Section 5.1

2) (a) Use six rectangles to find estimates of each type for the area under the given graph of \( f \) from \( x=0 \) to \( x=12 \).

(i) \( L_6 \)

(ii) \( R_6 \)

(iii) \( M_6 \)

(b) Is \( L_6 \) an underestimate or overestimate?

(c) Is \( R_6 \) an underestimate or overestimate?

(d) Which of \( L_6, R_6, \) and \( M_6 \) is the best estimate?

\[ \begin{align*}
(L_i) \quad & L_6 = 9(2) + 8.8(2) + 8.3(2) + 7.3(2) + 6(2) + 4(2) = 86.8 \\
& R_6 = 8.8(2) + 8.3(2) + 7.3(2) + 6(2) + 4(2) + 1(2) = 70.8 \\
& M_6 = 9(2) + 8.5(2) + 7.8(2) + 6.6(2) + 5(2) + 3.9(2) = 79.6
\end{align*} \]

(b) \( L_6 \) is an overestimate.

(c) \( R_6 \) is an underestimate.

(d) \( M_6 \) is the best estimate because \( f \) is decreasing and the midpoint is closer to the "average" value of \( f \) on each interval than the endpoints.
5) (a) Estimate the area under the graph of \( f(x) = 1 + x^2 \) from \( x = -1 \) to \( x = 2 \) using 3 rectangles and right endpoints. Thus improve your estimate by using 6 rectangles. Sketch the curve and the approximating rectangles.

(b) Repeat (a) using left endpoints.

(c) Repeat (a) using midpoints.

(d) From your sketches, which appears to be the best estimate?

(a) \( R_3 = f(0) + f(1) + f(2) = 1 + 2 + 5 = 8 \)

(b) \( L_3 = f(-1) + f(0) + f(1) = 2 + 1 + 2 = 5 \)

(c) \( M_3 = f(-\frac{1}{2}) + f(\frac{1}{2}) + f(\frac{3}{2}) = 5.75 \)

(d) \( M_6 \) appears to be the best estimate.
7) The speed of a runner increased steadily during the first 3 seconds of a race. Her speed at half seconds is given. Find upper and lower estimates for the distance traveled in these 3 seconds.

<table>
<thead>
<tr>
<th>t(s)</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v(t/\frac{1}{2}))</td>
<td>0</td>
<td>6.2</td>
<td>10.8</td>
<td>14.9</td>
<td>18.1</td>
<td>19.4</td>
<td>20.2</td>
</tr>
</tbody>
</table>

Since her speed is increasing, using right endpoints will overestimate the distance and left endpoints will underestimate it. This gives the upper estimate

\[ U = \frac{5}{2}(6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2) = 44.8 \text{ ft} \]

and the lower estimate

\[ L = \frac{5}{2}(0 + 6.2 + 10.8 + 14.9 + 18.1 + 19.4) = 34.7 \text{ ft} \]

10) Using the following table of times and velocities, estimate the height at 62 seconds.

<table>
<thead>
<tr>
<th>t(s)</th>
<th>0</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>32</th>
<th>59</th>
<th>62</th>
<th>125</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v(t/\frac{1}{2}))</td>
<td>0</td>
<td>185</td>
<td>319</td>
<td>441</td>
<td>742</td>
<td>1325</td>
<td>1445</td>
<td>4151</td>
</tr>
</tbody>
</table>

Using left endpoints we get the approximation

\[ h = 0 \cdot 10 + 185 \cdot 5 + 319 \cdot 5 + 441 \cdot 12 + 742 \cdot 25 + 1325 \cdot 3 = 30409 \text{ ft} \]

13) Find an expression for the area under the graph of \( f(x) = \sqrt{x}, \quad 12 \leq x \leq 16 \) as a limit.

In this case \( \Delta x = \frac{16-1}{n} = \frac{15}{n} \), so by Definition 2 we have

\[
A = \lim_{n \to \infty} \frac{n}{\Delta x} \sum_{i=1}^{n} f\left(1+i\cdot\frac{15}{n}\right) = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1+i\cdot\frac{15}{n}} \left(\frac{15}{n}\right)
\]
15) Determine a region whose area is

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i}{4n} \tan \left( \frac{i\pi}{4n} \right)$$

By definition 2, this is the area under the curve $f(x) = \tan(x)$ from $x = 0$ to $x = \pi/4$.

16) (a) Use definition 2 to find the area under the curve $y = x^3$ from 0 to 1 as a limit.
(b) Evaluate the limit in (a).

(a) Definition 2 gives

$$A = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{i}{n} \right)^3 \frac{1}{n} = \lim_{n \to \infty} \frac{1}{n^4} \sum_{i=1}^{n} \frac{i^3}{n}.$$

(b) We know $\sum_{i=1}^{n} i^3 = \left[ \frac{n(n+1)}{2} \right]^2$, so

$$A = \lim_{n \to \infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n \to \infty} \frac{1}{n^4} \left[ \frac{n^4 + 2n^3 + n^2}{4} \right] = \lim_{n \to \infty} \left( \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2} \right) = \frac{1}{4}.$$
20) (a) Let \( A_n \) be the area of a polygon with \( n \) equal sides inscribed in a circle of radius \( r \). Show that \( A_n = \frac{1}{2} n r^2 \sin \left( \frac{2\pi}{n} \right) \).
(b) Show that \( \lim_{n \to \infty} A_n = \pi r^2 \).
(c) The picture looks like this for \( n = 6 \).

In this fashion we can divide the polygon into \( n \) triangles of the form

Basic trigonometry tells us the height of this triangle is \( r \sin \left( \frac{\pi}{n} \right) \) and its base is \( 2r \cos \left( \frac{\pi}{n} \right) \), so its area is

\[
B_n = \frac{1}{2}bh = \frac{1}{2} r^2 \sin \left( \frac{\pi}{n} \right) \cos \left( \frac{\pi}{n} \right) = \frac{1}{2} r^2 \sin \left( \frac{2\pi}{n} \right),
\]

where we have used the identity \( \sin (2\theta) = 2\sin(\theta) \cos(\theta) \). Thus \( A_n = nB_n = \frac{1}{2} r^2 n \sin \left( \frac{2\pi}{n} \right) \).

(b) We compute

\[
\lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{1}{2} r^2 n \sin \left( \frac{2\pi}{n} \right) = \frac{1}{2} r^2 \lim_{n \to \infty} \frac{\sin \left( \frac{2\pi}{n} \right)}{\frac{1}{n}} = \frac{1}{2} r^2 \lim_{n \to \infty} \frac{\sin \left( \frac{2\pi}{n} \right)}{\frac{2\pi}{n}} = \pi r^2.
\]

where we have used the fact that \( \frac{2\pi}{n} \to 0 \) as \( n \to \infty \).