GLUING COMPLEXES IN A D-TOPOS

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Abstract. We prove two variations of the classical gluing result of Beilinson, Bernstein, and Deligne. We recast the problem of gluing in terms of filtered complexes in the total topos of a D-topos, in the sense of SGA 4, and prove our results using the filtered derived category.

1. Statements of results

1.1. Let D be a category and let T be a D-topos in the sense of [1, $\mathrm{V}^{\mathrm{bis}}$, 1.2.1]. For an object $d \in D$ we write T_d for the fiber of T over d (so T_d is a topos), and for each morphism $\delta : d \to e$ in D we write $f_\delta : T_e \to T_d$ for the corresponding morphism of topoi (see loc. cit.).

We write $\mathrm{Sh}(T)$ for the category of sheaves in T. The category $\mathrm{Sh}(T)$ can be described as the category of systems $(\{F_d\}_{d\in D}, \{\sigma_\delta\}_{\delta\in \mathrm{Mor}(D)})$, consisting of an object $F_d \in T_d$ for each $d \in D$ and for every morphism $\delta : d \to e$ in D a morphism $\sigma_\delta : f_\delta^* F_d \to F_e$ satisfying a natural compatibility with composition.

Let Λ be a sheaf of rings in T, and let Λ_d be its component in T_d. Let $\mathrm{Sh}(T, \Lambda)$ be the category of Λ-modules in $\mathrm{Sh}(T)$. For each $d \in D$ there is a restriction functor

$$e_d : \mathrm{Sh}(T) \to \mathrm{Sh}(T_d), \quad (\{F_d\}, \{\sigma_\delta\}) \mapsto F_d$$

which induces a functor (which we again denote by e_d)

$$\mathrm{Sh}(T, \Lambda) \to \mathrm{Sh}(T_d, \Lambda_d).$$

For an object $M \in \mathrm{Sh}(T, \Lambda)$ we often write M_d for $e_d M \in \mathrm{Sh}(T_d, \Lambda_d)$. For $* \in \{\emptyset, b, +, -\}$ we have the corresponding derived category $D^*(T, \Lambda)$ of $\mathrm{Sh}(T, \Lambda)$. We can also consider the triangulated subcategory $D^{(*)}(T, \Lambda) \subset D(T, \Lambda)$ consisting of complexes $M \in D(T, \Lambda)$ for which $M_d \in D^*(T_d, \Lambda_d)$ for all d.

Remark 1.2. As we discuss at the end of the introduction, the classical BBD gluing lemma [3, 3.2.4] can be formulated in terms of complexes in a Δ-topos, where Δ is the standard simplicial category. It is also interesting to consider the theory for $D = \Delta^{\mathrm{op}}$ (cosimplicial topoi), which arise naturally for example in [2, 4.7].

1.3. Given a category D and D-topos T, define Γ to be the category of systems $(\{M_d\}_{d\in D}, \{\varphi_\delta\})$, where

(i) $M_d \in D^b(T_d, \Lambda_d)$ is an object for each $d \in D$
(ii) for each morphism $\delta : c \to d$ we are given a morphism

$$\varphi_\delta : M_c \to Rf_{\delta *} M_d$$

compatible with compositions in D in the sense that for a triple

$$c \xrightarrow{\delta} d \xrightarrow{\epsilon} e$$

the diagram

$$\begin{array}{ccccc} M_c & \xrightarrow{\varphi_\delta} & Rf_{\delta *}M_d & \xrightarrow{\varphi_\epsilon} & Rf_{\delta *}Rf_{\epsilon *}M_e \\ & \searrow^{\varphi_{\epsilon\delta}} & & & \downarrow \simeq \\ & & & & Rf_{\epsilon\delta *}M_e \end{array}$$

commutes.

There is a functor

$$D^b(T, \Lambda) \rightarrow \Gamma, \quad M \mapsto (\{M_d\}, \{\varphi_\delta^{\mathrm{can}}\}) \tag{1.3.1}$$

sending a complex to its restrictions with the natural transition maps. The two basic problems we consider here are the following:

(i) Given $M, M' \in D^b(T, \Lambda)$, formulate conditions under which the map

$$\mathrm{Hom}_{D^b(T,\Lambda)}(M, M') \rightarrow \mathrm{Hom}_\Gamma((\{M_d\}, \{\varphi_\delta^{\mathrm{can}}\}), (\{M'_d\}, \{\varphi_\delta^{\mathrm{can}}\})) \tag{1.3.2}$$

is an isomorphism.

(ii) Given a system $(\{M_d\}, \{\varphi_\delta^{\mathrm{can}}\}) \in \Gamma$, formulate conditions on the system that imply that it is in the essential image of (1.3.1).

The main results are the following:

Theorem 1.4. *Let $M, M' \in D^b(T, \Lambda)$ be two objects such that for every morphism $\delta : c \rightarrow d$ in D we have*

$$\mathrm{Ext}^i_{D(T_c,\Lambda_c)}(M_c, Rf_{\delta *}M_d) = 0$$

for $i < 0$. Then the map (1.3.2) is an isomorphism.

Remark 1.5. Note that the condition in 1.4 could also be formulated as

$$\mathrm{Ext}^i_{D(T_d,\Lambda_d)}(\mathrm{L}f_\delta^* M_c, M_d)) = 0$$

for $i < 0$.

Theorem 1.6. *Let $(\{M_d\}, \{\varphi_\delta\})$ be an object of Γ such that there exists $a \leq b$ for which $M_d \in D^{[a,b]}(T_d, \Lambda_d)$ for all $d \in D$. Suppose one of the following conditions hold:*

(i) *for every diagram*

$$\begin{array}{ccc} d & & \\ \uparrow \delta & & \\ c & \xrightarrow{\gamma} & e \end{array}$$

we have

$$\mathrm{Ext}^i_{D(T_c,\Lambda_c)}(Rf_{\delta *}M_d, Rf_{\gamma *}M_e) = 0 \tag{1.6.1}$$

for $i < 0$; or

(ii) *for every diagram*

$$\begin{array}{ccc} d & & \\ \downarrow{\scriptstyle \delta} & & \\ c & \xleftarrow{\gamma} & e \end{array}$$

we have

$$\mathrm{Ext}^i_{D(T_c,\Lambda_c)}(\mathrm{L}f_\delta^*M_d, \mathrm{L}f_\gamma^*M_e) = 0 \tag{1.6.2}$$

for $i < 0$

Then $(\{M_d\}, \{\varphi_\delta\})$ *is in the essential image of* (1.3.1).

Remark 1.7. Jacob Lurie explained to us an alternate, though closely related, argument proving the above in a more general setting of infinity categories. Contemplation of his argument led, in particular, to removal of certain unnecessary assumptions in earlier drafts.

Example 1.8. The classical BBD gluing theorem [3, 3.2.4] can be viewed as a special case of 1.6.

Let (T, Λ) be a topos and let $U \in T$ be an object covering the final object of T. Let $M_U \in D^b(T|_U, \Lambda_U)$ be an object equipped with an isomorphism

$$\epsilon : \mathrm{pr}_1^*M_U \to \mathrm{pr}_2^*M_U$$

in $D(T|_{U\times U}, \Lambda_{U\times U})$ satisfying the cocycle condition over $U \times U \times U$. Suppose that

$$\mathscr{E}xt^i_{T|_U}(M_U, M_U) = 0 \tag{1.8.1}$$

for $i < 0$. Then we claim that (M, ϵ) is induced by a unique object of $D^b(T, \Lambda)$.

For this we apply 1.6 with $D = \Delta$ and consider the simplicial topos given by

$$T|_{U_\bullet},$$

where $U_\bullet$ is the coskeleton of $U \to *$. So the fiber of $T|_{U_\bullet}$ over $[n] \in \Delta$ is the topos $T|_{U^{n+1}}$. For each $[n] \in \Delta$ let $M_n \in D(T|_{U_n}, \Lambda_{U_n})$ be the pullback of M_U along the first projection $U_n \to U$. The isomorphism ϵ defines maps φ_δ (using the cocycle condition) so we get an object

$$(\{M_n\}, \{\varphi_\delta\}) \in \Gamma. \tag{1.8.2}$$

The vanishing condition (1.8.1) implies that condition (ii) in 1.6 holds. Indeed the vanishing of the local Ext-groups implies that $\mathscr{R}Hom(M, M) \in D^{\geq 0}(T, \Lambda)$ and therefore for all $[n] \in \Delta$ the complex

$$\mathrm{RHom}_{T|_{U_n}}(M_n, M_n)$$

is also concentrated in degrees ≥ 0. Combined with the observation that for every morphism $\delta : [n] \to [m]$ in Δ the induced map $\mathrm{L}f_\delta^*M_n \to M_m$ is an isomorphism we get condition (ii). Therefore the system (1.8.2) is induced by a unique object $M_\bullet \in D(T|_{U_\bullet}, \Lambda_{U_\bullet})$. The object $M_\bullet$ is cartesian by construction. Therefore, by [5, Tag 0D8I] the object $M_\bullet$ is induced by a unique object $M \in D(T, \Lambda)$.

Remark 1.9. Note that the cartesian condition only enters in at the very end of the argument and the prinicipal issue is to construct the object $M_\bullet$.

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2. Preliminaries on the filtered derived category

2.1. Let (T, Λ) be a ringed topos and let $DF(T, \Lambda)$ denote the filtered derived category of Λ-modules [4, V] (see also [5, Tag 05RX]). Objects of $DF(T, \Lambda)$ are complexes K equipped with a finite decreasing filtration, indexed by $\mathbf{Z}$. Note that this differs slightly from the definition in [5, Tag 05RX], but this distinction will not be important here. We also consider the subcategory $D^bF(T, \Lambda)$ (resp. $D^+F(T, \Lambda)$, $D^-F(T, \Lambda)$) of $DF(T, \Lambda)$ consisting of objects K such that for all i the complex $\mathrm{gr}^i_F K$ is a bounded (resp. bounded below, bounded above) complex.

Lemma 2.2. *Let $H \in DF(\mathscr{A})$ be an object in the filtered derived category of an abelian category $\mathscr{A}$ such that $\mathrm{gr}^n H \in D^{\geq n}(\mathscr{A})$ for all $n \geq 0$. Then for all $s \geq 0$ we have $F^s H \in D^{\geq s}(\mathscr{A})$ and the sequence*

$$0 \to H^s(F^s H) \to H^s(\mathrm{gr}^s H) \to H^{s+1}(\mathrm{gr}^{s+1} H),$$

obtained from the projection map $H^s(F^s H) \to H^s(F^s H/F^{s+2} H)$ and the distinguished triangle

$$\mathrm{gr}^{s+1} H \to F^s H/F^{s+2} H \to \mathrm{gr}^s H \to \mathrm{gr}^{s+1} H[1],$$

is exact.

Proof. We proceed by descending induction on s. By the definition of $DF(\mathscr{A})$ the result holds for s sufficiently big, so it suffices to show that if the result holds for $s + 1$ then it also holds for s. For this note first that by the distinguished triangle

$$F^{s+1} H \longrightarrow F^s H \longrightarrow \mathrm{gr}^s H \longrightarrow F^{s+1} H[1]$$

and the inductive hypothesis, which implies that $H^j(F^{s+1}) = 0$ for $j < s + 1$, we get that $F^s H \in D^{\geq s}(\mathscr{A})$ and an exact sequence

$$0 \to H^s(F^s H) \to H^s(\mathrm{gr}^s H) \to H^{s+1}(F^{s+1} H).$$

Since the map

$$H^{s+1}(F^{s+1} H) \to H^{s+1}(\mathrm{gr}^{s+1} H)$$

is injective, by the inductive hypothesis, we then get the result. $\square$

Lemma 2.3. *Let $E \in D^-F(T, \Lambda)$ and $E' \in D^+F(T, \Lambda)$ be objects such that*

$$\mathrm{Ext}^s(\mathrm{gr}^i E, \mathrm{gr}^j E') = 0$$

for all $i \leq j$ and $s < j - i$. Then the sequence

$$0 \longrightarrow \mathrm{Hom}_{DF(T,\Lambda)}(E, E') \to \oplus_i \mathrm{Hom}_{D(T,\Lambda)}(\mathrm{gr}^i E, \mathrm{gr}^i E') \to \oplus_i \mathrm{Hom}_{D(T,\Lambda)}(\mathrm{gr}^i E, \mathrm{gr}^{i+1} E'[1])$$

is exact and $\mathrm{RHom}_{DF(T,\Lambda)}(E, E') \in D^{\geq 0}(\mathrm{Ab})$.

Remark 2.4. The map

$$\oplus_i \operatorname{Hom}_{D(X)}(\mathrm{gr}^i E, \mathrm{gr}^i E') \to \oplus_i \operatorname{Hom}_{D(X)}(\mathrm{gr}^i E, \mathrm{gr}^{i+1} E'[1]) \tag{2.4.1}$$

is obtained as follows. Note that for all i there is a distinguished triangle

$$\mathrm{gr}^{i+1}E \longrightarrow F^i/F^{i+2} \longrightarrow \mathrm{gr}^i E \xrightarrow{\partial_i} \mathrm{gr}^{i+1}E[1],$$

and similarly for E' The map (2.4.1) is obtained by sending a collection of maps (φ_i) to the differences of the two ways of going around the squares

$$\begin{array}{ccc} \mathrm{gr}^i E & \xrightarrow{\partial_i} & \mathrm{gr}^{i+1}E[1] \\ \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_{i+1}[1]} \\ \mathrm{gr}^i E' & \xrightarrow{\partial_i} & \mathrm{gr}^{i+1}E'[1]. \end{array}$$

Proof of 2.3. We can reformulate our assumption on Ext-groups as follows. Recall [4, V, 1.4.9] that we have

$$H^s(\mathrm{gr}^n \mathrm{RHom}(E, E')) \simeq \oplus_i \mathrm{Ext}^s(\mathrm{gr}^i E, \mathrm{gr}^{i+n} E') \tag{2.4.2}$$

for all n and s. Using this, our assumption on Ext-groups can then be reformulated as saying that

$$H^s(\mathrm{gr}^n \mathrm{RHom}(E, E')) = 0$$

for $n \geq 0$ and $s < n$. Applying 2.2 with $H = \mathrm{RHom}(E, E')$ we conclude that the sequence

$$0 \to H^0(F^0\mathrm{RHom}(E, E')) \to H^0(\mathrm{gr}^0\mathrm{RHom}(E, E')) \to H^1(\mathrm{gr}^1\mathrm{RHom}(E, E'))$$

is exact, which gives the result when combined with (2.4.2) and [4, V, 1.4.6]. □

2.5. For an object $E \in D^bF(T, \Lambda)$ we get a bounded complex

$$\cdots \to P^s \to P^{s+1} \to \cdots$$

in $D(T, \Lambda)$ by setting $P^s := \mathrm{gr}^s E[s]$ and the maps $d_s : P^s \to P^{s+1}$ given by the ∂_i. Note that

$$d_{s+1} \circ d_s = 0.$$

This follows from the fact that $\partial_i : \mathrm{gr}^s E \to \mathrm{gr}^{s+1}E[1]$ factors through a map

$$\mathrm{gr}^s E \to F^{s+1}E/F^{s+3}E[1];$$

namely, the boundary map arising from the distinguished triangle

$$F^{s+1}E/F^{s+3}E \to F^sE/F^{s+3}E \to \mathrm{gr}^s E \to F^{s+1}E/F^{s+3}E[1].$$

Proposition 2.6. *Let $(P^\bullet, d_\bullet)$ be a bounded complex in $D(T, \Lambda)$ such that for $s \in \mathbf{Z}$ and $r \geq 0$ we have*

$$\mathrm{Ext}^i_{D(T,\Lambda)}(P^s, P^{s+r}) = 0 \tag{2.6.1}$$

for $i < 0$. Then there exists an object $E \in D^bF(T, \Lambda)$, unique up to unique isomorphism, inducing $(P^\bullet, d_\bullet)$ by the construction of (2.5).

Proof. For $(P^\bullet, d_\bullet)$ obtained from $E \in D^bF(T,\Lambda)$ the vanishing condition (2.6.1) is equivalent to

$$\mathrm{Ext}^s_{D(T,\Lambda)}(\mathrm{gr}^i E, \mathrm{gr}^j E) = 0$$

for $j \geq i$ and $s < j - i$. The uniqueness of E therefore follows from 2.3.

To construct $E \in D^bF(T,\Lambda)$ inducing a given $(P^\bullet, d_\bullet)$ we proceed by induction of the number of terms in $P^\bullet$. For an integer s let

$$\sigma_{\leq s} P^\bullet$$

be the complex in $D(T,\Lambda)$ with $(\sigma_{\leq s}P^\bullet)^i = P^i$ if $i \leq s$ and 0 if $i > s$. We then have a term-wise split exact sequence [5, Tag 014I] of complexes in $D(T,\Lambda)$

$$0 \to P^s[-s] \to \sigma_{\leq s}P^\bullet \to \sigma_{\leq s-1}P^\bullet \to 0.$$

This defines a distinguished triangle in $K(D(T,\Lambda))$; in particular, a map

$$\delta_s : \sigma_{\leq s-1}P^\bullet \to P^s[-s+1].$$

Concretely this is simply the map induced by

$$d_s : P^{s-1} \to P^s.$$

Note that the assumptions on $(P^\bullet, d_\bullet)$ are also satisfies by $(\sigma_{\leq s}P^\bullet, d_\bullet)$ and therefore by induction it suffices to show that if $(\sigma_{\leq s-1}P^\bullet, d_\bullet)$ is obtained from an object $E_{s-1} \in D^bF(T,\Lambda)$ then so is $(\sigma_{\leq s}P^\bullet, d_\bullet)$. For this it suffices, in turn, to show that δ_s is induced by a morphism in the filtered derived category

$$\tilde{\delta}_s : E_{s-1} \to (P^s[-s+1], G_s),$$

where for an integer q we write G_q for the filtration on $P^s[-s+1]$ for which $G^i_qP^s[-s+1]$ equals $P^s[-s+1]$ if $i \leq q$ and 0 otherwise. For this note that we have

$$\mathrm{Ext}^r(\mathrm{gr}^i(\sigma_{\leq s-1}E), \mathrm{gr}^j(P^s[-s+1], G_{s-1})) = 0$$

if $j \neq s-1$ or $i \geq s$, since in this case one of the factors is 0, and for $i \leq s-1$ we have

$$\mathrm{Ext}^r(\mathrm{gr}^i(\sigma_{\leq s-1}E), \mathrm{gr}^{s-1}(P^s[-s+1], G_{s-1})) = \mathrm{Ext}^{r-s+1+i}(P^i, P^s).$$

In particular, this vanishes if

$$r - s + 1 + i < 0, \quad \text{or equivalently,} \quad r < (s-1) - i.$$

Therefore by 2.3 the map δ_s lifts to a map

$$\delta'_s : E_{s-1} \to (P^s[-s+1], G_{s-1})$$

in the filtered derived category. Let

$$\tilde{\delta}_s : E_{s-1} \to (P^s[-s+1], G_s)$$

be the composition of δ'_s with the natural map

$$(P^s[-s+1], G_{s-1}) \to (P^s[-s+1], G_s).$$

□

3. Filtered complexes in a D-topos

3.1. The functor e_d discussed in 1.1 has both a right and left adjoint. The right adjoint

$$\lambda_d : \mathrm{Sh}(T_d, \Lambda_d) \to \mathrm{Sh}(T, \Lambda)$$

sends an object $N \in \mathrm{Sh}(T_d, \Lambda_d)$ to an object of $\mathrm{Sh}(T, \Lambda)$ whose e-component is given by

$$\prod_{\delta : e \to d} f_{\delta *} N.$$

The left adjoint

$$s_d : \mathrm{Sh}(T_d, \Lambda_d) \to \mathrm{Sh}(T, \Lambda)$$

sends an object $N \in \mathrm{Sh}(T_d, \Lambda_d)$ to an object of $\mathrm{Sh}(T, \Lambda)$ whose e-component is given by

$$\oplus_{\delta : d \to e} f_\delta^* N.$$

Since e_d is exact the functor λ_d takes injectives to injectives. For an object $M \in \mathrm{Sh}(T, M)$ the natural map $M \to \prod_d \lambda_d M_d$ is injective. Therefore if we choose for each d an inclusion $M_d \hookrightarrow J_d$ of M_d into an injective Λ_d-module J_d then we get an inclusion into an injective $M \hookrightarrow \prod_d \lambda_d J_d$. In particular, every injective object in $\mathrm{Sh}(T, \Lambda)$ is a direct summand of a sheaf of the form $\prod_d \lambda_d J_d$ with each J_d injective in $\mathrm{Sh}(T_d, \Lambda_d)$.

Notation 3.2. Let ND denote the nerve of D (a simplicial set). In degree k the elements of ND_k are the diagrams in D

$$\underline{d} : d_0 \to d_1 \to \cdots \to d_k$$

consisting of k composable morphisms. For $a \in D$ let $a \backslash ND_k$ denote the set of pairs $(\underline{d}, \rho)$ consisting of an object $\underline{d} \in ND_k$ and a morphism $\rho : a \to d_0$. Similarly let ND_k/b denote the set of pairs $(\underline{d}, \gamma)$ consisting of an object $\underline{d} \in ND_k$ together with a morphism $\gamma : d_k \to b$, and let $a \backslash ND_k/b$ denote the collection of triples $(\underline{d}, \rho, \gamma)$ consisting of an object $\underline{d} \in ND_k$ and morphisms $\rho : a \to d_0$ and $\gamma : d_k \to b$. Note that

$$a \backslash ND_k/b = \coprod_{\sigma : a \to b} (a \backslash ND_k/b)_\sigma,$$

where $(a \backslash ND_k/b)_\sigma$ denotes the subset of triples for which the induced morphism $a \to b$ is σ.

For $\underline{d} \in ND_k$ let $f_{\underline{d}} : T_{d_k} \to T_{d_0}$ be the morphism induced by the composition $d_0 \to d_k$.

3.3. For a sheaf $M \in \mathrm{Sh}(T, \Lambda)$ we can associate an augmented cosimplicial object of $\mathrm{Sh}(T, \Lambda)$

$$M \to \widetilde{C}(M),$$

as well as an augmented simplicial object $\mathrm{Sh}(T, \Lambda)$

$$\widetilde{L}(M) \to M$$

as follows.

The object $\widetilde{C}(M)$ is defined by

$$[k] \mapsto \prod_{\underline{d} \in ND_k} \lambda_{d_0}(f_{\underline{d}*} M_{d_k}).$$

So the restriction of $\widetilde{C}(M)$ to T_a is the cosimplicial object of $\mathrm{Sh}(T_a, \Lambda_a)$ given by

$$[k] \mapsto \prod_{\underline{d} \in a \backslash ND_k} \rho_* f_{\underline{d}*} M_{d_k}.$$

The transition maps are induced by the simplicial structure on ND. Note that

$$\widetilde{C}(M)_0 = \prod_{d \in D} \lambda_d M_d.$$

The adjunction maps $M \to \lambda_d M_d$ induce the augmentation $M \to \widetilde{C}(M)$.

The simplicial object $\widetilde{L}(M)$ is defined similarly by the formula

$$[k] \mapsto \oplus_{\underline{d} \in ND_k} s_{d_k}(f_{\underline{d}}^* M_{d_0}).$$

So for $b \in D$ the restriction of $\widetilde{L}(M)$ to T_b is given by

$$[k] \mapsto \oplus_{\underline{d} \in ND_k/b} \gamma^* f_{\underline{d}}^* M_{d_0}.$$

We have

$$\widetilde{L}(M)_0 = \oplus_{d \in D} s_d M_d,$$

and the augmentation $\widetilde{L}(M) \to M$ is induced by the adjunction maps $s_d M_d \to M$.

3.4. Let $C(M)$ (resp. $L(M)$) denote the normalized complex associated to $\widetilde{C}(M)$ (resp. $\widetilde{L}(M)$) so we have maps of complexes

(3.4.1) $$M \to C(M), \quad L(M) \to M.$$

We will prove that these maps are quasi-isomorphisms (see 3.8 below). For later purposes, however, we will show this using some slightly more general considerations.

3.5. Let $\mathscr{A}$ be an additive category with infinite products and let $F : E \to \mathscr{A}$ be a functor (we will apply this below with $E = a \backslash D$ or $E = D/a$ for $a \in D$ and $\mathscr{A} = \mathrm{Sh}(T_a, \Lambda_a)$ – hence the change in notation). For $e \in E$ write F_e for the value of F on e. We can then repeat the construction of $\widetilde{C}(M)$ above to get a cosimplicial object $\widetilde{C}(F)$ in $\mathscr{A}$ given by

$$[k] \mapsto \prod_{\underline{e} \in NE_k} F_{e_k}.$$

3.6. Suppose now that E has an initial object $b \in E$. For $k \geq 1$ define

$$h_k : \widetilde{C}(F)_k \to \widetilde{C}(F)_{k-1}$$

to be the map whose component in the factor corresponding to $d_0 \to \cdots \to d_{k-1}$ is the projection

$$\textstyle\prod_{e_0 \to \cdots \to e_k} F_{e_k} \longrightarrow F_{d_{k-1}}$$

onto the component given by

$$b \to d_0 \to \cdots \to d_{k-1}.$$

Define

$$h_0 : \widetilde{C}(F)_0 = \prod_{e \in E} F_e \to F_b$$

to be the projection onto the b-th factor, and let $d_{-1} : F_b \to C(F)$ to be the product of the maps $\sigma_e : F_b \to F_e$ given by the fact that b is the initial object in E.

Lemma 3.7. *For every $k \geq 0$ we have*

$$\mathrm{id}_{\widetilde{C}(F)_k} = d_{k-1}h_k + h_{k+1}d_k,$$

where $d_k : \widetilde{C}(F)_k \to \widetilde{C}(F)_{k+1}$ is given by the alternating sum of the maps $\delta_i : \widetilde{C}(F)_k \to \widetilde{C}(F)_{k+1}$ provided by the cosimplicial structure.

Proof. Fix

$$\underline{d} = (d_0 \to d_1 \to \cdots \to d_k) \in NE_k$$

and let us calculate the composition of the maps

$$\widetilde{C}(F)_k \to \widetilde{C}(F)_k$$

in question with the projection onto the $\underline{d}$-th factor of $\widetilde{C}(F)_k$. For $\underline{e} \in ND_k$ write

$$F_{\underline{e}}$$

for the factor of $\widetilde{C}(F)_k$ corresponding to $\underline{e}$ (so $F_{\underline{e}} = F_{e_k}$, but this notation reflects also which factor in the product we are considering).

Let $0 \leq i_0 \leq k$ be the smallest integer for which $d_i \neq b$. For both of the maps $d_{k-1}h_k$ and $h_{k+1}d_k$ the compositions in question factor through the projection from $\prod_{\underline{e} \in NE_k} F_{e_k}$ to the product of $F_{\underline{d}}$ with the factors of the form

$$F_{b \to d_0 \to \cdots \hat{d}_i \cdots \to d_k}$$

for $i \geq i_0$ (note here that there may be several different choices of i corresponding to the same factor). Thus it suffices to calculate the individual factors

$$F_{\underline{d}} \to F_{\underline{d}}, \quad F_{b \to d_0 \to \cdots \hat{d}_i \cdots \to d_k} \to F_{\underline{d}}$$

of our maps.

On the factor $F_{\underline{d}}$ the map $d_{k-1}h_k$ is given by (with the convention that if $i_0 = 0$ then the sum is 0)

$$\Big(\sum_{j=0}^{i_0 - 1} (-1)^j\Big) \cdot \mathrm{id}_{F_{\underline{d}}}$$

and the map $h_{k+1}d_k$ is given by

$$\Big(\sum_{j=0}^{i_0} (-1)^j\Big) \cdot \mathrm{id}_{F_{\underline{d}}},$$

so their sum is $\mathrm{id}_{F_{\underline{d}}}$.

To calculate the maps on a factor $F_{b \to d_0 \to \cdots \hat{d}_i \cdots \to d_k}$ let J denote the set of those elements j for which

$$(d_0 \to \cdots \hat{d}_j \cdots \to d_k) = (d_0 \to \cdots \hat{d}_i \cdots \to d_k).$$

On a factor $F_{b \to d_0 \to \cdots \hat{d}_i \cdots \to d_k}$ the map $d_{k-1}h_k$ is given by

$$\sum_{j \in J} (-1)^j$$

times the natural map

$$F_{b \to d_0 \to \cdots \hat{d}_i \cdots \to d_k} \simeq F_{d_0 \to \cdots \hat{d}_i \cdots \to d_k} \to F_{\underline{d}},$$

whereas the map $h_{k+1}d_k$ is given by

$$\sum_{j\in J}(-1)^{j+1}$$

times this map. The two maps therefore cancel on the factor of $F_{b\to d_0\to\cdots\hat{d}_i\cdots\to d_k}$. □

Corollary 3.8. *The maps* (3.4.1) *are quasi-isomorphisms.*

Proof. It suffices to verify that the maps restrict to quasi-isomorphisms over each T_a $(a\in D)$.

To prove that $M_a\to C(M)_a$ is a quasi-isorphism apply 3.7 with

$$F: a\backslash D\to \mathrm{Sh}(T_a,\Lambda_a)$$

sending $\delta: a\to d$ to $f_{\delta *}M_d$.

To get that $L(M)_a\to M_a$ is a quasi-isomorphism apply 3.7 to the functor

$$F:(D/a)^{\mathrm{op}}\to \mathrm{Sh}(T_a,\Lambda_a)^{\mathrm{op}}$$

sending

$$(\delta: d\to a)\mapsto f_\delta^*M_d.$$

□

3.9. We view $C(M)$ and $L(M)$ as filtered objects using the "stupid filtration", so for $k\geq 0$ we have

$$\mathrm{gr}^kC(M)=\prod_{\underline{d}\in ND_k}\lambda_{d_0}(f_{\underline{d}*}M_{d_k})[-k],\quad \mathrm{gr}^{-k}L(M)=\oplus_{\underline{d}\in ND_k}s_{d_k}(f_{\underline{d}}^*M_{d_0})[k].$$

Since the filtrations involved are infinite we cannot directly apply our results on the filtered derived category. To get around this, note that for all $n\in\mathbf{Z}$ the objects

$$C(M)/\mathrm{Fil}^n,\quad (\text{resp. } \mathrm{Fil}^nL(M))$$

define projective (resp. inductive) systems in $DF(T,\Lambda)$ and we have

$$C(M)\simeq \mathrm{holim}_n\, C(M)/\mathrm{Fil}^n,\quad L(M)\simeq \mathrm{hocolim}_{n\to-\infty}(\mathrm{Fil}^n(M))$$

in $D(T,\Lambda)$. Indeed this follows from noting that for any index p the map

$$\mathscr{H}^p(C(M)/\mathrm{Fil}^{n+1})\to\mathscr{H}^p(C(M)/\mathrm{Fil}^n)\ \ (\text{resp. } \mathscr{H}^p(\mathrm{Fil}^{n+1}(M))\to H^p(\mathrm{Fil}^{n+1}(M)))$$

is an isomorphism for n sufficiently large (resp. sufficiently negative); see also [5, Tag 0CQE].

We can extend $C(-)$ to $D^+(T,\Lambda)$ by applying the above construction termwise to bounded below injective complexes to get a functor

$$C(-): D^+(T,\Lambda)\to (\text{projective systems in } D^+F(T,\Lambda))$$

such that the composition of $C(-)$ with the forgetful functor followed by holim is the identity functor

$$D^+(T,\Lambda)\to D^+(T,\Lambda).$$

Similarly we define

$$L(-): D^-(T,\Lambda)\to(\text{inductive systems in } D^-F(T,\Lambda))$$

by considering flat complexes.

4. Proof of Theorem 1.4

We proceed with the notation of the theorem.

To show surjectivity of the map (1.3.1) fix a compatible collection of maps

$$\sigma_d : M_d \to M_{d'} \tag{4.0.1}$$

defining a morphism in Γ. We show that these maps are induced by a morphism in $D(T, \Lambda)$ as follows.

View M as an object of $DF(T, \Lambda)$ by setting $\mathrm{Fil}^i M = M$ for $i \leq 0$ and 0 otherwise. Consider the projective system $C(M')/\mathrm{Fil}^n$ of objects of $DF(T, \Lambda)$. We have $\mathrm{gr}^i M = 0$ for $i \neq 0$ and

$$\begin{aligned}\mathrm{Ext}^s_{D(T,\Lambda)}(\mathrm{gr}^0 M, \mathrm{gr}^j C(M')) &= \mathrm{Ext}^{s-j}_{D(T,\Lambda)}(M, \prod_{\underline{d} \in ND_j} \lambda_{d_0}(Rf_{\underline{d}*} M'_{d_k})) \\ &= \prod_{\underline{d} \in ND_j} \mathrm{Ext}^s_{D(T_{d_0}, \Lambda_{d_0})}(M_{d_0}, Rf_{\underline{d}*} M'_{d_k}).\end{aligned}$$

By assumption these groups vanish for $j \geq 0$ and $s < j$. Therefore the assumptions of 2.3 hold and it follows that the map

$$\sigma := \prod_{d \in D} \sigma_d : M \to \prod_{d \in D} \lambda_d M'_d$$

lifts uniquely to a compatible collection of morphisms $M \to C(M')/\mathrm{Fil}^n$ in the filtered derived category. Passing to the homotopy limit over n we then get a morphism $M \to C(M') \simeq M'$ in $D(T, \Lambda)$ inducing the σ_k proving the surjectivity.

For the injectivity, note that a morphism $\tau : M \to M'$ in $D(T, \Lambda)$ induces, by composition with the map $M' \to C(M')$, a compatible collection of maps in the filtered derived category $M \to C(M')/\mathrm{Fil}^n$ which recovers τ by passing to the homotopy limit. By the preceding discussion these maps in the filtered derived category are uniquely determined by the associated maps on graded pieces, which implies that τ is determined by its image in Γ. □

5. Proof of Theorem 1.6

5.1. **Proof under assumption (i).** The uniqueness follows from 1.4.

For the existence set

$$P^k := \prod_{\underline{d} \in ND_k} \lambda_{d_0}(Rf_{\underline{d}*} M_{d_k}) \in D(T, \Lambda),$$

and let $d_k : P^k \to P^{k+1}$ be the maps induced by the alternating sums of the φ_δ.

For a given

$$\underline{d} = (d_0 \to \cdots \to d_k) \in ND_k$$

and $s \leq k$ let

$$(ND_s)_{\underline{d}} \subset ND_s$$

be the images of $\underline{d}$ under the various degeneracy maps

$$ND_k \to ND_s.$$

So $(ND_s)_{\underline{d}}$ is a finite set and there is a projection

$$\pi^s_{\underline{d}} : P^s \to P^s_{\underline{d}} := \prod_{\underline{e}\in (ND_s)_{\underline{d}}} \lambda_{e_0}(Rf_{\underline{e}*}M_{e_s}).$$

Note that for $s \leq k$ the composition

$$P^{s-1} \xrightarrow{d_{s-1}} P^s \xrightarrow{\pi^s_{\underline{d}}} P^s_{\underline{d}}$$

factors through $\pi^{s-1}_{\underline{d}}$. This implies that if $\sigma_{\leq s}P^\bullet$ is defined as in the proof of 2.6, then we get a map of complexes in $D(T,\Lambda)$

$$\pi_{\underline{d}} : \sigma_{\leq s}P^\bullet \to \sigma_{\leq s}P^\bullet_{\underline{d}}.$$

Since $(ND_s)_{\underline{d}}$ is finite, our vanishing assumption (1.6.1) implies that

$$\mathrm{Ext}^i_{D(T,\Lambda)}(P^s_{\underline{d}}, P^{s+r}) = 0$$

for all $i < 0$, $r \geq 0$ and $s \leq k$. In particular, by 2.6 there exists for every $\underline{d} \in ND_k$ and $s \leq k$ a unique filtered complex $K^{(\underline{d})}_s \in DF(T,\Lambda)$ inducing the complex $\sigma_{\leq s}P^\bullet_{\underline{d}}$. Furthermore, for $\underline{d}' \in ND_{k'}$ for which $\underline{d} \in (ND_k)_{\underline{d}'}$ we have by 1.4 a unique morphism in $DF(T,\Lambda)$

$$\tau^s_{\underline{d},\underline{d}'} : K^{(\underline{d}')}_s \to K^{(\underline{d})}_s$$

inducing the natural map

$$\sigma_{\leq s}P^\bullet_{\underline{d}'} \to \sigma_{\leq s}P^\bullet_{\underline{d}}.$$

We now construct for each integer $n \geq 0$ an object $K_n \in DF(T,\Lambda)$ with maps

$$q_{n,\underline{d}} : K_n \to K^{(\underline{d})}_n$$

for $\underline{d} \in ND_k$ and $k \geq n$, and distinguished triangles

(5.1.1) $$P^{n+1}[-(n+1)] \longrightarrow K_{n+1} \xrightarrow{t_{n+1}} K_n \longrightarrow P^{n+1}[-n]$$

such that the following hold:

(i)

$$\mathrm{gr}^i K_n = \begin{cases} P^i[-i] & \text{if } i \leq n \\ 0 & \text{otherwise,} \end{cases}$$

(ii) the filtered structure on K_n induces the maps d_k (as in 2.5),

(iii) the map t_{n+1} induces an isomorphism

$$K_{n+1}/\mathrm{Fil}^{n+1} \simeq K_n$$

in $DF(T,\Lambda)$ compatible with the isomorphisms in (i).

(iv) For $k' \geq k$, $\underline{d}' \in ND_{k'}$ and $\underline{d} \in (ND_k)_{\underline{d}'}$ the diagram

$$\begin{array}{ccc} & K_n & \\ {\scriptstyle q_{n,\underline{d}}}\swarrow & & \searrow{\scriptstyle q_{n,\underline{d}'}} \\ K^{(\underline{d})}_n & \xrightarrow{\tau^n_{\underline{d},\underline{d}'}} & K^{(\underline{d}')}_n \end{array}$$

commutes.

For this we proceed by induction on n.

For $n = 0$ we take $K_0 = P^0$ with the $q_{0,\underline{d}}$ the projection maps.

To pass from n to $n+1$ note that K_{n+1} will be specified by a map

$$\alpha_n : K_n \to P^{n+1}[-n] = \prod_{\underline{d} \in ND_{n+1}} \lambda_{d_0}(Rf_{\underline{d}*}M_{d_k})[-n]$$

in $DF(T,\Lambda)$, where P^{n+1} is viewed as filtered with $\mathrm{Fil}^u P^{n+1} = P^{n+1}$ for $u \leq n+1$ and 0 for $u > n+1$. To give this map it suffices to specify for each $\underline{d} \in ND_{n+1}$ a map

$$\alpha_{n,\underline{d}} : K_n \to \lambda_{d_0}(Rf_{\underline{d}*}M_{d_k})[-n],$$

and we take for this map the composition of

$$K_n \xrightarrow{q_{n,\underline{d}}} K_n^{(\underline{d})}$$

and the map

$$K_n^{(\underline{d})} \to \lambda_{d_0}(Rf_{\underline{d}*}M_{d_k})[-n]$$

arising from $K_{n+1,\underline{d}}$. The above properties follow from the construction.

In particular, we get a tower of complexes

$$\cdots \to K_{n+1} \to K_n \to \cdots \to K_0,$$

with distinguished triangles as in (5.1.1). Let K denote the homotopy limit of the K_n. Precisely, K is defined to be the cocone of the map

$$1 - t : \prod_n K_n \to \prod_n K_n.$$

We claim that K is the desired object of $D(T,\Lambda)$.

By our assumptions there exists an integer n_0 such that $M_d \in D^{\geq n_0}(T_d, \Lambda_d)$ for all d. Using the distinguished triangle (5.1.1) we obtain that for every s there exists an integer r such that the map

$$\mathscr{H}^s(K_m) \to \mathscr{H}^s(K_{m-1})$$

is an isomorphism for $m \geq r$. This in turn implies that for every s the map

$$\mathscr{H}^s(1-t) : \mathscr{H}^s(\prod_n K_n) \to \mathscr{H}^s(\prod_n K_n)$$

is surjective with kernel $\mathscr{H}^s(K)$. From this we conclude that the projection map $K \to K_n$ induces an isomorphism $\mathscr{H}^s(K) \to \mathscr{H}^s(K_n)$ for n sufficiently big.

Fix $a \in D$. The restriction of P^n to T_a is the complex

$$P_a^n = \prod_{\underline{d} \in N(a \backslash D_k)} R\rho_* Rf_{d_0 *} M_{d_k}$$

with the transition maps induced by the φ_δ. If we view M_a as a filtered object with

$$\mathrm{Fil}^r M_a = \begin{cases} M_a & \text{for } r \leq 0 \\ 0 & \text{for } r > 0 \end{cases}$$

then by the vanishing (1.6.1) we have

$$\mathrm{Ext}^s_{D(T_a,\Lambda_a)}(\mathrm{gr}^i M_a, \mathrm{gr}^j K_n) = \mathrm{Ext}^{s-j}_{D(T_a,\Lambda_a)}(\mathrm{gr}^i M_a, P^j_a) = 0$$

for $s < j$ and all i. It follows from this and 2.3 that the natural map

$$M_a \to \mathrm{gr}^0 K_n = \prod_{\delta : a \to d} Rf_{\delta *} M_d$$

lifts uniquely to a morphism

$$\beta_{a,n} : M_a \to K_{n,a}$$

in $DF(T_a, \Lambda_a)$. By the uniqueness, the diagram

$$\begin{array}{ccc} M_a & \xrightarrow{\beta_{a,n}} & K_{n,a} \\ & \searrow^{\beta_{a,n-1}} & \downarrow t_n \\ & & K_{n-1,a} \end{array}$$

commutes. The maps $\beta_{a,n}$ therefore induce a map

$$\beta_a : M_a \to K_a.$$

To see that this map is an isomorphism note that the spectral sequence of the filtered complex K_a (see for example [5, Tag 012K]) takes the form

$$E_1^{p,q} = \mathscr{H}^q(P^p) \implies \mathscr{H}^{p+q}(K_a),$$

and the differentials

$$\mathscr{H}^q(P^p) \to \mathscr{H}^q(P^{p+1})$$

are induced by our given maps $P^p \to P^{p+1}$. From this it follows that the map β induces an isomorphism

$$\mathscr{H}^q(M_a) \to E_2^{0,q} = \mathrm{Ker}(\mathscr{H}^q(P^0) \to \mathscr{H}^q(P^1)),$$

and that $E_2^{p,q} = 0$ for $p \neq 0$. Indeed consider the functor

$$F : a\backslash D \to D(T_a, \Lambda_a), \quad (\delta : a \to d) \mapsto Rf_{\delta *} M_d,$$

and form the associated cosimplicial obejct $\widetilde{C}(F)$ in $D(T_a, \Lambda_a)$. Then we see that the complex $E_1^{*,q}$ is equal to the complex obtained by taking q-th cohomology sheaves level-wise of $\widetilde{C}(F)$ and then taking total complex. By 3.7 it follows that the natural map

$$\mathscr{H}^q(M_a) \to E_1^{*,q}$$

is a quasi-isomorphism.

We therefore obtain the desired isomorphisms $\beta : M_a \simeq K_a$, and by the construction these isomorphisms are compatible with the transitions maps φ_δ.

5.2. **Proof under assumption (ii).** Once again the uniqueness follows from 1.4.

For existence define for $k \geq 0$

$$P^{-s} := \oplus_{\underline{d} \in ND_s} s_{d_k}(\mathbf{L}f_{\underline{d}}^* M_{d_0})$$

and let

$$d_{-s} : P^{-s} \to P^{-s+1}$$

be the map obtained by taking the alternating sum of the maps given by the simplicial structure. We then get a complex, concentrated in degrees $(-\infty, 0]$, in $D(T, \Lambda)$.

For $\underline{d} \in ND_k$ we can also consider for $s \leq k$ the complex

$$P_{\underline{d}}^{-s} := \oplus_{\underline{e} \in (ND_s)_{\underline{d}}} s_{e_s}(\mathbf{L}f_{\underline{e}}^* M_{e_0}) \subset P^{-s}.$$

This defines a subcomplex

$$\sigma_{\geq -s} P_{\underline{d}}^\bullet \subset \sigma_{\geq -s} P^\bullet$$

in $D(T, \Lambda)$.

Our assumptions imply that for all $s, r \in \mathbf{Z}$ and $i < 0$

$$\mathrm{Ext}^i(P^r, P_{\underline{d}}^s) = 0.$$

By 2.6 the complex $P_{\underline{d}}^{-s}$ is induced by a unique filtered complex $F_{-s}^{(\underline{d})} \in DF(T, \Lambda)$. Moreover, for $k \leq k'$ and $\underline{d}' \in ND_{k'}$ for which $\underline{d} \in (ND_k)_{\underline{d}'}$ we have unique maps

$$\tau_{\underline{d}, \underline{d}'}^{-s} : F_{-s}^{(\underline{d})} \to F_{-s}^{(\underline{d}')}$$

in $DF(T, \Lambda)$ inducing the natural maps on complexes in $D(T, \Lambda)$.

We now construct for each integer $n \geq 0$ an object $F_{-n} \in DF(T, \Lambda)$ with maps

$$q_{n, \underline{d}} : F_{-n}^{(\underline{d})} \to F_{-n}$$

for $\underline{d} \in ND_k$ and $k \geq n$, and distinguished triangles

$$F_{-n} \xrightarrow{t_{n+1}} F_{-n-1} \longrightarrow P^{n+1}[n+1] \longrightarrow F_{-n}[1]$$

such that the following hold:

(i)

$$\mathrm{gr}^s F_{-k} = \begin{cases} P^s & \text{if } 0 \geq s \geq -k \\ 0 & \text{otherwise,} \end{cases}$$

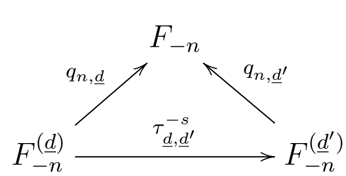
(ii) the filtered structure on F_{-n} induces the maps d_k (as in 2.5),

(iii) the map t_{n+1} induces an isomorphism

$$F_{-n} \simeq \mathrm{Fil}^{-n} F_{-n-1}$$

in $DF(T, \Lambda)$ compatible with the isomorphisms in (i).

(iv) For $k' \geq k$, $\underline{d}' \in ND_{k'}$ and $\underline{d} \in (ND_k)_{\underline{d}'}$ the diagram



commutes.

The objects F_{-n} are constructed by induction on n. For set $F_0 := P^0$.

To obtain F_{-n-1} given F_{-n}, note that F_{-n-1} is determined by a morphism

$$\alpha : P^{n+1}[n+1] \to F_{-n}[1],$$

or equivalently for each $\underline{d} \in ND_{n+1}$ a map

$$\alpha_{\underline{d}} : s_{d_k}(\mathbf{L}f^*_{\underline{d}}M_{d_0})[n+1] \to F_{-n}[1].$$

We take for $\alpha_{\underline{d}}$ the map induced by the natural map

$$F^{(\underline{d})}_{-n-1} \to F^{(\underline{d})}_{-n}[1] \to F_{-n}[1].$$

The above properties follow immediately from the construction.

We therefore get a sequence of objects in $D(T, \Lambda)$

$$F_0 \to F_{-1} \to F_{-2} \to \cdots .$$

We let K denote the homotopy colimit of this sequence.

We have

$$P^0 = \oplus_{d \in D} s_d M_d,$$

and therefore for $a \in D$

$$P^0_a = \oplus_{\delta : d \to a} \mathrm{L} f^*_\delta M_d.$$

The maps φ_δ therefore define a map

$$e_a : P^0_a \to M_a.$$

By 2.3 these maps are given by unique maps

$$e_{k,a} : F_{k,a} \to M_a$$

in $DF(T_a, \Lambda_a)$, where M_a is viewed as being filtered with $\mathrm{Fil}^i M_a = 0$ for $i > 0$ and $\mathrm{Fil}^0 M_a = M_a$. The uniqueness of the maps imply that $e_{k,a}$ restricts to $e_{k+1,a}$ on $F_{k+1,a}$ and we consequently get a map

$$e_a : K_a \to M_a.$$

By the same argument as in the proof under assumption (i), using the spectral sequence of a filtered complex and looking at cohomology sheaves, we find that e_a is an isomorphism compatible with the maps φ_δ.

This completes the proof of 1.6. □

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