MATH 748: HOMEWORK 9

(1) Find $\mu(K)$ for each quadratic number field K.

(2) Prove, without using Dirichlet's unit Theorem, that an imaginary quadratic number field has at most finitely many units.

(3) The continued fraction expansion for $\alpha \in \mathbb{R}$ is the writing of α as

$$a_0 + \frac{1}{a_1 + \frac{1}{\cdots}}$$

with each $a_i \in \mathbb{Z}$. To find the a_i, first let $[\alpha]$ be the greatest integer less than or equal to α, so that $a_0 = [\alpha]$. Let β be the reciprocal of the fractional part $\alpha - [\alpha]$, so that from above we have $\beta = a_1 + (1/(a_2 + \cdots))$. Thus $a_1 = [\beta]$. Continue in this manner to obtain the other a_i. If we truncate the expression above at the nth step, we obtain a rational number p_n/q_n. For instance, $p_0/q_0 = a_0/1$, $p_1/q_1 = a_0 + 1/a_1 = (a_0 a_1 + 1)/a_1$. The numbers p_n and q_n are called the convergents of α, and are given by the Fibonacci-like recurrences

$$p_{n+1} = a_{n+1} p_n + p_{n-1} \quad q_{n+1} = a_{n+1} q_n + q_{n-1}$$

with initial values p_0, p_1, q_0, q_1 as given above. The rational numbers p_n/q_n give successively better approximations of α. Now let $\alpha = \sqrt{d}$, where $d > 0$ is squarefree and $d \equiv 2, 3 \pmod 4$. The numbers p_n/q_n are nearly $\sqrt{d}$, meaning that $p_n^2/q_n^2 - d$ is small. Thus it should not be surprising that $p_n^2 - dq_n^2$ is a small integer. More surprisingly, there is the following result first proved by Lagrange: let $a^2 - db^2 = \pm 1$ for some $a, b \in \mathbb{Z}$. Then $a/b = p_n/q_n$ for some n. Since both p and q strictly increase with n, it follows that the smallest n with $p_n^2 - dq_n^2 = \pm 1$ gives us a fundamental unit in $\mathbb{Q}(\sqrt{d})$ (i.e. the fundamental unit that is > 1 is $p_n + q_n\sqrt{d}$). Thus we have an algorithm for finding a fundamental unit. See the text for further information about which convergent will actually give the fundamental unit. Using this, find the fundamental unit of the ring of integers in $\mathbb{Q}(\sqrt{11}), \mathbb{Q}(\sqrt{19})$, and $\mathbb{Q}(\sqrt{22})$. Don't use a computer, except to perform basic arithmetic to find the appropriate continued fraction expansions and to compute $p_n^2 - dq_n^2$.

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(5) Let $K = \mathbb{Q}(\sqrt{26})$ and let $\epsilon = 5 + \sqrt{26}$. Show

$$(2) = (2, \epsilon + 1)^2 \quad (5) = (5, \epsilon + 1)(5, \epsilon - 1) \quad (\epsilon + 1) = (2, \epsilon + 1)(5, \epsilon + 1).$$

Show that K has class number 2. Verify that ϵ is the fundamental unit. Deduce that all solutions in integers x, y to the equation $x^2 - 26y^2 = \pm 10$ are given by $x + \sqrt{26}y = \pm\epsilon^n(\epsilon \pm 1)$ for $n \in \mathbb{Z}$.