Add 1

a) $z(t) = 1 + i + \sqrt{2}e^{it}$, $\frac{\pi}{4} \le t \le \pi$

Smooth: $z'(t) = \sqrt{2}ie^{it} \neq 0$

$1 + i + \sqrt{2}e^{i\frac{\pi}{4}} = 1 + i + (1+i) = 2 + 2i$

$1 + i + \sqrt{2}e^{i\pi} = 1 - \sqrt{2} + i$

b) $z(t) = (1-t)i + t(-1-i)$

$= i + t(-1-2i)$

$z'(t) = (-1-2i) \neq 0$

$z(0) = i \quad z(1) = -1-i$

§ 46

1) a) $$\int_0^{\pi}\left(1+\frac{2}{2e^{i\theta}}\right)2ie^{i\theta}\,d\theta \qquad (2e^{i\theta})' = 2ie^{i\theta}$$

$$= 2\int_0^{\pi} ie^{i\theta}\,d\theta + 2i\int_0^{\pi} d\theta = 2\int_0^{\pi}(e^{i\theta})'\,d\theta + 2\pi i$$

$$= 2\left(e^{i\pi} - e^{0}\right) + 2\pi i$$

$$= -4 + 2\pi i$$

b) Same with $0 \rightsquigarrow \pi$, $\pi \rightsquigarrow 2\pi$

$$= 2(e^{i2\pi} - e^{i\pi}) + 2\pi i = 4 + 2\pi i$$

c) Sum of the previous, $4\pi i$

(or, a) with $\pi \rightsquigarrow 2\pi$ $= 2(e^{2\pi i} - e^{0}) + 4\pi i$
$= 4\pi i$)

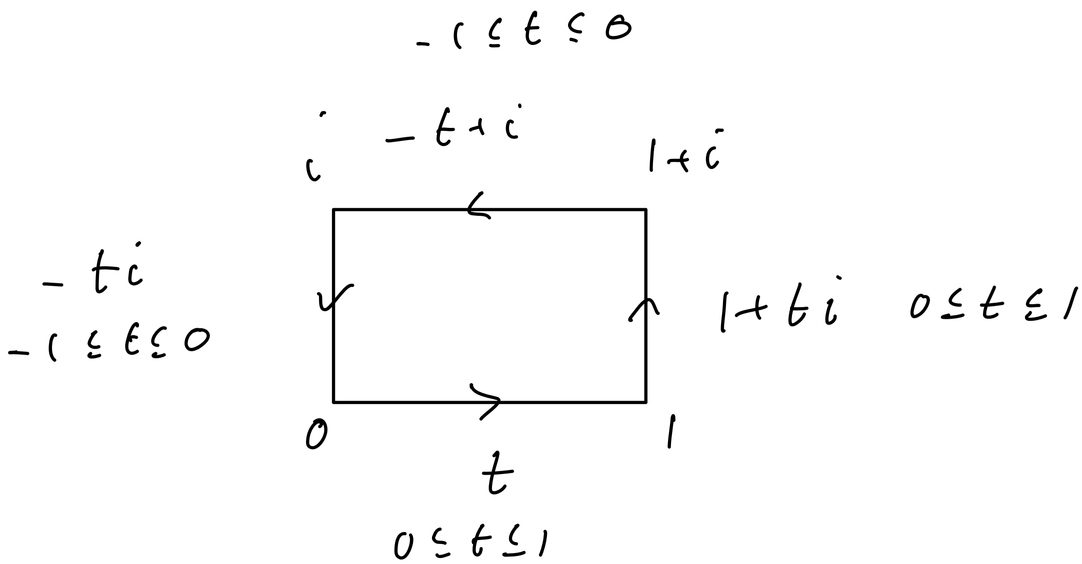
2) a) $\int_{\pi}^{2\pi} e^{i\theta}(ie^{i\theta})\,d\theta$ $\qquad (e^{i\theta})' = ie^{i\theta}$

$$= \int_{\pi}^{2\pi} \left(\frac{e^{2i\theta}}{2}\right)' d\theta = \frac{e^{4\pi i}}{2} - \frac{e^{2\pi i}}{2} = \frac{1}{2} - \frac{1}{2} = 0$$

b) $\int_{0}^{2} (x-1)(1)\,dx$ $\qquad (x)' = 1$

$$= \frac{x^2}{2} - x \Big|_0^2 = \frac{4}{2} - 2 = 0.$$

3)



$$\int_0^1 \pi e^{\pi t}(i)\,dt = e^{\pi} - e^{0} = e^{\pi} - 1 \qquad (t)' = 1$$

$$\int_0^1 \pi e^{\pi(1-ti)}(i)\,dt \qquad (1+ti)' = i$$

$$= \int_0^1 \left[-e^{\pi(1-ti)}\right]' dt = -\left(e^{\pi(1-i)} - e^{\pi}\right)$$

$$= 2e^{\pi}$$

$$\int_{-1}^0 \pi e^{\pi(-t-i)}(-i)\,dt = \qquad (-t+i)' = -1$$

$$\int_{-1}^0 \left[e^{\pi(-t-i)}\right]' dt = e^{-i\pi} - e^{\pi(1-i)}$$

$$= -1 + e^{\pi}$$

$$\int_{-1}^0 \pi e^{\pi t i}(-i) \qquad (-ti)' = -i$$

$$= \int_{-1}^0 \left[-e^{\pi t i}\right]' dt = -1 + e^{-\pi i} = -2$$

$$\text{Sum} = 4e^{\pi} - 4$$

4) $z(t) = t + it^3 \qquad -1 \le t \le 1 \qquad z' = 1 + 3t^2 i$

$y(t) = t^3 > 0$ iff $t > 0$

$$f \circ z(t) = \begin{cases} 1 & t < 0 \\ 4t^3 & t > 0 \end{cases}$$

$$\int_{-1}^{1} (f \circ z(t))\, z'(t)\,dt = \int_{-1}^{0} (1 + 3t^2 i)\,dt + \int_{0}^{1} 4t^3 (1 + 3t^2 i)\,dt$$

$$= 1 + it^3 \Big|_{-1}^{0} + t^4 \Big|_{0}^{1} + 2it^6 \Big|_{0}^{1}$$

$$= 2 + 3i$$

5)

$z(t), \quad t_1 \le t \le t_2 \quad z(t_1) = z_1$

$z(t_2) = z_2$

$$\int_{t_1}^{t_2} z'(t)\,dt = z(t_2) - z(t_1)$$

$$= z_2 - z_1$$

b) $z' = ie^{i\theta}$ $\mathrm{Log}(z(\theta)) = i\theta$

for $\theta \in (0, \pi)$, extends continuously to $\theta \in [0, \pi]$

$$\int_0^{\pi} e^{i(i\theta)}\, ie^{i\theta}\, d\theta = i\int_0^{\pi} e^{i\theta - \theta}\, d\theta$$

$$= i\int_0^{\pi} \left(\frac{e^{(i-1)\theta}}{i-1}\right)' d\theta = \frac{i}{i-1}\left[e^{(i-1)\pi} - 1\right]$$

$$= \frac{1-i}{2}\left[-e^{-\pi} - 1\right]$$

7) $$\int_0^{\pi/2} e^{(-1-2i)i\theta}\,(ie^{i\theta})\, d\theta$$

$$= i\int_0^{\pi/2} e^{2\theta}\, d\theta = \frac{i}{2}\left(e^{\pi} - e^{0}\right) = \frac{i}{2}\left(e^{\pi} - 1\right)$$

8) $z(t) = Re^{it}$, $-\pi \le t \le \pi$

$\mathrm{Log}(z(t)) = \ln R + it$ for $-\pi < t < \pi$

extends continuously to $\ln R + it$, $-\pi \le t \le \pi$.

$$\int_{-\pi}^{\pi} e^{(a-1)(\ln R + it)} \left(iRe^{it}\right) dt =$$

$$\left(e^{\ln R}\right)^{a-1} R \int_{-\pi}^{\pi} \left[\frac{1}{a} e^{ait}\right]' dt = R^a \left(\frac{e^{ai\pi} - e^{-ai\pi}}{a}\right)$$

$$= R^a \cdot \frac{2i \sin(a\pi)}{a}$$

9) a) $z(t) = e^{it}, \quad -\pi \le t \le \pi$

$\mathrm{Log}(z(t)) = it \quad \text{for } -\pi < t < \pi$

extends continuously to $it \quad -\pi \le t \le \pi.$

$$\int_{-\pi}^{\pi} e^{-\frac{3}{4}it}(ie^{it})\,dt = \int_{-\pi}^{\pi} ie^{\frac{it}{4}}\,dt = \int_{-\pi}^{\pi}\left(4e^{\frac{it}{4}}\right)'dt$$

$$= 4\left(e^{\frac{i\pi}{4}} - e^{-\frac{i\pi}{4}}\right) = 8i\sin\left(\frac{\pi}{4}\right) = \frac{8}{\sqrt{2}}i$$

b) $\log(z(t)) = \begin{cases} i(t+2\pi) & -\pi \le t < 0 \\ it & 0 < t \le \pi \end{cases}$

each portion extends continuously to 0.

$$\int_{-\pi}^{0} e^{-\frac{3}{4}(it+2\pi i)}(ie^{it})\,dt = \left(e^{-\frac{3}{2}\pi i}\right)4\left(1 - e^{-\frac{i\pi}{4}}\right)$$

$$= 4i\left(1 - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$\int_{0}^{\pi} e^{-\frac{3}{4}it}(ie^{it})\,dt = 4\left(e^{\frac{i\pi}{4}} - 1\right) = 4\left(\frac{1}{\sqrt{2}} - 1 + \frac{i}{\sqrt{2}}\right)$$

$\text{Sum} = -4 + 4i$ (note: can also change parametrization to $0 \to 2\pi$ to simplify integral)

13) $\displaystyle\int_{-\pi}^{\pi} R^{n-1} e^{i(n-1)\theta} (Rie^{i\theta})\, d\theta$

(if $n = 0$) $\displaystyle = \int_{-\pi}^{\pi} i \, d\theta = 2\pi i$

(if $n \neq 0$) $\displaystyle = R^n \int_{-\pi}^{\pi} \left[\frac{e^{i(n)\theta}}{n} \right]' d\theta$

$\displaystyle = \frac{R^n}{n} \left(e^{in\pi} - e^{in(-\pi)} \right)$

$\displaystyle = \frac{R^n}{n} \left((-1)^n - (-1)^n \right) = 0.$