MATH 115: NOTES ON CURVE THEORY 3

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1. Conics

Let k be a field with $2 \neq 0$ in k, and consider an equation of the form

$$y^2 = f(x),$$

where $f(x)$ is a polynomial over k of degree $d \geq 2$, say

$$f(x) = \sum_{i=0}^{d} a_i x^i.$$

Consider the corresponding homogenized equation

$$F = Y^2 Z^{d-2} - \sum_{i=0}^{d} a_i X^i Z^{d-i},$$

defining

$$V := V(F) \subset \mathbb{P}^2(k).$$

Notice that if $d \geq 3$ then the points at infinity of V is exactly the point $[0:1:0]$. Also the partial derivatives of F are given by

$$\frac{\partial F}{\partial X} = \sum_{i=0}^{d} i a_i X^{i-1} Z^{d-i}, \quad \frac{\partial F}{\partial Y} = 2YZ^{d-2}, \quad \frac{\partial F}{\partial Z} = (d-2)Y^2 Z^{d-3} - \sum_{i=0}^{d} (d-i) a_i X^i Z^{d-i-1}.$$

In particular if $d \geq 4$ then $[0:1:0]$ is a singular point of V.

On the other hand for $d = 2$ we can give a more detailed analysis as follows (the case $d = 3$ was discussed last time). Write

$$f(x) = ax^2 + bx + c.$$

The equations of the partial derivatives in this case become

$$\frac{\partial F}{\partial X} = 2aX + bZ, \quad \frac{\partial F}{\partial Y} = 2Y, \quad \frac{\partial F}{\partial Z} = bX + 2cZ.$$

At infinity we have points $[\alpha : \beta : 0]$, where

$$\beta^2 = a\alpha^2.$$

Therefore there exists a point at infinity if and only if a is a square in k, and in this case there are exactly two points with both α and β nonzero. Therefore all the points at infinity are nonsingular.

Furthermore if $[\alpha : \beta : 1] \in V$ is some other point which is singular, then we must have

$$\beta^2 = a\alpha^2 + b\alpha + c, \quad 2a\alpha + b = 0, \quad 2\beta = 0, \quad b\alpha + 2c = 0.$$

From this we get that $\beta = 0$, and that

$$f(\alpha) = 0.$$

Furthermore we get that $\alpha = -b/(2a)$. Plugging into the third equation we get that

$$-b^2/(2a) = 2c,$$

or equivalently that the discriminant

$$\Delta := b^2 - 4ac$$

is zero. Conversely, if $\Delta = 0$ then we find that $[-b/2a : 0 : 1]$ is a singular point.

2. Parametrizing points on a conic

Consider again a field k with $2 \neq 0$, and let

$$y^2 = ax^2 + bx + c$$

be an equation defining a conic $V \subset \mathbb{P}^2(k)$, and assume that the discriminant $b^2 - 4ac$ is nonzero.

In this case there are two cases.

Case 1: V is empty. There is not much to say in this case except that this can happen! For example consider

$$Y^2 = -X^2 - Z^2$$

in $\mathbb{P}^2(\mathbb{R})$.

Case 2: V is nonempty. In this case if $P_0 \in V$ is a point then we can use P_0 to define a bijection between V and $\mathbb{P}^1(k)$.

In this case we can define a bijection between V and $\mathbb{P}^1(k)$ as follows. Let

$$P_0 = [u_0 : v_0 : w_0],$$

and consider a line L in $\mathbb{P}^2(k)$ given by an equation

$$A_0X + A_1Y + A_2Z = 0. \tag{2.0.1}$$

The condition that this line passes through P_0 is then that

$$A_0u_0 + A_1v_0 + A_2w_0 = 0.$$

Note that the vector space of such vectors (A_0, A_1, A_2) is two dimensional. Fix two linearly independent vectors $\underline{\alpha} = (\alpha_0, \alpha_1, \alpha_2)$ and $\alpha' = (\alpha'_0, \alpha'_1, \alpha'_2)$ defining lines through P_0. For any point $M = [s : t] \in \mathbb{P}^1(k)$, define the line L_M through P_0 to be the line

$$(s\alpha_0 + t\alpha'_0)X + (s\alpha_1 + t\alpha'_1)Y + (s\alpha_2 + t\alpha'_2)Z = 0.$$

Note that the line L_M depends only on the point M and not the representative $[s : t]$.

Proposition 2.1. *Let L be a line in $\mathbb{P}^2(k)$. Then $L \cap V$ is either empty, consists of two distinct points, or one point and the last case occurs only if L is a tangent line of V.*

Proof. If L is the line at infinity of $\mathbb{P}^2(k)$ then the intersection $L \cap V$ is clearly either 0 or 2 points. Note that this is the line given by

$$Z = 0.$$

Next consider a line L given by an equation of the form

$$X - \alpha Z = 0.$$

In this case the intersection $L \cap V$ is given by solutions to the equation

$$y^2 = a(\alpha Z)^2 + b\alpha z^2 + cz^2 = (a\alpha^2 + b\alpha + c)z^2.$$

Note that $L \cap V$ has no points at infinity, for if $z = 0$, then we have both $x = \alpha z = 0$ and $y = 0$. So there are no intersection points at infinity. On the other hand if $z = 1$ then we are looking at solutions to the equation

$$y^2 = (a\alpha^2 + b\alpha + c).$$

If α is not a root of the equation ax^2+bx+c then this has either no solutions or two solutions. On the other hand, if α is a root of $ax^2 + bx + c$ then we have only 1 solution given by the point $[\alpha : 0 : 1]$. The tangent line at this point is given by

$$-(2a\alpha + b\alpha)X - (2c + b\alpha)Z = 0.$$

Now observe that

$$\alpha = \frac{b\alpha + 2c}{-2a\alpha - b}.$$

Indeed

$$-2a\alpha^2 - b\alpha = 2(b\alpha + c) - b\alpha = b\alpha + 2c.$$

Therefore L is the tangent line in this case.

Finally we have to consider a line given by an equation

$$y = mx + Bz.$$

Making a linear change of variable $x \mapsto x - (b/m)z$ we may assume that $B = 0$. In this case we are looking at solutions to the equation

$$m^2x^2 = ax^2 + bxz + cz^2.$$

If $z = 0$, then this has a solution if and only if $m^2 = a$ in which case there is exactly one solution and a is a square in k.

If $b = 0$ and $m^2 = a$ then L is the tangent line at the point $[1 : m : 0]$. Indeed the tangent line at this point is given by

$$-2aX + 2mY = 0,$$

which can also be written as

$$Y = mX.$$

If b is not equal to zero but $m^2 = a$ then there is exactly one point of intersection with $z = 1$.

Finally if $m^2 \neq a$, then we get either two or no solutions since we are solving a quadratic equation for x.

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