47.4

$$\left|\frac{2z^2-1}{z^4+5z^2+4}\right| = \left|\frac{2z^2-1}{(z^2+4)(z^2+1)}\right| \le \frac{2|z|^2+1}{\left||z|^2-4\right|\,\left||z|^2-1\right|}$$

$|2z^2-1| < 2|z|^2+1$ $\quad |z^2+4| \ge \left||z|^2-4\right|$ $\quad |z^2+1| \ge \left||z|^2-1\right|$

$$\int_{|z|=R} \frac{2z^2-1}{z^4+5z^2+4}\,dz \le \frac{2R^2-1}{(R^2-4)(R^2-1)}(2\pi R) = 2\pi\left(\frac{\frac{1}{R}-\frac{1}{R^4}}{1-\frac{5}{R^2}+\frac{4}{R^4}}\right)$$

$\to \frac{0}{1} = 0$ as $R \to \infty$.

49.3

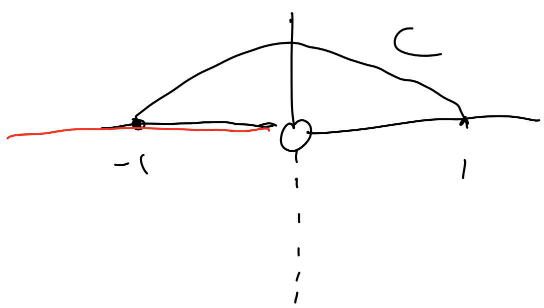
If $F(z) = (z-z_0)^n$, $F'(z) = (z-z_0)^{n-1}$ $\quad n \neq 0$, $z \neq z_0$

Thus $(z-z_0)^{n-1}$ has an antiderivative on $D = \mathbb{C} \setminus \{z_0\}$

and $\int_C (z-z_0)^{n-1}\,dz = 0$ for all closed contours C in D.

49.5

$f(z) = \exp(i\log(z))$ $\qquad \log(z) = \ln r + i\theta$, $\quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2}$



Then $f(z) = \exp(i\,\mathrm{Log}(z))$ for all z on C, and the integral we want to compute equals $\int_C f(z)\,dz$.

At all points of C, $f(z) = \left[\dfrac{\exp((i+1)\log(z))}{i+1}\right]'$

So
$$\int_C f(z)\,dz = \frac{\exp((i+1)\log(-1)) - \exp((i+1)\log(1))}{i+1}$$
$$= \frac{\exp((i+1)0) - \exp((i+1)i\pi)}{i+1}$$
$$= \frac{1 - e^{-\pi + i\pi}}{i+1} = (1+e^{-\pi})\left(\frac{1-i}{2}\right)$$

§3.2

a) $\dfrac{1}{3z^2+1}$ is analytic unless $3z^2+1=0$

$$z^2 = -\frac{1}{3}$$
$$z = \pm\frac{i}{\sqrt{3}}$$

Since $\frac{1}{\sqrt{3}} < 1$, those points lie inside the square,
so f is analytic between the circle and square

b) f is analytic unless $\frac{z}{2} = \pi k \Rightarrow z = 2\pi k$, $k \in \mathbb{Z}$
of those points $z=0$ is inside the square, and the rest are outside the circle, since $|2\pi k| \geq |2\pi| > 4$ for $k \neq 0$

So f is analytic between the circle and square

c) $f(z)$ is analytic unless $e^z = 1 \Rightarrow$

$z = \log(1) \Rightarrow z = 0 + 2\pi i k$, $k \in \mathbb{Z}$

$|z| = 2\pi|k|$, $k \in \mathbb{Z}$, see part b.

57, 1

Since $\frac{\pi i}{2}$ is interior to C

1) a) $= 2\pi i \, f\left(\frac{\pi i}{2}\right)$ where $f(z) = e^{-z}$

$= 2\pi i \, e^{-i\frac{\pi}{2}} = 2\pi i(-i) = 2\pi$.

b) Since $\sqrt{8} > 2$, $\frac{\cos(z)}{z^2+8}$ is analytic on and interior to C, so

$= 2\pi i \frac{\cos(0)}{0+8} = \frac{\pi i}{4}$

c) $\int_C \frac{z}{2z+1} dz = \frac{1}{2} \int_C \frac{z}{z-(-\frac{1}{2})} = \frac{2\pi i(-\frac{1}{2})}{2} = -\frac{\pi i}{2}$

d) $\cosh^{(3)}(0) = \frac{3!}{2\pi i} \int_C \frac{\cosh(z)}{z^{3+1}} dz$

So $\int_C \frac{\cosh(z)}{z^4} dz = \frac{2\pi i}{6} \sinh(0) = 0$

e) $= 2\pi i \left(\tan\left(\frac{z}{2}\right)\right)'(z_0) = \frac{2\pi i \sec^2\left(\frac{z_0}{2}\right)}{2}$

$= \pi i \sec^2\left(\frac{z_0}{2}\right)$

57,3

$2s^2-s-2$ is analytic on and interior to $|s|=3$,
so for z in the interior to $|s|=3$ (i.e. $|z|<3$)

$$\int \frac{2s^2-s-2}{s-z}\,dz = 2\pi i\,(2z^2-z-2)$$ and if $z=2$ we get $2\pi i\,(8-4)=8\pi i$.

If $|z|>3$, $\frac{2s^2-s-2}{s-z}$ is analytic on and interior to $|s|=3$, so $$\int_C \frac{2s^2-s-2}{s-z}\,dz = 0.$$

57,5

Since f is analytic, f' is analytic.

If z_0 is exterior to C, $\frac{f'(z)}{(z-z_0)}$ and $\frac{f(z)}{(z-z_0)^2}$ are analytic on and interior to C, so both integrals are 0. If z_0 is interior to C,

$$\int_C \frac{f(z)}{(z-z_0)^2} = 2\pi i\, f'(z_0) = \int_C \frac{f'(z)}{z-z_0}.$$

Also, $$\left(\frac{f(z)}{z-z_0}\right)' = \frac{f'(z)}{(z-z_0)} - \frac{f(z)}{(z-z_0)^2}$$ on C, so

Since C is closed $$0 = \int_C \left(\frac{f(z)}{z-z_0}\right)' = \int_C \frac{f'(z)}{(z-z_0)} - \int_C \frac{f(z)}{(z-z_0)^2}.$$

55.7 e^{az} is analytic on and interior to C, so

$$\frac{1}{2\pi i}\int_C \frac{e^{az}}{z-0}dz = e^{a\cdot 0} = 1.$$

Thus $$\int_0^{2\pi} \frac{e^{ae^{i\theta}}}{e^{i\theta}}(ie^{i\theta})\,d\theta = i\int_0^{2\pi} e^{a\cos\theta + ai\sin\theta}\,d\theta$$

$$= i\int_0^{2\pi} e^{a\cos\theta}\left(\cos(a\sin\theta) + i\sin(a\sin\theta)\right)d\theta = 2\pi i$$

Thus taking imaginary parts of each

$$\int_0^{2\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = 2\pi$$

59.1

$Re(f(z)) \le u_0$ $\quad g(z) = e^{f(z)}$ is entire

and $\quad |g(z)| = |e^{u(x,y)} e^{iv(x,y)}| = e^{u(x,y)} \le e^{u_0}$

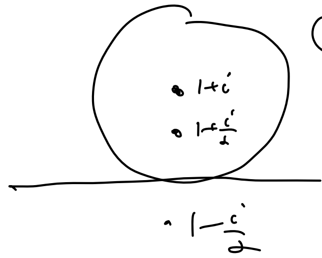
So $g(z)$ is entire and bounded, thus constant, say $g(z) = c$. Then

$f(z) = \log c$; since f is continuous it cannot "jump" between values of $\log c$, which differ by $2\pi i$.

So f is constant.

AMI

a) f is analytic on and interior to C, so the integral is 0



b) $f(z) = \dfrac{4\,Log(z)}{4z^2-8z+5} = \dfrac{4\,Log(z)}{(z-(1+\frac{i}{2}))(z-(1-\frac{i}{2}))}$

$4\,Log(z) \,/\, (z-(1-\frac{i}{2}))$ is analytic on and interior to C.

So $\displaystyle\int_C f(z) = 2\pi i \, \frac{4\,Log(1+\frac{i}{2})}{(1+\frac{i}{2}-(1-\frac{i}{2}))} = 8\pi\, Log(1+\tfrac{i}{2})$

c) $\dfrac{1}{z^2+2z+2} = \dfrac{1}{(z-(-1+i))(z-(-1-i))}$ is analytic on and interior to C

So the integral is 0.

d) $\dfrac{e^{2z}}{z^2-(2+2i)z+2i} = \dfrac{e^{2z}}{(z-(1+i))^2}$, e^{2z} is analytic on and interior to C

$\displaystyle\int_C \frac{e^z}{(z-(1+i))^2}\,dz = 2\pi i\,(e^z)'(1+i) = 2\pi i\, e^{1+i}$

Add pt 2

for all $R>0$, $z_0 \in \mathbb{C}$

f is analytic on and interior to $C_R = \{|z-z_0| = R\}$

for $z \in C_R$

$$|f(z)| \le C|z|^n \le C\big(|z-z_0| + |z_0|\big)^n \le C(R+|z_0|)^n$$

So by Cauchy's inequality

$$|f^{(n+1)}(z_0)| \le \frac{n!\,C\,(R+|z_0|)^n}{R^{n+1}} = \frac{n!\,C}{R}\left(1+\frac{|z_0|}{R}\right)^n$$

This $\to 0$ as $R \to \infty$, so choosing R large enough, $|f^{(n+1)}(z_0)|$ is less than any positive number, so $|f^{(n+1)}(z_0)| = 0$.

Thus $f^{(n)}$, $f^{(n-1)}$ is a linear polynomial, etc.