Math 185 001 **Final Exam** **May 8, 2023**

Name: ____

SID: ____

Instructions :

1. You have 170 minutes, 8:10am-11:00am. You may not need that much time.
2. No books, notes, or other outside materials are allowed.
3. There are ? questions on the exam for a total of ? points.
4. You need to show all of your work and justify all statements, unless otherwise noted. If you need more space, use the pages at the back of the exam or come get more paper at the front of the class. If you do so, please indicate which page your solution continues on.
5. Before you begin, take a quick look at all the questions on the exam, and start with the one you feel the most comfortable solving. It is more important to do the problems well that you know how to do, than it is to finish the whole exam.
6. While attempting any problem, do write something even if you are unable to solve it completely. You may get partial credit.

(Do not fill these in; they are for grading purposes only.)

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7	
Total	

Useful Formulae

1. **Cauchy Riemann equations** $f(z) = u(x,y) + iv(x,y)$

   Rectangular Coordinates
$$u_x = v_y \quad u_y = -v_x \quad f'(z) = u_x + iv_x$$
   Polar Coordinates
$$ru_r = v_\theta \quad u_\theta = -rv_r \quad f'(z) = e^{-i\theta}(u_r + iv_r)$$

2. **Cauchy Integral Formula** If f is analytic on and interior to a simple closed contour C, and z_0 is interior to C, then
$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz.$$

3. **Taylor/Laurent Theorem**

   Let $f(z)$ be analytic on the domain $r < |z - z_0| < R$. Then $f(z)$ is given by its Laurent series centered at z_0
$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$
   where
$$a_k = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{k+1}}dz$$
   for any simple closed contour C in that domain such that z_0 is interior to C.

   If $r = 0$, $\text{Res}_{z=z_0} f(z) = a_{-1}$.

   If $r = 0$ and f is analytic at z_0, then $a_k = 0$ for $k < 0$, $a_k = \frac{f^{(k)}(z_0)}{k!}$ for $k \geq 0$, and the series above is the Taylor Series centerd at z_0.

4. **Cauchy's Residue Theorem** If f is analytic on and interior to a simple closed curve C except at a finite set of points $z_1, ..., z_n$ interior to C then
$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z)$$
   If f is analytic on and exterior to a simple closed curve C then
$$\int_C f(z)dz = -\text{Res}_{z=\infty} f(z) = \text{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

5. **Residues at Poles**

   a) Let z_0 be an isolated singularity of $f(z)$. If $\phi(z) = (z-z_0)^m f(z)$ is analytic and nonzero at z_0 then $f(z)$ has a pole of order m at z_0 and
$$\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

   b) If p, q are analytic at z_0, $p(z_0) \neq 0, q(z_0) = 0$, and $q'(z_0) \neq 0$ then $f(z) = p(z)/q(z)$ has a simple pole at z_0 and
$$\text{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}$$

1. Prove that $f(z) = \bar{z}$ is not differentiable for any $z \in \mathbb{C}$.

$$\lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$ does not exist

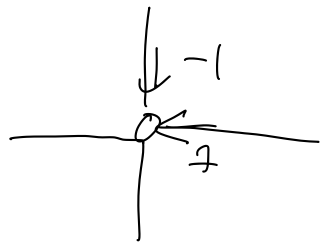
because if Δz is real $(\Delta z = \Delta x, \Delta y = 0)$

$$\lim_{\Delta x \to 0} \frac{\overline{\Delta x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1$$

but if Δz is imaginary $(\Delta z = i\Delta y, \Delta x = 0)$

$$\lim_{\Delta y \to 0} \frac{\overline{i\Delta y}}{i\Delta y} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = \lim_{\Delta y \to 0} -1 = -1$$

and the two limits are not equal



2. a) Find all $z \in \mathbb{C}$ such that $z^4 = -4$.

b) Give all the values of $(-2)^{i/2}$.

a) $-4 = 4e^{i\pi}$

$$z = (4)^{1/4} e^{i\left(\frac{\pi}{4} + \frac{2\pi k}{4}\right)}, \quad k \in \mathbb{Z}$$

$$= \sqrt{2}\, e^{i\frac{\pi}{4}}, \quad \sqrt{2}\, e^{i\frac{3\pi}{4}}, \quad \sqrt{2}\, e^{i\frac{5\pi}{4}}, \quad \sqrt{2}\, e^{i\frac{7\pi}{4}}$$

$$= 1+i, \quad -1+i, \quad -1-i, \quad 1-i$$

b) $-2 = 2e^{i\pi}$ $\qquad \log(-2) = \ln 2 + i(\pi + 2\pi k),\ k \in \mathbb{Z}$

$$-2^{i/2} = e^{\frac{i}{2}\log(-2)} = e^{-\frac{1}{2}(\pi + 2\pi k)} e^{\frac{i \ln(2)}{2}}$$

$$= e^{-\frac{\pi}{2} + \pi k} \cdot \left(\cos\left(\frac{\ln(2)}{2}\right) + i \sin\left(\frac{\ln(2)}{2}\right)\right), \quad k \in \mathbb{Z}.$$

3. Prove that $f(z) = \mathrm{Log}(x^3 - 3xy^2 + i(3x^2y - y^3))$ is analytic when $x > y > 0$

$\mathrm{Log}(w)$ is analytic unless $w \in \mathbb{R}$, $w \leq 0$.

$u + iv = (x^3 - 3xy^2) + i(3x^2y - y^3)$ is analytic everywhere because the partial derivs

$u_x = 3x^2 - 3y^2 \qquad u_y = -6xy$

$v_x = 6xy \qquad v_y = 3x^2 - 3y^2$

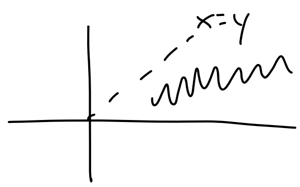
are continuous everywhere, and $u_x = v_y$, $u_y = -v_x$

If $x > y > 0$, $3x^2y > 3y^3 > y^3$, so $v > 0$

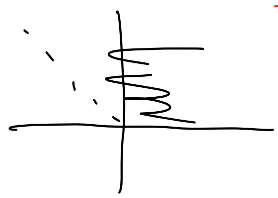
and $w = u + iv \notin \mathbb{R}$. Thus the composition is analytic when $x > y > 0$.

(OR: $w = u + iv = (x+iy)^3 = z^3$. If $x > y > 0$

then $z = re^{i\theta}$, $0 < \theta < \frac{\pi}{4}$



So $z^3 = r^3 e^{i3\theta} \notin \mathbb{R}$ because $0 < 3\theta < \frac{3\pi}{4}$)



4. Find an antiderivative for $f(z) = \sqrt{r}e^{i\theta/2}, r > 0, \theta \in (-\pi, \pi)$ which is analytic on the line sefment C from 1 to $-1+i$; justify your claim. Compute $\int_C f(z)dz$.

(idea: $f(z)$ is a branch of $z^{1/2}$;

If $g' = f(z)$, g should be a branch of $\frac{2}{3} z^{3/2}$)

Let $g(z) = \frac{2}{3} r^{\frac{3}{2}} e^{i\frac{3\theta}{2}}$, $r>0$ $-\pi<\theta<\pi$

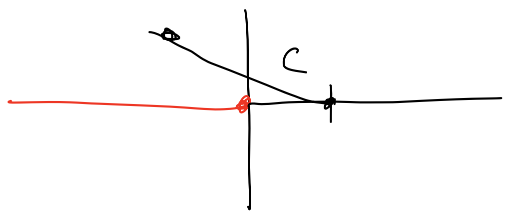
$= u+iv = \frac{2}{3} r^{\frac{3}{2}} \cos\left(\frac{3\theta}{2}\right) + i\,\frac{2}{3} r^{\frac{3}{2}} \sin\left(\frac{3\theta}{2}\right)$

is an analytic branch of $z^{\frac{3}{2}}$, and

$g'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(r^{\frac{1}{2}} \cos\left(\frac{3\theta}{2}\right) + i r^{\frac{1}{2}} \sin\left(\frac{3\theta}{2}\right)\right)$

$= e^{-i\theta} r^{1/2} e^{i\frac{3\theta}{2}} = r^{\frac{1}{2}} e^{i\frac{\theta}{2}} = f(z).$

$g(z)$ is analytic unless $z \in \mathbb{R}$, $z \le 0$; C contains no such z



$\int_C f(z) = g(-1+i) - g(1)$

$= g\left(\sqrt{2} e^{i\frac{3\pi}{4}}\right) - g\left(1 e^{i0}\right)$

$= \frac{2}{3} 2^{\frac{3}{4}} e^{i\frac{9\pi}{8}} - \frac{2}{3}.$

5. Find the singularity of $f(z) = (z^2 - 4)\cos\left(\frac{1}{z-2}\right)$. Identify whether it is a removable sigularity, pole of order m, or essential singularity. Find the residue of $f(z)$ at the singularity.

For all $\omega \in \mathbb{C}$, $\cos(\omega) = 1 - \frac{\omega^2}{2} + \frac{\omega^4}{4!} + \cdots$

Only singularity is $z=2$

For $0 < \underbrace{|z-2|}_{=\omega} < \infty$,

$$(z+2)(z-2)\left(1 - \frac{1}{2}\frac{1}{(z-2)^2} + \frac{1}{4!}\frac{1}{(z-2)^4} + \cdots\right)$$

$$= \left(4 + (z-2)\right)\left((z-2) - \frac{1}{2}\frac{1}{(z-2)} + \frac{1}{4!}\frac{1}{(z-2)^3} + \cdots\right)$$

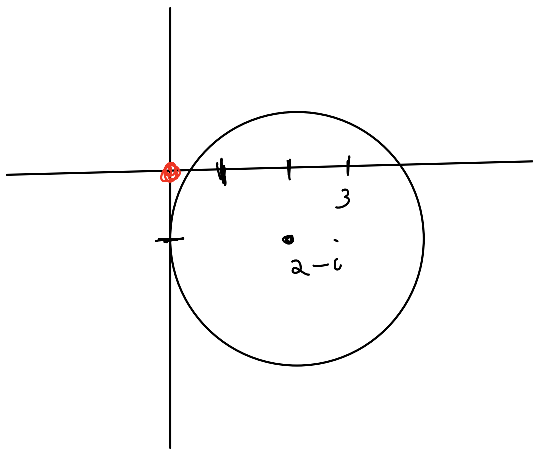
$$= (z-2) - \frac{1}{2} - \frac{2}{(z-2)} + \frac{1}{4!}\frac{1}{(z-2)^2} + \cdots$$

∞ many terms with negative exponent $\Rightarrow$ essential singularity

Residue = coefficient of $\frac{1}{z-2} = -2$

6. Compute
$$\int_C \frac{e^{1/z}+3z^2+2}{z^2-6z+9}dz$$
where C is the circle of radius 2 centerd at $2-i$.

$f(z) = e^{\frac{1}{z}} + 3z^2 + 2$ is analytic on $\mathbb{C}\setminus\{0\}$, so on and interior to C $\left(|2-i| = \sqrt{5} > 2 \right)$



By the Cauchy Integral formula, since 3 is interior to C $\left(|2-i-3| = \sqrt{2} < 2 \right)$

$$\int_C \frac{f(z)}{(z-3)^2}dz = \frac{2\pi i}{1!} f^{(2)}(3)$$

$$= 2\pi i \left(e^{\frac{1}{3}}\left(-\frac{1}{9}\right) + 6\cdot 3 \right)$$

$$= 2\pi i \left(18 - \frac{e^{\frac{1}{3}}}{9} \right).$$

7. Compute

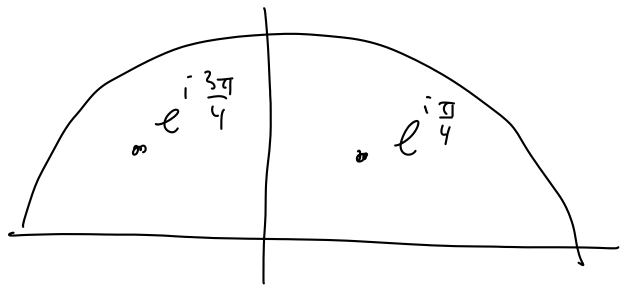
$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{x^4+1}dx.$$

Justify all steps of your computation.

$f(z) = \dfrac{z^2}{z^4+1}$ isolated singularities at
$(-1)^{1/4} = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$

$$\int_{-R}^{R}\frac{x^2}{x^4+1}dx + \int_{C_R} f(z)dz = 2\pi i\left(\operatorname*{Res}_{z=e^{i\frac{\pi}{4}}} f(z) + \operatorname*{Res}_{z=e^{i\frac{3\pi}{4}}} f(z)\right)$$

$(R > 1)$



$f(z) = \dfrac{p(z)}{q(z)} = \dfrac{z^2}{z^4+1}$; $p(e^{i\frac{\pi}{4}}) \neq 0$, $q(e^{i\frac{\pi}{4}}) = 0$,
$q'(e^{i\frac{\pi}{4}}) = 4(e^{i\frac{\pi}{4}}) \neq 0$

So $\displaystyle\operatorname*{Res}_{z=e^{i\frac{\pi}{4}}} = \frac{(e^{i\frac{\pi}{4}})^2}{4(e^{i\frac{\pi}{4}})^3} = \frac{1}{4}e^{-i\frac{\pi}{4}}$

similarly $\displaystyle\operatorname*{Res}_{z=e^{i\frac{3\pi}{4}}} = \frac{(e^{i\frac{3\pi}{4}})^2}{4(e^{i\frac{3\pi}{4}})^3} = \frac{1}{4}e^{-i\frac{3\pi}{4}} = \frac{1}{4}e^{i\frac{5\pi}{4}}$
$= -\frac{1}{4}e^{i\frac{\pi}{4}}$

$$2\pi i\left(\frac{1}{4}e^{-i\frac{\pi}{4}}-\frac{1}{4}e^{i\frac{\pi}{4}}\right)=\pi i\left(i\sin\left(-\frac{\pi}{4}\right)\right)=\frac{\pi}{\sqrt{2}}$$

If $|z|=R>1$, $\quad |z^4-1|\geq|R^4-1|=R^4-1$

$$|f(z)|=\frac{R^2}{|z^4-1|}\geq\frac{R^2}{R^4-1}$$

$$\left|\int_{C_R}f(z)\,dz\right|\leq\frac{(\pi R)\,R^2}{R^4-1}=\frac{\frac{\pi}{R}}{1-\frac{1}{R^4}}\to 0 \text{ as } R\to\infty$$

$$\lim_{R\to\infty}\left[\int_{-R}^{R}\frac{x^2}{x^4+1}\,dx+\int_{C_R}f(z)\,dz\right]=\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{x^4+1}\,dx$$

$$=\frac{\pi}{\sqrt{2}}$$