BONUS PROBLEM

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Let $k$ be a field and let $X = \mathbb{P}^n_k$. The purpose of this exercise is to give an alternate construction of the exact sequence

$$(0.0.1) \quad 0 \to \Omega^1_{X/k} \to \mathcal{O}^n_{X}(-1) \to \mathcal{O}_X \to 0.$$ 

(a) Let $T_X$ denote $\mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X)$. Show that giving the sequence 0.0.1 is equivalent to giving a sequence

$$(0.0.2) \quad 0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{n+1} \to T_X \to 0.$$ 

Let $\pi: \mathcal{O}_X^{n+1} \to \mathcal{O}_X(1)$ denote the the tauotological surjection over $X$, and let $\mathcal{O}_X(-1) \to \mathcal{O}_X^{n+1}$ be the inclusion obtained by applying $\mathcal{H}om(-, \mathcal{O}_X)$ to $\pi$. Tensoring with $\mathcal{O}_X(1)$ we obtain an inclusion $\alpha: \mathcal{O}_X \hookrightarrow \mathcal{O}_X(1)^{n+1}$. Let $Q$ be the cokernel. We show that $Q \simeq T_X$.

(b) Let $T_0 \hookrightarrow T$ be a square-zero closed immersion defined by an ideal $J \subset \mathcal{O}_T$, and let $x_0: T_0 \to X$ be a morphism corresponding to a surjection $\mathcal{O}^{n+1}_{T_0} \to L_0$. Show that the set of dotted arrows $x$ filling in the following diagram

$$
\begin{array}{ccc}
T_0 & \xrightarrow{x_0} & X \\
\downarrow & & \downarrow \\
T & & \\
\end{array}
$$

is canonically in bijection with the set of isomorphism classes of commutative diagrams of $\mathcal{O}_T$-modules

$$
\begin{array}{ccc}
\mathcal{O}_{T_0}^{n+1} & \xrightarrow{} & L \\
\downarrow & & \downarrow \\
\mathcal{O}_{T_0}^{n+1} & \xrightarrow{} & L_0, \\
\end{array}
$$

where $L$ is an invertible sheaf of $\mathcal{O}_T$-modules, and the horizontal arrows are surjections.

(c) Suppose $U \subset X$ is an affine open subset such that $\mathcal{O}_X(1)|_U$ is trivial, and let $F$ be a quasi-coherent sheaf on $U$. Show that for any commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_{U[F]}^{n+1} & \xrightarrow{} & L \\
\downarrow & & \downarrow \\
\mathcal{O}_U^{n+1} & \xrightarrow{} & L_0, \\
\end{array}
$$

as in (b), the invertible $\mathcal{O}_{U[F]}$-module $L$ is trivial.
(d) With notation and assumptions as in (c), let $L_0[F]$ denote $L_0 \otimes_{\mathcal{O}_U} \mathcal{O}_U[F]$ (so as an $\mathcal{O}_U$-module we have $L_0[F] \simeq L_0 \oplus L_0 \otimes F$). Show that there is a canonical bijection between dotted arrows filling in the following diagram

$\xymatrix{ \mathcal{O}_U^{n+1} \ar[d] \ar[r] & L_0 \ar[d] \ar[r] & L_0 \\
\mathcal{O}_U^{n+1} \ar[r] & L_0 }$

and $L_0^{n+1} \otimes F = \mathcal{O}_X(1)^{n+1}|_U \otimes F$.

(e) Continuing with the notation as in (d) show that the group of automorphisms of $L_0[F]$ inducing the identity on $L_0$ is canonically in bijection with $F$. Show that if $f \in F$ is an element with corresponding automorphism $a_f$, and if $\gamma : \mathcal{O}_U^{n+1} \rightarrow L_0[F]$ is a surjection corresponding to $v \in \mathcal{O}_X(1)^{n+1}|_U \otimes F$ then the composite surjection

$\xymatrix{ \mathcal{O}_U^{n+1} \ar[r]^{\gamma} & L_0[F] \ar[r]^{a_f} & L_0[F] }$

corresponds to $v$ plus the image of $f$ under the map

$F \xrightarrow{\alpha} \mathcal{O}_U \otimes F \xrightarrow{\alpha \otimes 1} L_0^{n+1} \otimes F$.

Deduce from this a canonical isomorphism $Q \otimes F \simeq T_{X/k} \otimes F$ over $U$. In particular, a canonical isomorphism $\epsilon_U : Q|_U \simeq T_X|_U$.

(f) Show that if $U, V \subset X$ are two open subsets as in (c), then the two isomorphisms $\epsilon_U, \epsilon_V : Q|_{U \cap V} \rightarrow T_X|_{U \cap V}$ are equal, so we obtain a global isomorphism $\epsilon : Q \simeq T_X$. 