§ 6.5

10) a) $\sinh(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$ all $z \in \mathbb{C}$

$\frac{\sinh(z)}{z^2} = \sum_{k=0}^{\infty} \frac{z^{2k-1}}{(2k+1)!}$ $|z| \neq 0$

$= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{z^{2k-1}}{(2k+1)!}$

$= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+3)!}$ reple k with $k+1$

b) $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$ for all $z \in \mathbb{C}$

$\sin(z^2) = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \frac{z^{14}}{7!} + \dots$ for all $z \in \mathbb{C}$

$\frac{\sin(z^2)}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \dots$ for all $|z| \neq 0$

11) If $0 < |z| < 4$ then $\left|\frac{z}{4}\right| < 1$ so

$\frac{1}{4z - z^2} = \frac{1}{4z}\left(\frac{1}{1-\frac{z}{4}}\right) = \frac{1}{4z} \sum_{k=0}^{\infty} \frac{z^k}{4^k}$

$= \frac{1}{4z} + \sum_{k=1}^{\infty} \frac{z^{k-1}}{4^{k+1}}$

$= \frac{1}{4z} + \sum_{k=0}^{\infty} \frac{z^k}{4^{k+2}}$ replace k with $k+1$

§6.8

1) $\sin(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$ for all $z \in \mathbb{C}$

$\sin\left(\frac{1}{z^2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!\, z^{4k+2}}$ for all $0<|z|$

$z^2 \sin\left(\frac{1}{z^2}\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!\, z^{4k}}$

4) If $0<|z|<1$, then

$$\frac{1}{z^2(1-z)} = \frac{1}{z^2} \sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} z^{k-2} = \frac{1}{z^2} + \frac{1}{z} + \sum_{k=2}^{\infty} z^{k-2}$$

$$= \frac{1}{z^2} + \frac{1}{z} + \sum_{k=0}^{\infty} z^k$$

If $1<|z|$, $\frac{1}{|z|}<1$, so

$$\frac{1}{z^2(1-z)} = \frac{1}{z^3} \frac{1}{\frac{1}{z}-1} = -\frac{1}{z^3}\left(\frac{1}{1-\frac{1}{z}}\right) = -\frac{1}{z^3} \sum_{k=0}^{\infty} \frac{1}{z^k}$$

$$= -\sum_{k=0}^{\infty} \frac{1}{z^{k+3}} = -\sum_{k=3}^{\infty} \frac{1}{z^k}$$

5) If $0<|z|<1$, then $\left|\frac{z}{2}\right| < \frac{1}{2} < 1$, so

$$\frac{1}{z-1} - \frac{1}{z-2} = -\left(\frac{1}{1-z}\right) + \frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right)$$

$$= -\sum_{k=0}^{\infty} z^k + \frac{1}{2} \sum_{k=0}^{\infty} \frac{z^k}{2^k} = \sum_{k=0}^{\infty} \left(2^{-k-1}-1\right) z^k$$

If $1 < |z| < 2$, $\frac{1}{|z|} < 1$, so

$$\frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} + \sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}}$$

$$= \frac{1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k} + \sum_{k}\frac{z^k}{2^{k+1}}$$

$$= \sum_{k=1}^{\infty}\frac{1}{z^{k+1}} + \sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}}$$

If $|z| > 2$, $\frac{1}{|z|} < \frac{1}{2}$, $\frac{2}{|z|} < 1$

$$\frac{1}{z}\frac{1}{(1-\frac{1}{z})} - \frac{1}{z}\left(\frac{1}{1-\frac{2}{z}}\right) = \frac{1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k} - \frac{2^k}{z^k}$$

$$= \sum_{k=0}^{\infty}\frac{1-2^k}{z^{k+1}}$$

$$= \sum_{k=1}^{\infty}\frac{1-2^{k-1}}{z^k}$$

§9.2

1) $$\left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2} = \sum_{k=0}^{\infty}(z^k)' = \sum_{k=1}^{\infty} k z^{k-1} = \sum_{k=0}^{\infty}(k+1)z^k$$

$$\left(\frac{1}{(1-z)^2}\right)' = \frac{2}{(1-z)^3} = \sum_{k=1}^{\infty}(k+1)k z^{k-1} = \sum_{k=0}^{\infty}(k+2)(k+1)z^k$$

4) For all $z \neq 0$,

$$\frac{1-\cos z}{z^2} = \frac{1}{z^2}\left(1 - \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k)!}\right)$$

$$= \frac{1}{z^2}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1} z^{2k}}{(2k)!}\right)$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^{2k-2}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k} z^{2k}}{(2k+2)!}$$

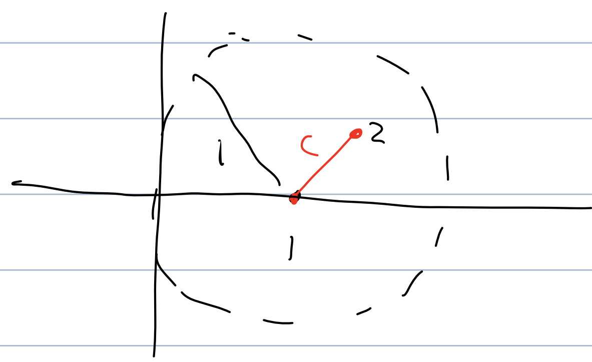
Since this is a power series with nonnegative exponents, it converges on $|z| < R$ for some R; we conclude $R = \infty$. Thus the power series converges to an entire function which is $\frac{(-1)^0}{(0+2)!} = \frac{1}{2}$ at $z = 0$.

b) Let $|z-1| < 1$. Let C be given by

$z(t) = 1 + t(z-1) \quad 0 \le t \le 1$;

then $\sum_{n=0}^{\infty} (-1)^n (w-1)^n$ converges uniformly on C, so

$$\int_C \frac{1}{w}\, dw = \sum_{n=0}^{\infty} (-1)^n \int_C (w-1)^n$$



$Log(w)$ is analytic on C, and

$(Log(w))' = \frac{1}{w}$, $\left(\frac{(w-1)^{n+1}}{n+1}\right)' = (w-1)^n$

Using the antiderivatives to compute the integrals we have

$$Log(z) - Log(1) = \sum_{n=0}^{\infty} (-1)^n \left[\frac{(z-1)^{n+1}}{n+1} - 0 \right]$$

$$Log(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n}$$

§77

1) a) for $0<|z|<1$, $\frac{1}{z+z^2} = \frac{1}{z}\left(\frac{1}{1-(-z)}\right) = \frac{1}{z}\left(1-z+z^2+\dots\right)$

$= \frac{1}{z} - 1 + z + \dots$ $\quad \operatorname{Res}_{z=0} \frac{1}{z+z^2} = 1$

b) for $0<|z|$, $z\cos\left(\frac{1}{z}\right) = z\left(1 - \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{4!}\left(\frac{1}{z}\right)^4 + \dots\right)$

$= z + \frac{(-1/2)}{z} + \frac{1}{4!}\frac{1}{z^3} + \dots$

$\operatorname{Res}_{z=0} z\cos\left(\frac{1}{z}\right) = -\frac{1}{2}$

c) for $0<|z|$, $\frac{z-\sin z}{z} = \frac{1}{z}\left(z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right)\right)$

$= \frac{1}{z}\left(\frac{z^3}{3!} - \frac{z^5}{5!} + \dots\right)$

$= \frac{z^2}{3!} - \frac{z^4}{5!} + \dots$

$\operatorname{Res}_{z=0} \frac{z-\sin z}{z} = 0.$

d) hard

$\frac{1}{\sin(z)} = \frac{1}{z - \frac{z^3}{6} + \frac{z^5}{120} + \dots} = \frac{1}{z}\left(\frac{1}{1-\left[\frac{z^2}{6} - \frac{z^4}{120} + \dots\right]}\right) = \frac{1}{z}\left(\frac{1}{1-w}\right)$

$= \frac{1}{z}\left(1 + w + w^2 + \dots\right) = \frac{1}{z}\left(1 + \frac{z^2}{6} - \frac{z^4}{120} + \frac{z^4}{36} + \dots\right)$

$$\frac{1}{z^4}\left(\frac{\cos(z)}{\sin(z)}\right) = \frac{1}{z^4}\left(1-\frac{z^2}{2}+\frac{z^4}{24}+\cdots\right)\left(\frac{1}{z}+\frac{z}{6}+\frac{7}{360}z^3+\cdots\right)$$

$$= \frac{1}{z^4}\left(\cdots+z^3\left(\frac{7}{360}-\frac{1}{12}+\frac{1}{24}\right)+\cdots\right)$$

$$= \cdots\left(\frac{7-30+15}{360}\right)\frac{1}{z}+\cdots = -\frac{1}{45}\frac{1}{z}$$

for $0<|z|<1$

e) $f(z)=\frac{1}{z^4}\left(z+\frac{z^3}{3!}+\cdots\right)\left(1+z^2+z^4+\cdots\right)$

$$= \frac{1}{z^4}\left(z+\left(1+\frac{1}{6}\right)z^3+\cdots\right) = \frac{1}{z^3}+\frac{7}{6}\frac{1}{z}+\cdots$$

$$\operatorname*{Res}_{z=0} f(z) = \frac{7}{6}$$

Add 1

$z_1, \dots z_n$ are not contained in $|z-z_0|<R$, so $f(z)=\frac{1}{(z-z_1)\cdots(z-z_n)}$ is analytic on that disc and the Taylor series converges there. Thus the radius of convergence is $\geq R$.

Now suppose the radius of convergence is $R_1>R$. Let z_m be the closest point in $z_1,\dots,z_n$ to z_0, so $|z_0-z_m|=R<R_1$. Thus the Taylor series converges to a continuous function at z_m. But it converges to $f(z)$ on $|z-z_0|<R$, and $\lim_{z\to z_m} f(z)$ does not exist, a contradiction.

Add 2

Since f is analytic on $|z-z_0| < R$

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2}(z-z_0)^2 + \dots \quad \text{for } |z-z_0| < R$$

$$\frac{f(z)}{(z-z_0)} = \frac{f(z_0)}{z-z_0} + f'(z_0) + \frac{f''(z_0)}{2}(z-z_0) + \dots$$

$$\operatorname*{Res}_{z=z_0} \frac{f(z)}{z-z_0} = f(z_0).$$