MATH 748: HOMEWORK 4 (FINAL)

(1) Milne 3-4 (note relation to HW 1 #2, but I do not recommend you show this ring is isomorphic to that one)

(2) Let I be a non-zero ideal of a Dedkind domain A. Show that in every ideal class of $Cl(A)$ there is an integral ideal relatively prime to I.

(3) Show that every ideal I in a Dedekind domain A can be generated by two elements.

(4) Let $K = \mathbb{Q}(\sqrt{35})$ and $\omega = 5 + \sqrt{35}$. Verify the ideal equations $(2) = (2, \omega)^2$ and $(5) = (5, \omega)^2$ and $(\omega) = (2, \omega)(5, \omega)$. Show that the class group of K contains an element of order 2. (Make sure to verify that ideals are non-principal if you claim it.)

(5) Use sage to compute the class groups of the real quadratic fields $\mathbb{Q}(\sqrt{D})$ for square-free $2 \leq D \leq 50$. Make a table of the frequencies of each group.

(6) Let p be an odd prime and $K = \mathbb{Q}(\zeta_p)$ where ζ_p is a primitive pth root of unity. Determine $[K : \mathbb{Q}]$. Calculate $Nm_{K/\mathbb{Q}}(\pi)$ and $\mathrm{Tr}_{K/\mathbb{Q}}(\pi)$ where $\pi = 1 - \zeta_p$.
   (a) By considering traces $\mathrm{Tr}_{K/\mathbb{Q}}(\zeta_p^j \alpha)$ show that $\mathbb{Z}[\zeta_p] \subset \mathcal{O}_K \subset \frac{1}{p}\mathbb{Z}[\zeta_p]$.
   (b) Show that $\frac{1-\zeta_p^r}{1-\zeta_p^s}$ is a unit in $\mathcal{O}_K$ for all $r, s \in \mathbb{Z}$ coprime to p, and that $\pi^{p-1} = up$ where u is a unit in $\mathcal{O}_K$.
   (c) Prove that the natural map $\mathbb{Z} \to \mathcal{O}_K/(\pi)$ is surjective. Deduce that for any $\alpha \in \mathcal{O}_K$ and $m \geq 1$ there exist $a_0, a_1, \ldots, a_{m-1} \in \mathbb{Z}$ such that

$$\alpha \equiv a_0 + a_1\pi + \cdots + a_{m-1}\pi^{m-1} \pmod{\pi^m \mathcal{O}_K}.$$

   (d) Deduce that $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$.