MATH 748: HOMEWORK 5 (FINAL)

(1) Let $d \in \mathbb{Z}$ be square-free and p a prime number not dividing $2d$. Let $K = \mathbb{Q}(\sqrt{d})$. Show that the ideal $p\mathcal{O}_K$ is prime if and only if the congruence $x^2 \equiv d \pmod{p}$ has no solutions.
(2) Milne 3-3
(3) Give an example of a domain B, a non-zero prime ideal $\wp$ in B, and a subring A of B such that $\wp \cap A = 0$. (Note that in class in our proof for extensions of Dedekind domains, we used integrality of B over A to show that this can't happen in that case.)
(4) Let $K = \mathbb{Q}(\sqrt{-m})$ where $m > 0$ is the product of distinct primes $p_1, \dots, p_k$ and $m \not\equiv 3 \pmod{4}$. Show that $(p_i) = \wp_i^2$ where $\wp_i = (p_i, \sqrt{-m})$. Show that just two of the ideals $\prod_i \wp_i^{r_i}$ with $r_i \in \{0, 1\}$ are principal. Deduce that the class group Cl_K contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{k-1}$.
(5) Let $K = \mathbb{Q}(\theta)$ where θ is a root of $X^3 - 4X + 7$. Determine (by hand) the ring of integers and discriminant of K. You may look up and use (if you quote) the formula for the discriminant of a cubic polynomial . Determine (by hand) the factorization into prime ideals of $p\mathcal{O}_K$ for $p = 2, 3, 5, 7, 11$.
(6) Use sage to compute the ring of integers and discriminant of $K = \mathbb{Q}(\alpha)$ where α is a root of $f(X) = X^3 + X^2 - 2X + 8$. Which primes ramify? Which primes of $\mathbb{Z}$ can you determine the factorization of in $\mathcal{O}_K$ by looking at how $f(X)$ factors mod p? For the primes for which you can not do the above, use sage to compute their factorization.