§5.9

2) Since $f(z)$ is analytic and nonzero on R, $g(z) = \frac{1}{f(z)}$ is analytic on R. Since R is compact and $|g(z)|$ is continuous, $|g(z)|$ achieves its maximum at $z_0 \in R$. Let D be the set of interior points of R; then D is a domain (§12, def of "region"). If $z_0 \in D$ then is constant on D by the max. modulus princ. and thus on R by continuity. So $z_0 \in R \setminus D$

The max value of $|g(z)|$ is the min value of $|f(z)|$.

5) $e^{f(z)}$ is analytic and nonzero and non-constant* on R, so by 2) $|e^{f(z)}| = e^{u}$ achieves its minimum at $z_0 \in R \sim D$ and not on D; since e is strictly increasing as a real valued function, u achieves its min. at z_0 and not in D.

* If e^{f} is constant, $|e^{f}| = e^{u}$ is constant, so u is constant (e^{x} is 1-1 as a real function), $f' = 0$ by C.R. eqns and f is constant.

§61

3) Let $\varepsilon > 0$. Choose N such that $n > N \Rightarrow |z_n - z| < \varepsilon$. Then $||z_n| - |z|| \leq |z_n - z| < \varepsilon$, so $|z_n| \to |z|$.

9) a) Choose $N \in \mathbb{N}$ such that $n > N \Rightarrow |z_n - z| < 1$. Then $n > N \Rightarrow |z_n| = |z + (z_n - z)| \leq |z| + 1$.

So for all n, either $z_n = z_1, \ldots, z_{N-1}$ or $|z_n| \leq |z| + 1$; so $|z_n| \leq \max\{|z_1|, \ldots, |z_{N-1}|, |z|+1\}$ for all n.

b) Since $x_n \to x$ and $y_n \to y$, $|x_n| \leq M_1$ for all n and $|y_n| \leq M_2$ for all n. Thus $|z_n| \leq |x_n| + |y_n| \leq M_1 + M_2$ for all n.

§65

2) a) $(e^z)' = e^z$, so $f^{(n)}(1) = e^1 = e$ for all n and the Taylor series centered at 1 is $\sum_{n=0}^{\infty} \frac{e}{n!}(z-1)^n$.

b) $e^z = e\, e^{z-1} = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$.

3) If $|z| < \sqrt{2}$, $|z^4| = |z|^4 < \sqrt{2}^4 = 4$, so $\left|\frac{z^4}{4}\right| < 1$ and

$$\frac{z}{z^4+4} = \frac{z}{4} \frac{1}{1 - \left(-\frac{z^4}{4}\right)} = \frac{z}{4} \sum_{k=0}^{\infty} \left(-\frac{z^4}{4}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^{k+1}} z^{4k+1}$$

5) 6 8 9

$$\sinh(z) = -\sinh(z+\pi i) = -\sinh(z+\pi i - 2\pi i)$$
$$= -\sinh(z-\pi i)$$
$$= -\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(z-\pi i)^{2k+1}$$

6) $\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$ is analytic on $\mathbb{C} \setminus \{\frac{\pi}{2}i + \pi k i \mid k \in \mathbb{Z}\}$

$\Rightarrow$ analytic on $|z| < \frac{\pi}{2}$

$\Rightarrow$ Series centered at 0 converges on $|z| < \frac{\pi}{2}$. If it converged on a larger disc it would converge to a function continuous at $\frac{\pi}{2}i$, but $|\tanh(z)| \to \infty$ as $z \to \frac{\pi}{2}i$.

$\tanh(0) = 0$ $\quad (\tanh)'(0) = \frac{1}{\cosh^2(0)} = 1$ $\quad \tanh^{(2)}(0) = \frac{-2\sinh(0)}{\cosh^3(0)} = 0$

$$\tanh^{(3)}(0) = \frac{-2}{\cosh^2(0)} + \frac{6\sinh^2(0)}{\cosh^4(0)} = -2$$

$$\tanh(z) = z - \frac{2}{3!}z^3 + \ldots = z - \frac{1}{3}z^3 + \ldots$$

8) a) $\cos(z) = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(iz)^k}{k!} + \frac{(-iz)^k}{k!}\right)$ $\begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{2(iz)^k}{k!} & \text{if } k \text{ is} \end{cases}$

$$= \sum_{k=0}^{\infty} \frac{(iz)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$$

b) $f^{(2k+1)}(0) = \pm\sin(0) = 0$ $\quad f^{(2k)} = (-1)^k \cos(0) = (-1)^k$

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$$

9) $\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$ all $z \in \mathbb{C}$

$\sin(z^2) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k+2}}{(2k+1)!}$ all $z \in \mathbb{C}$

Since $f(z) = \sin(z^2)$ is entire, $f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)\, z^m}{m!}$

and by uniqueness of series representations

$f^{(m)}(0) = 0$ unless $m = 4k+2$ for $k \in \mathbb{Z}$

i.e., m is even (and $m \equiv 2 \mod 4$.)

and if $m = 4n$, $\frac{m-2}{4} = n - \frac{1}{2} \notin \mathbb{Z}$.