m)}) = \mathrm{lt}_y^{\mathrm{na}}(K_\beta, u_{r,\beta}^{(0,m)}).$$

For this note that by 4.11 again, we have for $n \geq m$

$$\mathrm{lt}_y^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(n,m)}) \in \iota_\alpha(E), \quad \mathrm{lt}_y^{\mathrm{na}}(K_\beta, u_{r,\beta}^{(n,m)}) \in \iota_\beta(E),$$

and

$$\mathrm{lt}_y^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(n,m)}) = \mathrm{lt}_y^{\mathrm{na}}(K_\beta, u_{r,\beta}^{(n,m)}).$$

Statement (ii) therefore follows from (i) applied to the I-system with $c_r^{(0,m)}$-structure

$$\{(K_\alpha, \epsilon_\alpha^{(0,m)}, u_{r,\alpha}^{(0,m)})\}.$$

$\square$

4.21. **Strongly compatible systems.**

4.22. As in subsection 4.12 let $\mathbb{F}_q$ be a finite field of characteristic p, let E be a field of characteristic 0, and let I be a set of pairs $(\ell_\alpha, \iota_\alpha)$, where ℓ_α is a prime different from p and $\iota_\alpha : E \hookrightarrow \overline{\mathbb{Q}}_{\ell_\alpha}$ is an inclusion of fields.

Let $c : C_0 \to X_0 \times X_0$ be a correspondence over $\mathbb{F}_q$ with C_0 and X_0 separated finite type Deligne-Mumford stacks over $\mathbb{F}_q$ (but now we make no quasi-finiteness assumption on c_2).

Definition 4.23. A *strongly compatible I-system of Weil complexes with c-structure* (or just *strongly compatible system* if no confusion seems likely to arise) is a collection of Weil complexes with c-structure $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}_{\alpha \in I}$ satisfying the following:

(i) The collection $\{(K_\alpha, \epsilon_\alpha)\}_{\alpha\in I}$ is a compatible system of Weil complexes on X_0.

(ii) For every $n, m \geq 0$, $r \geq 1$, and proper component $Z \subset \mathrm{Fix}(c_r^{(n,m)})$ we have for all $\alpha \in I$
$$\mathrm{lt}_Z(K_\alpha, u_{r,\alpha}^{(n,m)}) \in \iota_\alpha(E)$$
and for any two indices $\alpha, \beta \in I$ we have
$$\iota_\alpha^{-1}(\mathrm{lt}_Z(K_\alpha, u_{r,\alpha}^{(n,m)})) = \iota_\beta^{-1}(\mathrm{lt}_Z(K_\beta, u_{r,\beta}^{(n,m)})).$$

Proposition 4.24. *Let $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}_{\alpha\in I}$ be a strongly compatible system. Then the dual system $\{(DK_\alpha, \epsilon_\alpha^t, u_\alpha^t)\}_{\alpha\in I}$ is a strongly compatible system on X_0 with the correspondence $c^t : C_0 \to X_0 \times X_0$.*

Proof. That $\{(DK_\alpha, \epsilon_\alpha^t)\}_{\alpha\in I}$ is a compatible system of Weil complexes is Gabber's result [6, Theorem 2].

To verify (ii) in the definition, it suffices to observe that we have a canonical identification $\mathrm{Fix}(c_r^{(n,m)}) \simeq \mathrm{Fix}((c^t)_r^{(m,n)})$, and that for any proper component $Z \subset \mathrm{Fix}(c_r^{(n,m)})$ we have by [7, III, 5.1.6] an equality
$$\mathrm{lt}_Z(K_\alpha, u_r^{(n,m)}) = \mathrm{lt}_Z(DK_\alpha, (u^t)_r^{(m,n)}).$$
□

Proposition 4.25. *Assume that c_2 is quasi-finite. Then any strongly compatible system $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}_{\alpha\in I}$ is a compatible system in the sense of 4.17.*

Proof. By [17, 2.2.4 (b) and 2.1.3], there exists for any $r \geq 1$ an integer n_0 such that for $n \geq n_0$ the fixed point scheme $\mathrm{Fix}(c_r^{(n,0)})$ consists of a finite set of points and for every $z \in \mathrm{Fix}(c_r^{(n,0)})$ we have
$$\mathrm{lt}_z(K_\alpha, u_{r,\alpha}^{(n,0)}) = \mathrm{lt}_z^{\mathrm{na}}(K_\alpha, u_{r,\alpha}^{(n,0)})$$
for all $\alpha \in I$. The result therefore follows from the definitions and 4.20. □

5. Six operations for compatible systems

Granting conjecture 1.8, we explain in this section how to develop a theory of six operations for compatible systems.

5.1. **Duality.**

Theorem† 5.2. *Let $c : C_0 \to X_0 \times X_0$ be a correspondence as in 4.12 with both c_1 and c_2 quasi-finite, and let $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ be a compatible system of Weil complexes with c-structure. Then $\{(DK_\alpha, \epsilon^t_\alpha, u^t_\alpha)\}$ is a compatible system of Weil complexes with c^t-structure.*

Proof. This follows immediately from 4.24 and 4.25. □

5.3. **Pullback f^*.**

5.4. Consider a commutative diagram of separated Deligne-Mumford stacks over a finite field $\mathbb{F}_q$ of characteristic p

$$\begin{tikzcd}
 & C_0 \arrow[dl, "c_1"'] \arrow[dr, "c_2"] \arrow[dd, "g"] & \\
X_0 \arrow[dd, "f"] & & X_0 \arrow[dd, "f"] \\
 & D_0 \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\
Y_0 & & Y_0,
\end{tikzcd}$$

with c_2 and d_2 quasi-finite, and let $\ell \neq p$ be a prime. Fix as usual an algebraic closure $\mathbb{F}_q \hookrightarrow k$, and assume that the map

$$\chi : C \to D \times_{d_2, Y, f} X$$

is a universal homeomorphism. In this case we have a base change isomorphism

$$bc : c_{2!}g^* \simeq f^*d_{2!}.$$

Remark 5.5. A key case is when f is a closed embedding, and C_0 is the fiber product of the diagram

$$\begin{tikzcd}
 & X_0 \times X_0 \arrow[d] \\
D_0 \arrow[r, "d"] & Y_0 \times Y_0,
\end{tikzcd}$$

which is the scheme-theoretic intersection $d_1^{-1}(X) \cap d_2^{-1}(X)$. In this case, the condition that χ is a universal homeomorphism is equivalent to the condition that $d_2^{-1}(X)_{\mathrm{red}} \subset d_1^{-1}(X)_{\mathrm{red}}$.

5.6. Let (K, ϵ, u) be an ℓ-adic Weil complex with d-structure on Y. Define

$$f^*\epsilon : F_X^*(f^*K) \to f^*K$$

to be the isomorphism

$$F_X^* f^*K \xrightarrow{\simeq} f^*F_Y^*K \xrightarrow{\epsilon} f^*K,$$

and let

$$f^*u : c_{2!}c_1^*f^*K \to f^*K$$

be the morphism

$$c_{2!}c_1^*f^*K \xrightarrow{\simeq} c_{2!}g^*d_1^*K \xrightarrow{bc} f^*d_{2!}d_1^*K \xrightarrow{u} f^*K.$$

Lemma 5.7. *The data $(f^*K, f^*\epsilon, f^*u)$ is a Weil complex with c-structure on X.*

Proof. The condition that the diagram

$$\begin{tikzcd}
c_{2!}c_1^*F_X^*f^*K \arrow[r, "f^*\epsilon"] \arrow[d, "\simeq"] & c_{2!}c_1^*f^*K \arrow[dd, "f^*u"] \\
F_X^*c_{2!}c_1^*f^*K \arrow[d, "f^*u"] & \\
F_X^*f^*K \arrow[r, "f^*\epsilon"] & f^*K
\end{tikzcd}$$

commutes, is equivalent to the commutativity of the big outside diagram in the following:

(5.7.1)
$$\begin{tikzcd}
c_{2!}c_1^*F_X^*f^*K \arrow[r, "F_X^*f^*\simeq f^*F_Y^*"] \arrow[d, "\simeq"] & c_{2!}c_1^*f^*F_Y^*K \arrow[rr, "\epsilon"] \arrow[d, "c_1^*f^*\simeq g^*d_1^*"] & & c_{2!}c_1^*f^*K \arrow[d, "c_1^*f^*\simeq g^*d_1^*"] \\
F_X^*c_{2!}c_1^*f^*K \arrow[d, "c_1^*f^*\simeq g^*d_1^*"] & c_{2!}g^*d_1^*F_Y^*K \arrow[rr, "\epsilon"] \arrow[d, "bc"] & & c_{2!}g^*d_1^*K \arrow[d, "u"] \\
F_X^*c_{2!}g^*d_1^*K \arrow[d, "bc"] & f^*d_{2!}d_1^*F_Y^*K \arrow[r, "\epsilon"] \arrow[d, "\simeq"] & f^*d_{2!}d_1^*K \arrow[dr, "u"] & c_{2!}g^*d_2^!K \arrow[d, "bc"] \\
F_X^*f^*d_{2!}d_1^*K \arrow[r, "F_X^*f^*\simeq f^*F_Y^*"] & f^*F_Y^*d_{2!}d_1^*K \arrow[d, "u"] & & f^*d_{2!}d_2^!K \arrow[d, "d_{2!}d_2^!\to\mathrm{id}"] \\
& f^*F_Y^*K \arrow[rr, "\epsilon"] & & f^*K.
\end{tikzcd}$$

This implies the lemma as all the small inside diagram in (5.7.1) clearly commute. $\square$

Let

$$\gamma : \mathrm{Fix}(c) \to \mathrm{Fix}(d)$$

be the natural map.

Lemma 5.8. *For every $y \in \mathrm{Fix}(c)(k)$, we have*

$$\mathrm{lt}_y^{\mathrm{na}}(f^*K, f^*u) = \mathrm{lt}_{\gamma(y)}^{\mathrm{na}}(K, u).$$

Proof. Let x denote $c_1(y) = c_2(y)$. The lemma follows by noting that the diagram

$$\begin{tikzcd}
K_x \arrow[r, hook, "y"] & \bigoplus_{z\in c_2^{-1}(x)}(c_1^*f^*K)_z \arrow[r, "\simeq"] & (c_{2!}c_1^*f^*K)_x \arrow[d, "bc"] \arrow[dr, "f^*u"] & \\
& & (f^*d_{2!}d_1^*K)_x \arrow[r, "u"] \arrow[d, "\simeq"] & K_x \\
K_{f(x)} \arrow[uu, "f^*"] \arrow[r, hook, "\gamma(y)"] & \bigoplus_{w\in d_2^{-1}(f(x))}(d_1^*K)_w \arrow[r, "\simeq"] & (d_{2!}d_1^*K)_{f(x)} \arrow[r, "u"] & K_{f(x)} \arrow[u, "f^*"]
\end{tikzcd}$$

commutes. □

Proposition 5.9. *Let $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ be an I-compatible system of Weil complexes with d-structure on Y. Then*

$$\{(f^*K_\alpha, f^*\epsilon_\alpha, f^*u_\alpha)\}$$

is a compatible system of Weil complexes with c-structure on X.

Proof. The fact that $\{(f^*K_\alpha, f^*\epsilon_\alpha)\}_{\alpha \in I}$ is a compatible system of Weil complexes follows from [6, Theorem 2].

The rationality and independence of ℓ of the local terms follows from 5.8 and the observation that

$$f^*(u_r^{(n,0)}) = (f^*u)_r^{(n,0)}.$$

□

5.10. **Extraordinary inverse image $f^!$.**

5.11. We can dualize the results of the previous section. Consider a commutative diagram of separated Deligne-Mumford stacks over $\mathbb{F}_q$

$$\begin{tikzcd}
& C_0 \arrow[dl, "c_1"'] \arrow[dr, "c_2"] \arrow[dd, "g"] & \\
X_0 \arrow[dd, "f"] & & X_0 \arrow[dd, "f"] \\
& D_0 \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\
Y_0 & & Y_0,
\end{tikzcd}$$

with c_1 and d_1 quasi-finite, and let $\ell \neq p$ be a prime. Assume that the map

$$\chi : C \to X \times_{f,Y,d_1} D$$

is a universal homeomorphism. This implies that there is a base change morphism

$$bc : c_1^* f^! \to g^! d_1^*$$

adjoint to the composite

$$g_! c_1^* f^! \xrightarrow{g_! c_1^* \simeq d_1^* f_!} d_1^* f_! f^! \xrightarrow{f_! f^! \to \mathrm{id}} d_1^*.$$

For an ℓ-adic Weil complex with d-structure (K, ϵ, u), define

$$f^!\epsilon : F_X^* f^! K \to f^! K$$

to be the composite isomorphism

$$F_X^* f^! K \xrightarrow{\simeq} f^! F_Y^* K \xrightarrow{\epsilon} f^! K,$$

and let

$$f^! u : c_1^* f^! K \to c_2^! f^! K$$

be the composite morphism

$$c_1^* f^! K \xrightarrow{bc} g^! d_1^* K \xrightarrow{u} g^! d_2^! K \xrightarrow{\simeq} c_2^! f^! K.$$

Proposition† 5.12. *Let $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ be a compatible system of Weil complexes with d-structure on Y. Then $\{(f^! K_\alpha, f^! \epsilon_\alpha, f^! u_\alpha)\}$ is a compatible system of Weil complexes with c-structure on X.*

Proof. This follows from 5.2 and 5.9, together with the observation that under the canonical isomorphism

$$f^! K_\alpha \simeq D f^* D(K_\alpha)$$

the morphism $f^! \epsilon_\alpha$ (resp. $f^! u_\alpha$) corresponds to $(f^*(\epsilon_\alpha^t))^t$ (resp. $(f^*(u_\alpha^t))^t$), which is immediate from the definitions. □

5.13. **Compactly supported cohomology $f_!$.**

5.14. Consider a commutative diagram of separated Deligne-Mumford stacks of finite type over $\mathbb{F}_q$

$$\begin{tikzcd}
 & C_0 \arrow[dl, "c_1"'] \arrow[dr, "c_2"] \arrow[dd, "g"] & \\
X_0 \arrow[dd, "f"] & & X_0 \arrow[dd, "f"] \\
 & D_0 \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\
Y_0 & & Y_0,
\end{tikzcd}$$

with c_2 and d_2 quasi-finite. Let ℓ be a prime number not equal to p, and let $K \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ be a complex with a c-structure

$$u : c_1^* K \to c_2^! K.$$

Let P_1 denote the fiber product

$$P_1 := D \times_{d_1, Y, f} X,$$

let $\epsilon_1 : P_1 \to X$ be the projection, and let

(5.14.1)
$$C \xrightarrow{\beta} P_1 \xrightarrow{h} D$$

be the natural factorization of g.

Assume that the map β in 5.14.1 is proper. Then we get a d-structure

$$f_! u : d_1^* f_! K \to d_2^! f_! K$$

on $f_! K$ from the composite

$$d_{2!} d_1^* f_! K \xrightarrow{d_1^* f_! \simeq h_! \epsilon_1^*} d_{2!} h_! \epsilon_1^* K \xrightarrow{\mathrm{id} \to \beta_* \beta^*} d_{2!} h_! \beta_* \beta^* \epsilon_1^* K \xrightarrow{\beta_* \simeq \beta_!} d_{2!} h_! \beta_! c_1^* K \xrightarrow{\simeq} f_! c_{2!} c_1^* K \xrightarrow{u} f_! K.$$

5.15. Suppose now that (K, ϵ, u) is a Weil complex with c-structure on X, and define

$$f_!\epsilon : F_Y^* f_! K \to f_! K$$

to be the composite isomorphism

$$F_Y^* f_! K \xrightarrow{\simeq} f_! F_X^* K \xrightarrow{\epsilon} f_! K.$$

Lemma 5.16. *The data $(f_!K, f_!\epsilon, f_!u)$ is a Weil complex with d-structure on Y.*

Proof. To see that the diagram

$$\begin{tikzcd}
d_{2!}d_1^*F_Y^*f_!K \arrow[r, "f_!\epsilon"] \arrow[d, "\simeq"] & d_{2!}d_1^*f_!K \arrow[dd, "f_!u"] \\
F_Y^*d_{2!}d_1^*f_!K \arrow[d, "f_!u"] & \\
F_Y^*f_!K \arrow[r, "f_!\epsilon"] & f_!K
\end{tikzcd}$$

commutes, note that this is equivalent to showing that the big outside diagram in the following diagram commutes:

$$\begin{tikzcd}
d_{2!}d_1^*F_Y^*f_!K \arrow[r, "\simeq"] \arrow[d, "\simeq"] & d_{2!}d_1^*f_!F_X^*K \arrow[r, "\epsilon"] \arrow[d, "bc"] & d_{2!}d_1^*f_!K \arrow[d, "bc"] \\
F_Y^*d_{2!}d_1^*f_!K \arrow[d, "bc"] & d_{2!}g_{1!}\epsilon_1^*F_X^*K \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}\to\beta_*\beta^*"] & d_{2!}g_{1!}\epsilon_1^*K \arrow[d, "\mathrm{id}\to\beta_*\beta^*"] \\
F_Y^*d_{2!}g_{1!}\epsilon_1^*K \arrow[d, "\mathrm{id}\to\beta_*\beta^*"] & d_{2!}g_{1!}\beta_*\beta^*\epsilon_1^*F_X^*K \arrow[r, "\epsilon"] \arrow[d, "\beta_*\simeq\beta_!"] & d_{2!}g_{1!}\beta_*\beta^*\epsilon_1^*K \arrow[d, "\beta_*\simeq\beta_!"] \\
F_Y^*d_{2!}g_{1!}\beta_*\beta^*\epsilon_1^*K \arrow[d, "\beta_*\simeq\beta_!"] & d_{2!}g_{1!}\beta_!c_1^*F_X^*K \arrow[r, "\epsilon"] \arrow[d, "\simeq"] & d_{2!}g_{1!}\beta_!c_1^*K \arrow[d, "\simeq"] \\
F_Y^*d_{2!}g_{1!}\beta_!c_1^*K \arrow[d, "\simeq"] & f_!c_{2!}c_1^*F_X^*K \arrow[r, "\epsilon"] \arrow[d, "\simeq"] & f_!c_{2!}c_1^*K \arrow[dd, "u"] \\
F_Y^*f_!c_{2!}c_1^*K \arrow[r, "\simeq"] \arrow[d, "u"] & f_!F_X^*c_{2!}c_1^*K \arrow[d, "u"] & \\
F_Y^*f_!K \arrow[r, "\simeq"] & f_!F_X^*K \arrow[r, "\epsilon"] & f_!K.
\end{tikzcd}$$

This follows from observing that all the small inside diagrams in this diagram commute. ☐

5.17. Let $i : Z \hookrightarrow Y$ be a closed subscheme such that

$$d_1^{-1}(Z)_{\mathrm{red}} = d_2^{-1}(Z)_{\mathrm{red}}.$$

Let

$$a : X_Z \hookrightarrow X$$

be the inverse image of Z, and let

$$e : D_Z \to Z \times Z$$

and

$$b : C_Z \to X_Z \times X_Z$$

be the pullbacks of D and C respectively (so $D_Z = d_1^{-1}(Z) \cap d_2^{-1}(Z)$, and $C_Z = c_1^{-1}(X_Z) \cap c_2^{-1}(X_Z)$). Let

$$h : X_Z \to Z$$

be the base change of f.

Lemma 5.18. *Let (K, ϵ, u) be an ℓ-adic Weil complex with c-structure on X.*

(i) The diagrams

(5.18.1)
$$\begin{array}{ccc} F_Z^* i^* f_! K & \xrightarrow{i^* f_! \epsilon} & i^* f_! K \\ \downarrow{\scriptstyle i^* f_! \simeq h_! a^*} & & \downarrow{\scriptstyle i^* f_! \simeq h_! a^*} \\ F_Z^* h_! a^* K & \xrightarrow{h_! a^* \epsilon} & h_! a^* K \end{array}$$

and

(5.18.2)
$$\begin{array}{ccc} e_{2!} e_1^* i^* f_! K & \xrightarrow{i^* f_! u} & i^* f_! K \\ \downarrow{\scriptstyle i^* f_! \simeq h_! a^*} & & \downarrow{\scriptstyle i^* f_! \simeq h_! a^*} \\ e_{2!} e_1^* h_! a^* K & \xrightarrow{h_! a^* u} & h_! a^* K \end{array}$$

commute.

(ii) For any $n \geq 0$ and fixed point $z \in \mathrm{Fix}(e^{(n)}) \subset \mathrm{Fix}(c^{(n)})$ we have

$$\mathrm{lt}_z^{\mathrm{na}}(f_! K, f_! u^{(n)}) = \mathrm{lt}_z^{\mathrm{na}}(h_! a^* K, h_! a^* u^{(n)}).$$

Proof. For the commutativity of (5.18.1) note that the diagram

$$\begin{array}{ccccccc} F_Z^* i^* f_! K & \xrightarrow{\simeq} & i^* F_Y^* f_! K & \xrightarrow{\simeq} & i^* f_! F_X^* K & \xrightarrow{\epsilon} & i^* f_! K \\ \downarrow{\scriptstyle i^* f_! \simeq h_! a^*} & & & & \downarrow{\scriptstyle i^* f_! \simeq h_! a^*} & & \downarrow{\scriptstyle i^* f_! \simeq h_! a^*} \\ F_Z^* h_! a^* K & \xrightarrow{\simeq} & h_! F_{X_Z}^* a^* K & \xrightarrow{\simeq} & h_! a^* F_X^* K & \xrightarrow{\epsilon} & h_! a^* K \end{array}$$

commutes.

For the commutativity of (5.18.2) it suffices, by expanding out the definitions of $i^*f_!u$ and $h_!a^*u$, to show that the following diagram commutes:

(5.18.3)
$$\begin{tikzcd}
e_{2!}e_1^*i^*f_!K \arrow[r, "\simeq"] \arrow[d, "i^*f_!\simeq h_!a^*"] & e_{2!}\sigma^*d_1^*f_!K \arrow[r, "e_{2!}\sigma^*\simeq i^*d_{2!}"] \arrow[dr, "d_1^*f_!\simeq g_{1!}\epsilon_1^*"] & i^*d_{2!}d_1^*f_!K \arrow[r, "d_1^*f_!\simeq g_{1!}\epsilon_1^*"] & i^*d_{2!}g_{1!}\epsilon_1^*K \arrow[d, "\mathrm{id}\to\beta_*\beta^*"] \\
e_{2!}e_1^*h_!a^*K \arrow[d, "e_1^*h_!\simeq\nu_{1!}\tau^*"] & e_{2!}\nu_{1!}t^*\epsilon_1^*K \arrow[dl, "\simeq"'] & e_{2!}\sigma^*g_{1!}\epsilon_1^*K \arrow[l, "\nu_{1!}t^*\simeq\sigma^*g_{1!}"'] \arrow[d, "\mathrm{id}\to\beta_*\beta^*"] & i^*d_{2!}g_{1!}\beta_*\beta^*\epsilon_1^*K \arrow[d, "\beta_*\simeq\beta_!"] \\
e_{2!}\nu_{1!}\tau^*a^*K \arrow[d, "\mathrm{id}\to\gamma_*\gamma^*"] & e_{2!}\nu_{1!}t^*\beta_*\beta^*\epsilon_1^*K \arrow[dl, "bc"'] & e_{2!}\sigma^*g_{1!}\beta_*\beta^*\epsilon_1^*K \arrow[l, "bc"] \arrow[ur, "bc"] \arrow[d, "\simeq"] & i^*d_{2!}g_{1!}\beta_!c_1^*K \arrow[d, "\simeq"] \arrow[ddl, "\simeq"'] \\
e_{2!}\nu_{1!}\gamma_*\gamma^*\tau^*a^*K \arrow[d, "\gamma_*\simeq\gamma_!"] & e_{2!}\nu_{1!}t^*\beta_!c_1^*K \arrow[dl, "bc"'] \arrow[d, "\simeq"] & e_{2!}\sigma^*g_{1!}\beta_!c_1^*K \arrow[l, "bc"'] \arrow[ur, "bc"] & i^*f_!c_{2!}c_1^*K \arrow[d, "u"] \arrow[ddl, "bc"] \\
e_{2!}\nu_{1!}\gamma_!b_1^*a^*K \arrow[d, "\simeq"] & (e_2\nu_1)_!t^*\beta_!c_1^*K \arrow[r, "bc"] \arrow[d, "bc"] & (d_2g_1)_!\beta_!c_1^*K & i^*f_!K \arrow[d, "bc"] \\
h_!b_{2!}b_1^*a^*K \arrow[r, "\simeq"] & h_!b_{2!}\tilde{a}^*c_1^*K \arrow[r, "bc"] & h_!a^*c_{2!}c_1^*K \arrow[r, "u"] & h_!a^*K.
\end{tikzcd}$$

Here the notation is as follows. Let Q_1 denote the fiber product $X_Z \times_{h,Z,e_1} D$ and let

$$\gamma : C_Z \to Q_1$$

be the natural map. Let $\tilde{a} : C_Z \hookrightarrow C$ be the inclusion, so we have cartesian squares

$$\begin{tikzcd}
Q_1 \arrow[r, "\nu_1"] \arrow[d, "\tau"] & D_Z \arrow[d, "e_1"] \\
X_Z \arrow[r, "h"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
D_Z \arrow[r, "\sigma"] \arrow[d, "e_2"] & D \arrow[d, "d_2"] \\
Z \arrow[r, "i"] & Y,
\end{tikzcd}$$

and

$$\begin{tikzcd}
Q_1 \arrow[r, "t"] \arrow[d, "\nu_1"] & P_1 \arrow[d, "g_1"] \\
D_Z \arrow[r, "\sigma"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
C_Z \arrow[r, "\tilde{a}"] \arrow[d, "\gamma"] & C \arrow[d, "\beta"] \\
Q_1 \arrow[r, "t"] & P_1.
\end{tikzcd}$$

Part (i) of the lemma then follows by noting that (5.18.3) commutes, as each of the small inside squares commutes by standard properties of the base change isomorphisms.

Part (ii) of the lemma follows from (i) and the definition of the naive local terms. □

Theorem 5.19. *Let $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}_{\alpha\in I}$ be a compatible system of Weil complexes with c-structure on X. Then $\{(f_!K_\alpha, f_!\epsilon_\alpha, f_!u_\alpha)\}$ is a compatible system of Weil complexes with d-structure on Y.*

Proof. Let $z \in \mathrm{Fix}(d_r^{(n,0)})(k)$ be a fixed point for some $n \geq 0$ and $r \geq 1$. We have to show that

$$\mathrm{lt}_z^{\mathrm{na}}(f_!K, f_!u_{r,\alpha}^{(n,0)}) \in \iota_\alpha(E)$$

for all $\alpha \in I$, and that the resulting elements of E agree. Replacing d by $d_r^{(n,0)}$, c by $c_r^{(n,0)}$, and possibly making a field extension $\mathbb{F}_q \to \mathbb{F}_{q^r}$, we may assume that $n = 0$, $r = 1$, and that z is defined over $\mathbb{F}_q$. Let $y_0 \in Y_0(\mathbb{F}_q)$ be the image of z. After removing

$$(d_1^{-1}(y) \cup d_2^{-1}(y)) - \{z\}$$

from D and replacing C by the corresponding inverse image, we may further assume that $d_1^{-1}(y)_{\mathrm{red}} = d_2^{-1}(y)_{\mathrm{red}} = \{z\}$.

Let

$$i : \mathrm{Spec}(\mathbb{F}_q) \hookrightarrow Y_0$$

be the inclusion defined by y_0, let X_y denote $f^{-1}(y)$, and let C_z denote $g^{-1}(z)$. We then have a correspondence

$$e : C_z \to X_y \times X_y,$$

with e_1 proper (even finite). Let $\{(L_\alpha, \eta_\alpha, v_\alpha)\}$ be the compatible system of Weil complexes with e-structure on X_y obtained from $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ by pullback 5.9. By 5.18 we then have

$$\mathrm{lt}_z^{\mathrm{na}}(f_! K_\alpha, f_! u_\alpha) = \mathrm{tr}(R\Gamma_c(v_\alpha) | R\Gamma_c(X_y, L_\alpha)).$$

Now by Fujiwara's theorem 1.1, there exists an integer n_0 such that for all $n \geq n_0$ we have

$$\mathrm{lt}_z^{\mathrm{na}}(f_! K_\alpha, f_! u_\alpha) \in \iota_\alpha(E),$$

and such that these numbers all agree (in fact they are given by a sum of local terms of L_α). By 4.20 we therefore get the theorem. $\square$

5.20. **Ordinary cohomology f_*.**

5.21. Consider a commutative diagram of separated Deligne-Mumford stacks of finite type over $\mathbb{F}_q$

$$\begin{tikzcd}
& C_0 \arrow[dl, "c_1"'] \arrow[dd, "g"] \arrow[dr, "c_2"] & \\
X_0 \arrow[dd, "f"] & & X_0 \arrow[dd, "f"] \\
& D_0 \arrow[dl, "d_1"] \arrow[dr, "d_2"'] & \\
Y_0 & & Y_0,
\end{tikzcd}$$

with c_1 and d_1 quasi-finite. Let ℓ be a prime number not equal to p, and let $K \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ be a complex with a c-structure

$$u : c_1^* K \to c_2^! K.$$

Let P_2 denote the fiber product

$$P_2 := D \times_{d_2, Y, f} X,$$

let $\epsilon_2 : P_2 \to X$ be the projection, and let

$$C \xrightarrow{\beta} P_2 \xrightarrow{h} D \tag{5.21.1}$$

be the natural factorization of g.

Assume that the map β in 5.21.1 is proper.

We then get a d-structure f_*u on f_*K defined as the composite morphism

$$d_1^*f_*K \longrightarrow g_*c_1^*K \xrightarrow{u} g_*c_2^!K \xrightarrow{\simeq} h_*\beta_*\beta^!\epsilon_2^!K \xrightarrow{\beta_*\simeq\beta_!} h_*\beta_!\beta^!\epsilon_2^!K \xrightarrow{\beta_!\beta^!\to \mathrm{id}} h_*\epsilon_2^!K \xrightarrow{bc} d_2^!f_*K.$$

If

$$\epsilon : F_X^*K \to K$$

is a Weil complex structure on K, we also get a Weil complex structure $f_*\epsilon$ on f_*K from the composite

$$F_Y^*f_*K \xrightarrow{\simeq} f_*F_X^*K \xrightarrow{\epsilon} f_*K.$$

Proposition† 5.22. *Let $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ be a compatible system of Weil complexes with c-structure on X. Then $\{(f_*K_\alpha, f_*\epsilon_\alpha, f_*u_\alpha)\}$ is a compatible system of Weil complexes with d-structure on Y.*

Proof. This follows from 5.19 and 5.2, and the observation that under the canonical isomorphism

$$f_*K_\alpha \simeq Df_!DK_\alpha$$

the map $f_*\epsilon_\alpha$ (resp. f_*u_α) corresponds to $(f_!(\epsilon_\alpha^t))^t$ (resp. $(f_!(u_\alpha^t))^t$), which is immediate from the definitions. □

6. Group theoretic considerations

6.1. Basic definitions.

6.2. Fix a field k_0 and an algebraic closure $k_0 \hookrightarrow k$. Let X_0/k_0 be a finite type separated geometrically normal Deligne-Mumford stack over k_0, and let $\mathscr{V}_0$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X_0 (we change notation here using calligraphic font for the sheaf and roman font for its stalk below). Let X be the base change of X_0 to k, and let $\mathscr{V}$ denote the pullback of $\mathscr{V}_0$ to X.

Let $\langle \mathscr{V}_0\rangle_\otimes$ (resp. $\langle \mathscr{V}\rangle_\otimes$) denote the Tannakian subcategory of the category of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on X_0 (resp. X) generated by $\mathscr{V}_0$ (resp. $\mathscr{V}$).

Let $x \in X_0(k_0)$ be a point, and let $\bar x : \mathrm{Spec}(k) \to X_0$ be the induced geometric point. Let V denote the stalk $\mathscr{V}_{0,\bar x}$ so we get a representation

$$\rho : \pi_1(X_0, \bar x) \to GL(V).$$

We refer to the Zariski closure of this homomorphism as the *arithmetic monodromy group* and denote it by G_{arithm}. The algebraic group G_{arithm} is the Tannaka dual of the category $\langle \mathscr{V}_0\rangle_\otimes$ with respect to the fiber functor

$$\omega_{\bar x} : \langle \mathscr{V}_0\rangle_\otimes \to \mathrm{Vec}_{\overline{\mathbb{Q}}_\ell}$$

sending a lisse sheaf $\mathscr{W}$ to its stalk $\mathscr{W}_{\bar x}$. We also define the *geometric monodromy group*, denoted G_{geom}, as the Zariski closure of the image of $\pi_1(X, \bar x)$ under the homomorphism ρ. This group is the Tannaka dual of the category $\langle \mathscr{V}\rangle_\otimes$ with respect to the fiber functor taking stalk at $\bar{x}$. We have an inclusion $G_{\mathrm{geom}} \hookrightarrow G_{\mathrm{arithm}}$ Tannaka dual to the restriction functor $\langle \mathscr{V}_0\rangle_\otimes \to \langle \mathscr{V}\rangle_\otimes$.

If $k_0 = k$ then $G_{\mathrm{geom}} = G_{\mathrm{arithm}}$ and we refer to this group simply as the *monodromy group* of $\mathscr{V}$.

We will also consider the image of $\pi_1(X_0, \bar{x})$ (resp. $\pi_1(X_k, \bar{x})$) in $GL(V)$, and we will refer to this group as the *pro-finite arithmetic monodromy group* (resp. *pro-finite geometric monodromy group*), and if $k_0 = k$ then we call these groups simply the *pro-finite monodromy group.*

6.3. Now suppose further that k_0 is a finite field with q elements. We then have an exact sequence

$$1 \to \pi_1(X, \bar{x}) \to \pi_1(X_0, \bar{x}) \to \widehat{\mathbb{Z}} \to 1,$$

and a section $\widehat{\mathbb{Z}} \to \pi_1(X_0, \bar{x})$ induced by the point x. Let $F_x \in G_{\mathrm{arithm}}$ be the image of $1 \in \widehat{\mathbb{Z}}$ under the composite

$$\widehat{\mathbb{Z}} \to \pi_1(X_0, \bar{x}) \to G_{\mathrm{arithm}}.$$

Let $k_0 \subset \mathbb{F}_{q^n} \subset k$ be a subextension, and let $y \in X_0(\mathbb{F}_{q^n})$ be a point. Denote by $\bar{y} \in X(k)$ the resulting geometric point. We then have two fiber functors

$$\omega_{\bar{y}}, \omega_{\bar{x}} : \langle \mathscr{V}_0\rangle_\otimes \to \mathrm{Vec}_{\overline{\mathbb{Q}}_\ell},$$

and by choosing an isomorphism between these fiber functors we get from the q^n-power Frobenius automorphism of $k/\mathbb{F}_{q^n}$ an element $F_y \in G_{\mathrm{arithm}}$. The conjugacy class of this element is independent of the choice of isomorphism of fiber functors, and is referred to as the *Frobenius element of* y (or sometimes we abusively refer to F_y as the Frobenius element of y). By the Čebotarev density theorem [16, Theorem 7], the set of elements $F_y \in G_{\mathrm{arithm}}$ obtained from the closed points of X_0 in this way are Zariski dense.

6.4. **Group theoretic calculations of local terms.**

6.5. Let k be an algebraically closed field, and let

$$c : C \to X \times X$$

be a correspondence, with X and C smooth connected k-schemes of the same dimension, and c_1 and c_2 generically quasi-finite. Let $\mathscr{V}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X with c-structure given by an isomorphism $u : c_1^*\mathscr{V} \to c_2^*\mathscr{V}$. Let $y \in C(k)$ be a fixed point mapping to $x \in X(k)$, let $V := \mathscr{V}_x$ denote the stalk of $\mathscr{V}$ at x, and let $G \subset GL(V)$ be the monodromy group of $\mathscr{V}$, which we assume is connected.

6.6. Let H_j denote the monodromy group of $c_j^*\mathscr{V}$ on C with respect to the base point y. We then get a diagram

(6.6.1)
$$\begin{tikzcd} H_1 \arrow[r, "\sigma_u"] \arrow[d, "c_{1*}"] & H_2 \arrow[d, "c_{2*}"] \\ G & G, \end{tikzcd}$$

where σ_u is the isomorphism defined by u.

Lemma 6.7. *The maps c_{1*} and c_{2*} in (6.6.1) are isomorphisms.*

Proof. Fix $j \in \{1, 2\}$. Let $I \subset GL(V)$ denote the image of $\pi_1(X, x) \to GL(V)$, and let $I_j \subset GL(V)$ denote the image of the composite map

$$\pi_1(C, y) \xrightarrow{c_{j*}} \pi_1(X, x) \longrightarrow GL(V),$$

so G (resp. H_j) is the Zariski closure of I (resp. I_j). Fix an algebraic closure $k(C) \hookrightarrow \Omega$ of the function field $k(C)$, and let $c_j^* : k(X) \hookrightarrow k(C)$ be the inclusion of fields induced by c_j. Then $\mathrm{Aut}(\Omega/k(C))$ injects into $\mathrm{Aut}(\Omega/k(X))$ and the quotient is the set of embeddings of $k(C) \hookrightarrow \Omega$ over $k(X)$. In particular, the subgroup $\mathrm{Aut}(\Omega/k(C))$ is a subgroup of finite index in $\mathrm{Aut}(\Omega/k(X))$. Since we have a commutative diagram

$$\begin{tikzcd} \mathrm{Aut}(\Omega/k(C)) \arrow[r, two heads] \arrow[d] & I_j \arrow[d] \\ \mathrm{Aut}(\Omega/k(X)) \arrow[r, two heads] & I \end{tikzcd}$$

with surjective horizontal arrows, it follows that I_j is of finite index in I.

In particular, there exists an integer N such that map of schemes

$$G \to G, g \mapsto g^N$$

has image contained in H_j. On the other hand, this map induces multiplication by N on the tangent space at the identity, and therefore is étale in a neighborhood of the identity. It follows that the inclusion $H_j \subset G$ is étale in a neighborhood of the identity, whence H_j contains a dense open subset of G. Since G is assumed connected this implies that $H_j = G$. $\square$

6.8. Define

$$\alpha : G \to G$$

to be the automorphism given by the composite

$$G \xrightarrow{c_{1*}^{-1}} H_1 \xrightarrow{\sigma_u} H_2 \xrightarrow{c_{2*}} G.$$

Observe that with this definition the map $u_y : V \to V$ has the property that

$$u(g * v) = \alpha(g) * u(v)$$

for $g \in G$.

6.9. Let $y' \in \mathrm{Fix}(c)$ be a second fixed point mapping to $x' \in X(k)$, and let G' (resp. I', I'_j, H'_j) be the corresponding groups defined using y' and x'. Let $\gamma : y \to y'$ be an étale path between y and y' in C, and let $c_{j*}(\gamma)$ ($j = 1, 2$) be the induced étale paths between x and x' in X. Let

$$\epsilon_\gamma \in I$$

be the element $c_{1*}(\gamma)^{-1} c_{2*}(\gamma)$, and let V' denote the stalk $\mathscr{V}_{x'}$. We have a commutative diagram

$$\begin{tikzcd} G' \arrow[d, "c_{1*}(\gamma)"] & H'_1 \arrow[l, "c_{1*}"'] \arrow[r, "\sigma_u"] \arrow[d, "\gamma"] & H'_2 \arrow[r, "c_{2*}"] \arrow[d, "\gamma"] & G' \arrow[d, "c_{2*}(\gamma)"] \\ G & H_1 \arrow[l, "c_{1*}"'] \arrow[r, "\sigma_u"] & H_2 \arrow[r, "c_{2*}"] & G, \end{tikzcd}$$

which implies that the diagram

$$\begin{tikzcd} G' \arrow[rr, "\alpha'"] \arrow[d, "c_{1*}(\gamma)"] & & G' \arrow[d, "c_{1*}(\gamma)"] \\ G \arrow[r, "\alpha"] & G \arrow[r, "g \mapsto \epsilon_\gamma g \epsilon_\gamma^{-1}"] & G \end{tikzcd}$$

commutes. Furthermore, since the diagram

$$\begin{tikzcd} V \arrow[d, "c_{1*}(\gamma)"] & (c_1^*\mathscr{V})_y \arrow[l, "\simeq"'] \arrow[r, "u_y"] \arrow[d, "\gamma"] & (c_2^*\mathscr{V})_y \arrow[r, "\simeq"] \arrow[d, "\gamma"] & V \arrow[d, "c_{2*}(\gamma)"] \\ V' & (c_1^*\mathscr{V})_{y'} \arrow[l, "\simeq"'] \arrow[r, "u_{y'}"] & (c_2^*\mathscr{V})_{y'} \arrow[r, "\simeq"] & V' \end{tikzcd}$$

commutes we have

$$\mathrm{tr}(u_{y'}|V_{x'}) = \mathrm{tr}(\epsilon_\gamma u_y|V_x). \tag{6.9.1}$$

Lemma 6.10. *If $h \in \pi_1(C, y)$ is an element, then in G we have*

$$c_{1*}(h)^{-1}\epsilon_\gamma\alpha(c_{1*}(h)) = \epsilon_{\gamma h}.$$

Proof. This is immediate from the definition. □

Remark 6.11. Note that this is consistent with (6.9.1) since

$$\mathrm{tr}(\epsilon_{\gamma h}u_y) = \mathrm{tr}(c_{1*}(h)^{-1}\epsilon_\gamma\alpha(c_{1*}(h))u_y) = \mathrm{tr}(c_{1*}(h)^{-1}\epsilon_\gamma u_y c_{1*}(h)) = \mathrm{tr}(\epsilon_\gamma u_y).$$

6.12. **Density of fixed points.**

6.13. Let $\mathbb{F}_q$ be a finite field and let $c : C_0 \to X_0 \times X_0$ be a correspondence over $\mathbb{F}_q$ with C_0 and X_0 smooth of the same dimension d and c_1 and c_2 dominant (and hence generically quasi-finite). Fix an algebraic closure $\mathbb{F}_q \hookrightarrow k$ and let C (resp. X) denote the base change of C_0 (resp. X_0) to k.

Let $\mathscr{V}_0$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X_0 and let $u : c_1^*\mathscr{V} \to c_2^*\mathscr{V}$ be an isomorphism over C which commutes with Frobenius in the sense that the diagram

$$\begin{tikzcd} c_1^*F_X^*\mathscr{V} \arrow[r, "c_1^*\epsilon"] \arrow[d, "\simeq"] & c_1^*\mathscr{V} \arrow[r, "u"] & c_2^*V \\ F_C^*c_1^*\mathscr{V} \arrow[r, "F_C^*u"] & F_C^*c_2^*\mathscr{V} \arrow[r, "\simeq"] & c_2^*F_X^*\mathscr{V} \arrow[u, "c_2^*\epsilon"'] \end{tikzcd} \tag{6.13.1}$$

commutes, where $\epsilon : F_X^*\mathscr{V} \to \mathscr{V}$ is the natural Weil sheaf structure on $\mathscr{V}$.

Let $z \in \mathrm{Fix}(c)$ be a fixed point defined over $\mathbb{F}_q$, so z is also a fixed point of $c^{(n)}$ for any $n \geq 0$. Let $x \in X(k)$ be the image of z, and let G_{arithm} (resp. G_{geom}) be the arithmetic (resp. geometric) monodromy group of $\mathscr{V}$ with respect to the base point x. Assume that G_{arithm} and G_{geom} are connected. Let $I \subset G_{\mathrm{geom}}$ be the image of $\pi_1(X, x)$, so I is a profinite group.

For any $n \geq 0$ and fixed point $z' \in \mathrm{Fix}(c^{(n)})$ we get as in 6.9 an element $\epsilon_\gamma \in I$ by choosing an étale path γ between z' and z. Let $S \subset I$ be the subset of elements obtained in this way.

Proposition 6.14. *The subset $S \subset I$ is dense in the profinite topology, and therefore also dense in G_{geom} with the Zariski topology.*

Proof. Since $\mathscr{V}_0$ is defined over $\mathbb{F}_q$, there exists a finite extension L of $\mathbb{Q}_\ell$ with ring of integers $\mathscr{O}$ and a lisse sheaf of $\mathscr{O}$-modules $\mathscr{W}_0$ on X_0 such that $\mathscr{W}_0 \otimes_{\mathscr{O}} \overline{\mathbb{Q}}_\ell \simeq \mathscr{V}_0$. After possibly rescaling u and extending L we may also assume that u defines a morphism (not necessarily isomorphism) $c_1^*\mathscr{W} \to c_2^*\mathscr{W}$ which commutes with the Frobenius morphism $\epsilon : F_X^*\mathscr{W}_0 \to \mathscr{W}_0$. Let W_n denote $\mathscr{W}/\mathfrak{m}_{\mathscr{O}}^{n+1}\mathscr{W}$ and let I_n denote the image of the corresponding representation $\pi_1(X, x) \to GL(W_{n,x})$ so $I = \varprojlim_n I_n$.

Fix $m \geq 1$ and let $\rho \in I_m$ be an element. To prove the proposition we find an integer $n \geq 0$, a fixed point $z' \in \mathrm{Fix}(c^{(n)})$ and an étale path γ between z' and z such that $\epsilon_\gamma \in I$ maps to ρ in I_m.

Let $(Y, y) \to (X, x)$ be the pointed finite étale cover corresponding to the representation $\pi_1(X, x) \to I_m$, so Y/X is a Galois cover with group I_m, and let D denote the fiber product $C \times_{X\times X} (Y \times Y)$, so D is a $I_m \times I_m$-torsor over C. Let $\tilde{z} \in D$ be the point defined by z and the lifting $(y, y) \in Y \times Y$ of (x, x). Let $D' \subset D$ be the connected component containing $\tilde{z}$ so we have a diagram of correspondences

$$\begin{tikzcd} & D' \arrow[dl, "d_1'"'] \arrow[dd] \arrow[dr, "d_2'"] & \\ Y \arrow[dd] & & Y \arrow[dd] \\ & C \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ X & & X, \end{tikzcd}$$

and $\tilde{z} \in \mathrm{Fix}(d')$.

If $z' \in \mathrm{Fix}(c)$ is another fixed point, then the images in I_m of the elements ϵ_γ obtained from paths between z and z' can be described as follows. Let $\tilde{z}' \in D'$ be a lifting of z'. Then since Y/X is an I_m-torsor there exists a unique element $\delta_{\tilde{z}'} \in I_m$ such that $d_1'(\tilde{z}') = \delta_{\tilde{z}'} \cdot d_2'(\tilde{z}')$. The elements $\delta_{\tilde{z}'}$ for different choices of $\tilde{z}'$ are equal to the images of the ϵ_γ. Let $D_\rho \subset D$ be the connected component $(\mathrm{id} \times \rho)(D')$, so we have another correspondence

$$d_\rho : D_\rho \to Y \times Y$$

over c. There is a commutative diagram

$$\begin{tikzcd} D' \arrow[r, "1\times\rho"] \arrow[d] & D_\rho \arrow[d] \\ Y \times Y \arrow[r, "1\times\rho"] & Y \times Y, \end{tikzcd}$$

which identifies the set of points $w \in D'$ such that $d_1'(w) = \rho \cdot d_2'(w)$ with $\mathrm{Fix}(d_\rho)$.

After possibly replacing $\mathbb{F}_q$ by a field extension, we may assume that D', Y, and D_ρ are all defined over $\mathbb{F}_q$. This reduces us to showing that the correspondence $d_\rho : D_\rho \to Y \times Y$ has a fixed point after possibly twisting by Frobenius. This follows from [8, Theorem 1.1]. $\square$

Remark 6.15. The proof of 6.14 in fact shows a stronger statement. For $m \geq 1$ let I_m be as in the proof and let $\rho \in I_m$ be an element. Then there exists an integer n_0 such that for every $n \geq n_0$ there exists a fixed point $z' \in \mathrm{Fix}(c^{(n)})$ and an étale path γ between z and z' such that ϵ_γ maps to ρ in I_m.

6.16. **Nonvanishing of traces.**

6.17. Let K be an algebraically closed field of characteristic 0, and let G/K be a reductive connected group over K. Let V be an irreducible representation of G, and let $\alpha : G \to G$ be an automorphism. Let $T \subset G$ be a maximal torus, and assume that α restricts to the identity map on T. Let $u : V \to V$ be an isomorphism of vector spaces such that for every $g \in G$ and $v \in V$ we have $u(gv) = \alpha(g)u(v)$. Define a function

$$\tau : G \to \mathbb{A}^1, \quad g \mapsto \mathrm{tr}(gu : V \to V).$$

Lemma 6.18. *There exists a dense open subset $U \subset G$ such that $U \cap T \neq \emptyset$ and such that for every $g \in U$ we have $\tau(g) \neq 0$.*

Proof. It suffices to show that the open subset $\tau^{-1}(\mathbb{A}^1 - \{0\}) \subset G$ is nonempty and intersects T. Equivalently, to exhibit a single element $g \in T$ for which $\tau(g) \neq 0$. Let X be the character group of T and let $\chi_1, \dots, \chi_r$ be the characters occurring in V, viewed as a representation of T. Let $v \in V$ be a highest weight vector in V (with respect to some Borel), let $L \subset V$ be the line spanned by v, and assume that χ_1 is the character of T acting on L. Then u must take L to L so if we write the function τ as an element of $k[X]$ it is equal to

$$a\chi_1 + \sum_{\chi \neq \chi_1} \alpha_\chi \cdot \chi,$$

with $a \neq 0$. In particular this function is nonzero so there exists an element of T on which it takes nonzero value. $\square$

Corollary 6.19. *For every $m \geq 1$ there exists a dense open subset $U \subset G$ such that $U \cap T \neq 0$ and for every $g \in U$ we have $\mathrm{tr}(g^m u^m) \neq 0$.*

Proof. By applying the preceding lemma to G, α^m, and u^m we find a dense open subset $W \subset G$ such that $W \cap T \neq 0$ and for every $g \in W$ we have $\mathrm{tr}(gu^m) \neq 0$. Then let $U \subset G$ be the preimage of W under the morphism of schemes $G \to G$ sending g to g^m. $\square$

6.20. This group theoretic situation arises as follows. Let $\mathbb{F}_q$ be a finite field, and let $c : C_0 \to X_0 \times X_0$ be a correspondence over $\mathbb{F}_q$ with C_0 and X_0 smooth of the same dimension and c_1 and c_2 quasi-finite. As usual let k be an algebraic closure of $\mathbb{F}_q$ and let C (resp. X) denote the base change of C_0 (resp. X_0) to k. Let $\mathscr{V}_0$ be an irreducible lisse $\overline{\mathbb{Q}}_\ell$–sheaf on X_0 and let $(\mathscr{V}, \epsilon)$ be the corresponding Weil sheaf on X. Let $u : c_1^*\mathscr{V} \to c_2^*\mathscr{V}$ be an isomorphism such that the diagram (6.13.1) commutes. Let $z \in \mathrm{Fix}(c)$ be a fixed point defined over $\mathbb{F}_q$ mapping to $x \in X(k)$, let G_{arithm} (resp. G_{geom}) be the arithmetic (resp. geometric) monodromy group of $\mathscr{V}_0$ and assume these groups are connected. Let V be the stalk of $\mathscr{V}$ at x, and let $u_z : V \to V$ be the automorphism given by the stalk of u at z. Let $f : V \to V$ be the automorphism defined by the Frobenius at x (an element of G_{arithm}). Since u commutes with ϵ the two automorphisms u_z and f of V commute. As in 6.8 we get an automorphism $\alpha : G_{\mathrm{arithm}} \to G_{\mathrm{arithm}}$ such that for every $g \in G_{\mathrm{arithm}}$ and $v \in V$ we have

$u_z(g\cdot v)=\alpha(g)\cdot u_z(v)$. Note also that $\alpha(f)=f$. Let $f=f^s\cdot f^u$ be the decomposition of f into unipotent and semisimple part. Then we must also have $\alpha(f^s)=f^s$. In particular, if f^s generates a maximal torus T in G_{arithm} then we are in the setting of 6.17.

6.21. **Compatibility and irreducible factors.**

6.22. Let k be the algebraic closure of a finite field $\mathbb{F}_q$ and let

$$c : C_0 \to X_0 \times X_0$$

be a correspondence over a $\mathbb{F}_q$, with C_0 and X_0 smooth of the same dimension and the maps c_1 and c_2 generically quasi-finite. Let (V,ϵ) be a pure Weil sheaf of weight 0 on X, and suppose given two c-structures

$$u, \tilde u : c_1^*V \to c_2^*V$$

which commute with Frobenius. Fix representatives $V_1,\dots,V_r$ for the isomorphism classes of irreducible subsheaves of V. Since V is pure we have (see [4, 3.4.1 (iii)])

$$V=\oplus_i T_i\otimes_{\overline{\mathbb{Q}}_\ell} V_i,$$

where

$$T_i=\mathrm{Hom}(V_i,V).$$

Assume further that ϵ preserves this decomposition (we denote the restriction of ϵ to $T_i\otimes V_i$ by ϵ_i), and that u and $\tilde u$ are the sum of isomorphisms

$$u_i,\tilde u_i : T_i\otimes V_i\to T_i\otimes V_i.$$

Proposition 6.23. *Suppose that for every $n\geq 0$ and fixed point $y\in \mathrm{Fix}(c^{(n)})$ the characteristic polynomials of $u_y^{(n)}$ and $\tilde u_y^{(n)}$ are equal. Then for every i the characteristic polynomials of $u_{i,y}^{(n)}$ and $\tilde u_{i,y}^{(n)}$ are equal.*

Proof. We prove the proposition by contradiction. So let $y\in\mathrm{Fix}(c^{(n)})$ be a fixed point mapping to $x\in X(k)$, and suppose that $u_{i,y}^{(n)}$ and $\tilde u_{i,y}^{(n)}$ have different characteristic polynomials. We then show that there exists an integer n' and a fixed point $y'\in\mathrm{Fix}(c^{(n')})$ such that the characteristic polynomials of $u_{y'}^{(n')}$ and $\tilde u_{y'}^{(n')}$ are not equal.

After replacing c by $c^{(n)}$ we may assume that $n=0$. Since the characteristic polynomials are different, there exists an integer r such that

$$\mathrm{tr}(u_{i,y}^r)\neq \mathrm{tr}(\tilde u_{i,y}^r).$$

We can further reduce to the case when $r=1$ as follows. Let

$$d : D_0\to X_0\times X_0$$

be the r-fold composition of c with itself, and let $z\in D$ be the fixed point defined by y. Choose a proper generically finite morphism $D_0'\to D_0$ with D_0' smooth (this is possible by [3, 4.1]), and let $z'\in D'$ be a lifting of z. We can further arrange that the two projections $d_i : D'\to X$ are generically quasi-finite. Then z' is a fixed point of the correspondence

$$d' : D_0'\to X_0\times X_0$$

and u^r (resp. $\tilde u^r$) defines an action $v : d_1^*V\to d_2^*V$ (resp. $\tilde v : d_1^*V\to d_2^*V$) commuting with Frobenius such that

$$v_{z'}=u_y^r : V_x\to V_x,\quad \tilde v_{z'}=u_{y'}^r : V_x\to V_x.$$

This therefore reduces the proof to the case when $r = 1$.

Since our sheaf is defined over $\mathbb{F}_q$ the geometric monodromy group G_{geom} sits in an exact sequence

$$1 \to G_{\mathrm{geom}} \to \widetilde{G} \to \mathbb{Z} \to 1,$$

where $\widetilde{G}$ is as in [4, 1.3.7]. The Frobenius at y defines an element $F \in \widetilde{G}$ mapping to $1 \in \mathbb{Z}$ such that the morphism on G_{geom} induced by Frobenius is equal to conjugation by F

$$c_F : G_{\mathrm{geom}} \to G_{\mathrm{geom}},$$

and this automorphism commutes with α. Let I be the profinite monodromy group of V and let $\pi : I \to G_{\mathrm{geom}}$ be the natural map. Recall that π is injective with Zariski dense image.

The two functions

$$\mathrm{tr}(- \cdot u_y),\ \mathrm{tr}(- \cdot \tilde{u}_y) : G_{\mathrm{geom}} \to \mathbb{A}^1$$

are distinct since the i-components of these functions differ, and therefore there exists an open set $K \subset I$ such that for every $\gamma \in K$ we have

$$\mathrm{tr}(\gamma \cdot u_y) \neq \mathrm{tr}(\gamma \cdot \tilde{u}_y).$$

Let $\mathscr{O}$ be a finite extension of $\mathbb{Z}_\ell$ with an embedding $\mathscr{O} \hookrightarrow \overline{\mathbb{Q}}_\ell$ such that there exists a lisse sheaf of $\mathscr{O}$-modules M such that $V = M \otimes_{\mathscr{O}} \overline{\mathbb{Q}}_\ell$, and such that there exists an integer t such that $\ell^t u$ and $\ell^t \tilde{u}$ induce morphisms $c_1^* M \to c_2^* M$. Let N be an integer such that there exists $\tau \in I$ such that

$$\mathrm{tr}(\tau \cdot \ell^t u_y) \not\equiv \mathrm{tr}(\tau \cdot \ell^t \tilde{u}_y) \pmod{\ell^N \mathscr{O}}.$$

Let n_0 be an integer such that $F_y^{n_0}$ acts trivially modulo $\ell^N \mathscr{O}$. Then it suffices to find a fixed point $z' \in \mathrm{Fix}(c^{(nn_0)})$ for some $n \geq 0$ and an étale path γ between z and z' such that the induced element ϵ_γ has the same image as τ in $GL(M/\ell^N M)$. The existence of such a fixed point follows from 6.15. □

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