Disclaimer: You are responsible for all the material indicated on the syllabus. The questions below are intended only as a way to get more practice, and to give an indication of the type of problems the could appear on the exam.

(1) (a) State each of the following theorems (taking care to include all necessary hypothesis): The theorem relating the derivative of a function to the derivative of its inverse, l'Hospital's rule, extreme value theorem (and know how to deduce Fermat's theorem from this), the Mean value theorem and Rolle's theorem,

(b) For each of the theorems in (a), give an example to show that the theorem is false when omitting one of the hypotheses.

Solution: See book.

(2) Derive the change of base formula for logarithms.

Solution: See book.

(3) A cube of ice results on a hot plate. It melts in such a way that its shape at every moment is a cube, and the rate of decrease of the volume is a constant multiple of the area of the face resting on the hot plate. If after 5 minutes the volume of the cube is one eighth of its initial volume, how much longer is required before the cube melts entirely?

Solution: The volume \( V \) of a cube with side length \( r \) is \( r^3 \), and the surface area \( S \) of any of its sides is equal to \( r^2 \). We therefore have the equation

\[
V = S^{3/2}.
\]

We are given that

\[
\frac{dV}{dt} = kS = kV^{2/3}
\]

for some constant \( k \) (note that since the volume is decreasing the constant \( k \) is negative). Therefore the derivative of \( V^{1/3} \) is given by

\[
\frac{d}{dt} (V^{1/3}) = \frac{1}{3} V^{-2/3} \frac{dV}{dt} = \frac{1}{3} V^{-2/3} kV^{2/3} = k/3.
\]

Therefore

\[
V^{1/3} = (k/3)t + c
\]

for some constant \( c \). We conclude that

\[
V = ((k/3)t + c)^3.
\]

Now

\[
V(5) = (5k/3 + c)^3 = \frac{1}{8} \cdot c^3.
\]
Therefore
\[8(5k/3 + c)^3 = (10k/3 + 2c)^3 = c^3.\]
We conclude that
\[10k/3 + 2c = c,\]
which gives
\[c = -10k/3.\]
Therefore the function \(V\) is given by
\[V(t) = ((k/3)t - 10k/3)^3.\]
We get \(V(t) = 0\) when \(t = 10\). The cube is gone after an additional 5 minutes.

(4) Compute the following limits:

(a) \(\lim_{x \to 0^-} \frac{\cos^2(x) - 1}{\sin(2x)}\).

This is an indeterminate form \(0/0\) so we use l’Hospital’s rule. Taking derivatives of numerator and denominator gives the limit
\[\lim_{x \to 0^-} \frac{-2\cos(x)\sin(x)}{2\cos(2x)} = 0.\]
Therefore the answer is 0.

(b) \(\lim_{x \to \infty} \frac{x - e^x}{x}\).

Since \(\lim_{x \to \infty} e^{-x} = 0\), the answer is 0.

(c) \(\lim_{x \to 1^+} \left(\frac{x}{x - 1} - \frac{1}{\ln x}\right)\).

This is an indeterminate form \(\infty - \infty\). We can put this in the more standard form, but finding a common denominator to write the limit as
\[\lim_{x \to 1^+} \frac{x \ln x - x + 1}{(x - 1)\ln x}.\]
This is an indeterminant form \(0/0\), so we consider the limit of the ratio’s of the derivatives
\[\lim_{x \to 1^+} \frac{\ln x + 1 - 1}{\ln x + (1 - 1/x)} = \lim_{x \to 1^+} \frac{\ln x}{\ln x + (1 - 1/x)}.\]
This is again in the indeterminate form \(0/0\), so we apply l’Hospital’s rule again
\[\lim_{x \to 1^+} \frac{1/x}{1/x - 1/x^2} = \lim_{x \to 1^+} \frac{x}{x - 1} = \infty.\]
We conclude that
\[\lim_{x \to 1^+} \left(\frac{x}{x - 1} - \frac{1}{\ln x}\right) = \infty.\]

(5) (a) Compute the derivative of \(y = x^x\).

Write \(y = e^{x\ln x}\). Then \(y' = e^{x\ln x}(\ln x + 1) = x^x(\ln x + 1).\)

This can also be done by logarithmic differentiation: We have \(\ln y = \ln(x^x) = x\ln x\).
Therefore
\[y'/y = \ln x + 1\]
which gives
\[y' = y(\ln x + 1) = x^x(\ln x + 1).\]

(b) Compute the derivative of \(y = \frac{x\sqrt{x - 1}}{e^{2x}}\).
Again this can be either done directly, or using logarithmic differentiation. We have
\[
\ln y = \ln \left( \frac{x\sqrt{x-1}}{e^{2x}} \right) = \ln(x\sqrt{x-1}) - \ln(e^{2x}) = \ln x + \frac{1}{2} \ln(x-1) - 2x.
\]
Therefore
\[
y'/y = \frac{1}{x} + \frac{1}{2(x-1)} - 2.
\]
We conclude that
\[
y' = \frac{x\sqrt{x-1}}{e^{2x}} \left( \frac{1}{x} + \frac{1}{2(x-1)} - 2 \right).
\]

(6) (a) Define the function \( \sin^{-1}(x) \).
(b) Prove the formula
\[
\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}.
\]
See book.

(7) A motionless submarine is sitting 300 meters below sea level. A ship is sailing on the surface of the ocean, moving at a constant speed toward the point of the water surface which is exactly above the submarine. Using its sonar, the submarine measures that the distance between the submarine and the ship is decreasing at a rate of 10 m/s when this distance is 500 meters. What is the speed of the ship?

(8) Find the area of the largest rectangle that can be inscribed in the triangle formed by the \( x \) and \( y \) axis and the line \( y = -x + 2 \).

(9) Let
\[
g(x) = \frac{x+1}{(x-1)^2}.
\]
Draw a graph of this function, indicating where it is increasing, decreasing, concave up, concave down, points of inflection, asymptotes, etc.
\[ x = \sqrt{y^2 - (300)^2} \]

\[
\frac{dx}{dt} = \frac{1}{2} \frac{1}{\sqrt{y^2 - (300)^2}} (2y) \frac{dy}{dt} = \text{constant}. \quad C.
\]

When \( y = 500 \) get

\[
\frac{500}{\sqrt{(500)^2 - (300)^2}}. \quad (10) = C.
\]

\[
\frac{11}{100 \sqrt{125 - 9}}. \quad (10) = \frac{-50}{4} = -12.5 \text{ m/s}.
\]

So ship has velocity 12.5 m/s.

\[ A(x) = \text{area} = x(2-x) = 2x - x^2 \]

\[ A'(x) = 2 - 2x \]
So $A'(x) = 0$ when $x = 1$.

$A'(x) > 0$ for $x < 1$, $A'(x) < 0$ for $x > 1$.

So $x = 1$ is a global max. (First Der. Test)

Therefore max. area is $A(1) = 1$. 
A. Domain \((-\infty, 1) \cup (1, \infty)\)

B. \(y\)-intercept: \((0,1)\)

\(x\)-intercept: \((-1, 0)\).

C. No symmetries

D. \(\lim_{x \to -\infty} \frac{x+1}{(x-1)^2} = 0\), \(\lim_{x \to 0} \frac{x+1}{(x-1)^2} = 0\), \(\lim_{x \to 1+} \frac{x+1}{(x-1)^2} = \infty\)

\(\lim_{x \to 1^-} \frac{x+1}{(x-1)^2} = \infty\).

So horizontal asymptote at \(y = 0\).

Vertical asymptote at \(x = 1\).

E. \(g'(x) = \frac{(x-1)^2 - 2(x+1)(x-1)}{(x-1)^4}\)

\(= \frac{(x-1) - 2(x+1)}{(x-1)^3} = -\frac{x-3}{(x-1)^3} = -\frac{(x+3)}{(x-1)^3}\)

So \(g'(x) < 0\) on \((-\infty, -3)\) decreasing

\(g'(x) > 0\) on \((-3, 1)\) increasing

\(g'(x) < 0\) on \((1, \infty)\) decreasing

F. \(g'(x) = 0\) when \(x = -3\). By first derivative test \(x = -3\) is a local minimum.

Note that \(g(-3) = \frac{-2}{16} = \frac{-1}{8}\)

G. \(g''(x) = -\left[\frac{(x-1)^3 - 3(x-1)^2(x+3)}{(x-1)^6}\right] = -\left[\frac{(x-1)-3(x+3)}{(x-1)^4}\right]\)

\(= -\left[\frac{-2x-10}{(x-1)^4}\right] = \frac{2(x+5)}{(x-1)^4}\)
$g''(x) < 0 \quad \text{for} \quad x < -5 \quad \text{concave down}$

$g''(x) > 0 \quad \text{for} \quad x > -5 \quad \text{concave up}$