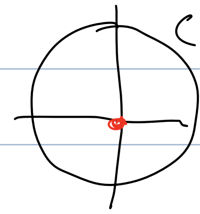
§77

2) a) $f(z) = \dfrac{e^{-z}}{z^2}$ isolated sing. at 0



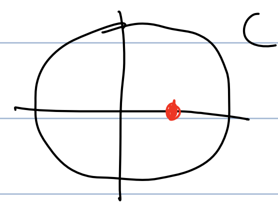
$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} f(z)$$

$$f(z) = \frac{1}{z^2}\left(1 - z + \frac{z^2}{2} + \dots\right) = \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2} + \dots \quad \text{for all } z$$

$$\operatorname*{Res}_{z=0} f(z) = -1$$

$$\int_C f(z)\,dz = -2\pi i$$

b) $f(z) = \dfrac{e^{-z}}{(1-z)^2}$ isolated singularity at $z=1$

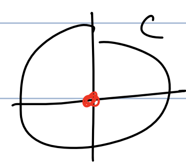


$$f(z) = \frac{e^{-(z-1)-1}}{(z-1)^2} = e^{-1}\,\frac{e^{-(z-1)}}{(z-1)^2}$$

$$= \frac{1}{e(z-1)^2}\left(1 - (z-1) + \dots\right) = \frac{1/e}{(1-z)^2} + \frac{-1/e}{\quad} + \dots$$

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=1} f(z) = -\frac{2\pi i}{e}$$

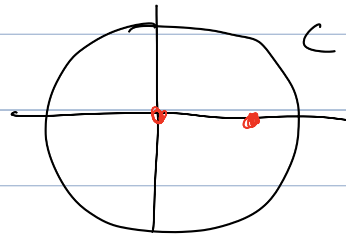
c) $f(z) = z^2 e^{\frac{1}{z}}$ isolated singularity at 0



$$f(z) = z^2\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots\right) = z^2 + z + \frac{1}{2} + \frac{1}{6z} + \dots$$

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} f(z) = 2\pi i\left(\frac{1}{6}\right) = \frac{\pi i}{3}$$

d) $f(z) = \frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)}$



analytic exterior to C

$$\int_C f(z)dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z)$$

$$= 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\,\frac{\frac{1}{z}+1}{\frac{1}{z^2}-\frac{2}{z}} = \frac{\frac{1}{z}+1}{1-2z} = \frac{1+z}{z(1-2z)}$$

$$= \frac{1}{z}\,\frac{1}{1-2z} + \underbrace{\frac{1}{1-2z}}_{\text{analytic near 0, contributes only } \geq 0 \text{ powers of } z\text{, ignore}}$$

for $|2z|<1$ $$= \frac{1}{z}\left(1+2z+4z^2+\dots\right)$$

$$\operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 1, \quad \int_C f(z)dz = 2\pi i$$

4) a) analytic outside C, $\left(\mathbb{C} - \{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}\right)$

$$\int_C f(z)dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\,\frac{1/z^5}{1-\frac{1}{z^3}} = \frac{1}{z^7-z^4} = \frac{-1}{z^4}\,\frac{1}{(1-z^3)}$$

$(|z|<1)$ $$= -\frac{1}{z^4}\left(1+z^3+\dots\right) = -\frac{1}{z^4} - \frac{1}{z}$$

$$\int_C f(z)dz = -2\pi i.$$

b) Analytic on $\mathbb{C} \setminus \{\pm i\}$, so outside C

$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \frac{1}{1+\frac{1}{z^2}} = \frac{1}{z^2+1}$ analytic at 0,

So $\operatorname{Res}_{z=\infty} f(z) = 0$, $\int_C f(z)dz = 0$.

c) $\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} z = \frac{1}{z}$, $\operatorname{Res}_{z=\infty} f(z) = -2\pi i$

$\int_C f(z)dz = 2\pi i$.

6) Chose $R > \max\{|z_1|, \dots, |z_n|\}$, and let C be the circle $|z| = R$. Then

$$\int_C f(z)\,dz = \sum_{k=1}^{n} 2\pi i \operatorname{Res}_{z=z_k} f(z).$$

But f is also analytic exterior to C so

$$\int_C f(z)dz = -2\pi i \operatorname{Res}_{z=\infty} f(z).$$

§79

1) a) $z \exp\left(\frac{1}{z}\right) = z\left(\sum_{k=0}^{\infty} \frac{1}{k!\, z^k}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} z^{1-k}$

∞ many nonzero terms with negative exponents $\Rightarrow$ 0 is an essential singularity

b) isolated sing. at $z=-1$

$$\frac{z^2}{z+1} = \frac{(z+1-1)^2}{z+1} = \frac{(z+1)^2}{z+1} - \frac{2(z+1)}{z+1} + \frac{1}{z+1}$$

$$= (z+1) - 2 + \frac{1}{z+1}$$

One nonzero term w/ negative exponent $\Rightarrow$ pole

c) $$\frac{z - \frac{z^3}{6} + \cdots}{z} = 1 - \frac{z^2}{6} + \cdots$$ no negative exponents $\Rightarrow$ removable sing at 0

d) $$\frac{1 - \frac{z^2}{2} + \cdots}{z} = \frac{1}{z} - \frac{z}{2} + \cdots$$ one nonzero term with negative exponent $\Rightarrow$ pole at 0.

e) $$\frac{1}{(z-3)^3}$$ one term with negative exponent $\Rightarrow$ pole at 3.

2) a) $$\frac{1-\cosh(z)}{z^3} = \frac{-\frac{z^2}{2} - \frac{z^4}{24}}{z^3} = \frac{-1/2}{z} + -1/24\,(z) + \cdots$$

pole of order 1, Res $= -1/2$

b) $$\frac{1-\left(1+2z+\frac{4z^2}{2}+\frac{8z^3}{6}+\frac{16z^4}{24}+\frac{32z^5}{120}+\cdots\right)}{z^4}$$

$$= \frac{-2}{z^3} + \frac{-2}{z^2} + \frac{-4/3}{z} + -2/3 + \cdots$$

Pole of order 3, Res $= -4/3$

c) $\dfrac{\exp(2z-2+2)}{(z-1)^2} = \dfrac{e^2\left(1+2(z-1)+\frac{4(z-1)^2}{2}+\frac{8(z-1)^3}{6}+\dots\right)}{(z-1)^2}$

$= \dfrac{e^2}{(z-1)^2} + \dfrac{2e^2}{z-1} + 2e^2 + \dots$

Pole of order 2, $\quad$ Res $= 2e^2$

§81

1) a) isolated singularities at $z = \pm 3i$

$z = 3i$

$\phi(z) = (z-3i)\dfrac{z+1}{z^2+9} = \dfrac{z+1}{z+3i}$ analytic near $z=3i$, $\phi(3i) = \dfrac{3i+1}{6i} = \dfrac{3-i}{6}$

Simple Pole, $\quad$ Res $= \phi(3i) = \dfrac{3-i}{6}$

$z = -3i$

$\phi(z) = (z+3i)\dfrac{z+1}{z^2+9} = \dfrac{z+1}{z-3i}$ analytic near $-3i$, $\phi(-3i) = \dfrac{-3i+1}{-6i} = \dfrac{3+i}{6}$

$=$ Res.

b) isolated singularity at $z=1$, $\quad \phi(z) = z^2+2$, $\phi(1) = 3$

simple pole, $\quad$ Res $= \phi(1) = 3$.

c) isolated sing at $z = -\frac{1}{2}$

$\phi(z) = (z+\frac{1}{2})^3 \dfrac{z^3}{(2z+1)^3} = \dfrac{z^3}{8}$, $\quad \phi(-\frac{1}{2}) = -\dfrac{1}{64}$

Pole of order 3; $\quad$ Res $= \dfrac{\phi^{(2)}(-\frac{1}{2})}{2!} = \dfrac{1}{2}\left(\dfrac{6(-\frac{1}{2})}{8}\right) = -\dfrac{3}{16}$

d) Singularities $z = \pm i\pi$

$z = i\pi$

$$\phi(z) = (z - i\pi)\frac{e^z}{z^2+\pi^2} = \frac{e^z}{z+i\pi}, \quad \phi(i\pi) = \frac{-1}{2\pi i} = \frac{i}{2\pi}$$

simple pole, residue $= \phi(i\pi) = \frac{i}{2\pi}$

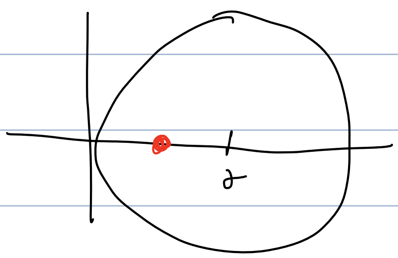
$z = -i\pi$

$$\phi(z) = (z+i\pi)\frac{e^z}{z^2+\pi^2} = \frac{e^z}{z-i\pi}, \quad \phi(-i\pi) = \frac{-1}{-2i\pi} = -\frac{i}{2\pi}$$

$= \text{Res}.$

4) Singularities $1, \pm 3i$

a)



$|z-2| = 2$

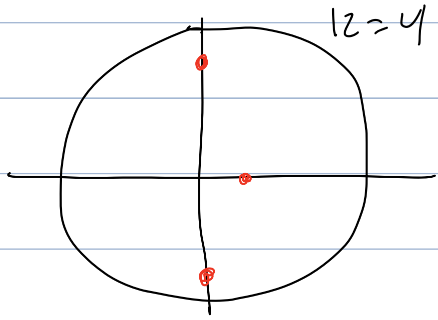
near $z=1$, $\phi(z) = (z-1)f(z)$

$$= \frac{3z^3+2}{z^2+9}, \quad \phi(1) = \frac{1}{2} = \text{Res}$$

$$\int_c f(z)\,dz = \pi i$$

b)

$|z| = 4$



analytic outside C

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\frac{\frac{3}{z^3}+2}{\left(\frac{1}{z}-1\right)\left(\frac{1}{z^2}+9\right)} = \frac{3+2z^3}{(1-z)(1+9z^2)z^2}$$

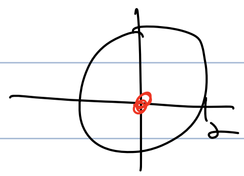
near $z=0$, $\phi(z) = z^2 \times \rightarrow = \frac{3+2z^3}{(1-z)(1+9z^2)}$ analytic, $\phi(0) = 3$

Pole of order 2

$\text{Res} \, \frac{1}{z^2} f\left(\frac{1}{z}\right) = \phi'(0) = 3$, $\int_{|z|=4} f(z)\,dz = 6\pi i$

5) Singularities $z=0$, $z=-4$

a)

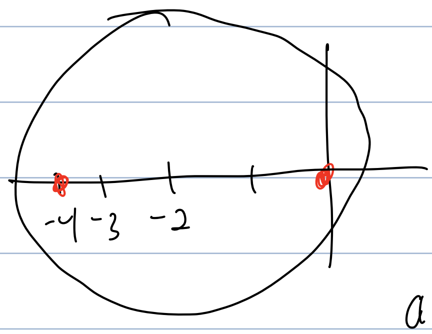


near $z=0$, $\phi(z) = z^3 f(z) = \frac{1}{z+4}$, $\phi(0) = \frac{1}{4} \neq 0$

$$\text{Res} = \frac{\phi^{(2)}(0)}{2} = \frac{2}{2} \frac{1}{(0+4)^3} = \frac{1}{64}$$

$$\int_{|z|=2} f(z)\,dz = \frac{\pi i}{32}$$

b)



$\text{Res}_{z=0} f(z) = \frac{1}{64}$

near -4,

$\phi(z) = (z+4) f(z) = \frac{1}{z^3}$, $\phi(-4) = -\frac{1}{64}$

Simple pole, $\text{Res}_{z=0} f(z) = \phi(-4) = -\frac{1}{64}$

Sum of Res $= 0$, $\int f(z)\,dz = 0$.

6) $\int_C \frac{\cosh(\pi z)\,dz}{z(z^2+1)}$ $\quad C = \{|z| = 2\}$

$f(z) = \frac{\cosh(\pi z)}{z(z+i)(z-i)}$ has simple poles at $0, \pm i$

at 0, $\phi(z) = \dfrac{\cosh(\pi z)}{z^2+1}$, $\phi(0) = 1$, $\underset{z=0}{\text{Res}}\, f(z) = 1$

at i, $\phi(z) = \dfrac{\cosh(\pi z)}{z(z+i)}$, $\phi(i) = \dfrac{\cosh(i\pi)}{i(2i)} = \dfrac{\cos(\pi)}{-2} = \dfrac{1}{2}$

at $-i$, $\phi(z) = \dfrac{\cosh(\pi z)}{z(z-i)}$ $\phi(-i) = \dfrac{\cosh(-i\pi)}{(-i)(-2i)} = \dfrac{\cos(\pi)}{-2} = \dfrac{1}{2}$

$$\int_C \frac{\cosh(\pi z)}{z(z^2+1)}\, dz = 2\pi i\left(1 + \frac{1}{2} + \frac{1}{2}\right) = 4\pi i.$$