# §33

3) $i^3 = -i \quad \Theta = -\frac{\pi}{2} \quad \mathrm{Log}(-i) = \ln 1 - \frac{\pi}{2} i = -\frac{\pi}{2} i$

$3\,\mathrm{Log}(i) = 3\left(\ln 1 + \frac{\pi}{2} i\right) = \frac{3\pi}{2} i$

4) $i^2 = -1, \quad \arg(-1) = \pi + 2\pi k, \quad k \in \mathbb{Z}$

$\theta = \pi \in \left(\frac{3\pi}{4}, \frac{11\pi}{4}\right) \Rightarrow \log(-1) = i\pi$

$\arg(i) = \frac{\pi}{2} + 2\pi k, \quad k \in \mathbb{Z}$

$\theta = \frac{5\pi}{2} \in \left(\frac{3\pi}{4}, \frac{11\pi}{4}\right) \Rightarrow \log(i) = i\frac{5\pi}{2}$

$\Rightarrow 2\log(i) = i\,5\pi$

9) If $f(z)$ $(r>0,\ \alpha<\theta<\alpha+2\pi)$
is analytic and $e^{f(z)}=z$ then
$1=(e^{f(z)})'=e^{f(z)}(f'(z))=z\,f'(z)$
Since $z\neq 0$, $f'(z)=\frac{1}{z}$.

10) a) $w=z-i$ is entire
$\text{Log}(w)$ is analytic as long as $w\neq u$, $u\in\mathbb{R}$, $u\leq 0$
So $\text{Log}(z-i)$ is analytic (chain rule) unless
$z-i=u \iff x+i(y-1)=u$
$\iff x=u\leq 0$ AND $y=1$

b) Similar to part a, $\text{Log}(z+4)$ is analytic unless $y=0$, $x\leq -4$
and $\frac{1}{z^2+i}$ is analytic unless
$z^2+i=0 \Rightarrow z^2=-i \Rightarrow z^2=e^{-i\frac{\pi}{2}}$
$\Rightarrow z=\pm e^{-i\frac{\pi}{4}}=\pm\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)$

11) Directly: $u(x,y)=\ln(x^2+y^2)$
$u_x=\dfrac{2x}{x^2+y^2}$ $\quad u_{xx}=\dfrac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2}=\dfrac{2y^2-2x^2}{(x^2+y^2)^2}$

$u_y=\dfrac{2y}{x^2+y^2}$ $\quad u_{yy}=\dfrac{2x^2-2y^2}{(x^2+y^2)^2}=-u_{xx}$.

using analytic functions: let $z = x + iy \neq 0$.
Choose a branch $f(z)$ of $\log(z)$
defined for $r > 0$, $\alpha < \theta < \alpha + 2\pi$ where
$\alpha \neq \arg(z^2)$. Then $f(z^2)$ is analytic at $z$,
and $f(z^2) = \ln(r^2) + 2i\theta$
$= \ln(x^2+y^2) + 2i\theta$, so
$\ln(x^2+y^2)$ is Harmonic at $(x,y)$.

§ 34 R 2,3,4,7

1) $z_1 = r_1 e^{i\Theta_1}$ $z_2 = r_2 e^{i\Theta_2}$ $\Theta_1, \Theta_2 \in (-\pi, \pi]$.
$z_1 z_2 = r_1 r_2 e^{i(\Theta_1 + \Theta_2)}$, $\Theta_1 + \Theta_2 \in (-2\pi, 2\pi]$.
Since $\mathrm{Arg}(z_1 z_2) \in (-\pi, \pi]$, and
$\mathrm{Arg}(z_1 z_2) = \Theta_1 + \Theta_2 + 2\pi K$, $K \in \mathbb{Z}$,
$-1 \leq K \leq 1$, so $K = -1, 0,$ or $1$ and

$\mathrm{Log}(z_1 z_2) = \ln(|z_1||z_2|) + i\,\mathrm{Arg}(z_1 z_2)$
$= \ln|z_1| + \ln|z_2| + i\Theta_1 + i\Theta_2 + 2\pi K i$
$= \mathrm{Log}(z_1) + \mathrm{Log}(z_2) + 2\pi K i$

§36 R1,3

1) a) $(1+i)^i = e^{i\log(1+i)}$

$\log(1+i) = \ln\sqrt{2} + i\frac{\pi}{4} + 2\pi i k, \quad k \in \mathbb{Z}$

$(1+i)^i = e^{-\frac{\pi}{4} - 2\pi k} e^{i\frac{1}{2}\ln 2} \quad k \in \mathbb{Z}$

b) $i^{2i} = e^{2i\log(i)} = e^{2i(i\frac{\pi}{2} + 2\pi i k)} \quad k \in \mathbb{Z}$

$= e^{-\pi - 4\pi k}, \; k \in \mathbb{Z}$

$\frac{1}{i^{2i}} = e^{\pi + 4\pi k}, \; k \in \mathbb{Z}$

6) $z^a = e^{a\ln(z)} = e^{a\ln(r) + ia\Theta}$

Sine $a\ln(r)$, $a\Theta \in \mathbb{R}$,

$|z^a| = e^{a\ln(r)} = e^{a(\ln(r) + i\Theta)}, \quad \Theta = 0 = \mathrm{Arg}(r)$

$= \text{P.V. } r^a$

$= \text{P.V. } |z|^a$

8) $e^{c_1 \mathrm{Log}(z)} e^{c_2 \mathrm{Log}(z)} = e^{(c_1 + c_2)\mathrm{Log}(z)}$

$\dfrac{e^{c_1 \mathrm{Log}(z)}}{e^{c_2 \mathrm{Log}(z)}} = e^{(c_1 - c_2)\mathrm{Log}(z)}$

$(e^{c\,\mathrm{Log}(z)})^n = e^{c\log(z) + \ldots + c\,\mathrm{Log}(z)} = e^{nc\,\mathrm{Log}(z)} = z^{nc}$

9) $c^{f(z)} = e^{f(z)\log(c)}$

$$\left(c^{f(z)}\right)' = e^{f(z)\log(c)} \log(c) f'(z)$$

$$= c^{f(z)} \log(c) f'(z)$$

## §42

2) a) $(1+it)^2 = \left(\frac{(1+it)^3}{3i}\right)'$

$$\int_0^1 (1+it)^2 = \frac{(1+i)^3 - 1}{3i} = \frac{3i-3-i}{3i}$$

$$= \frac{2}{3} + i$$

b) $\left(\int_1^2 \left(\frac{1}{t^2} - 1\right) dt\right) + i\left(\int_1^2 -\frac{2}{t}\, dt\right)$

$$= -t^{-1}\Big|_1^2 - 1 + i\left(-2\ln t\Big|_1^2\right)$$

$$= -\frac{1}{2} - i\ln 4$$

c) $e^{i2t} = \left(\frac{1}{2i}e^{i2t}\right)'$

$$\int_0^{\pi/6} e^{i2t}\, dt = \frac{1}{2i}\left(e^{i\frac{\pi}{3}} - e^0\right) = \frac{1}{2i}\left(\frac{1}{2} - 1 + \frac{\sqrt{3}}{2}i\right)$$

$$= \frac{\sqrt{3}}{4} + \frac{i}{4}$$

d) $(e^{-2t}) = \left(-\frac{e^{-2t}}{2}\right)'$ (variable is $t$)

$$\int_0^\infty e^{-2t}\,dt = \lim_{r\to\infty} \int_0^r e^{-2t}\,dt =$$

$$= \lim_{r\to\infty} -\left(\frac{e^{-2r} - e^0}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{2} \lim_{r\to\infty} e^{-xr} e^{-iyr} = \frac{1}{2}$$

$|e^{-xr} e^{-iyr}| = e^{-xr} \to 0$ as $r \to \infty$, $x > 0$

3) If $m \neq n$, $e^{i(m-n)\theta} = \left(\frac{e^{i(m-n)\theta}}{m-n}\right)'$

$$\int_0^{2\pi} e^{i(m-n)\theta}\,d\theta = \frac{e^{i(m-n)2\pi} - e^0}{m-n} = 0$$

~~If~~ $m = n$, $e^{i(m-n)\theta} = 1$, $\int_0^{2\pi} 1\,d\theta = 2\pi$.

4) $e^{(1+i)t} = e^{t}e^{it} = e^{t}\cos(t) + i\, e^{t}\sin(t)$

$$\int_0^{\pi} e^{(1+i)t}\,dt = \int_0^{\pi} e^{t}\cos(t)\,dt + i\int_0^{\pi} e^{t}\sin(t)\,dt$$

$$e^{(1+i)t} = \left(\frac{e^{(1+i)t}}{1+i}\right)'$$

$$\int_0^{\pi} e^{(1+i)t}\,dt = \frac{e^{(1+i)\pi} - e^{0}}{1+i}$$

$$= \frac{e^{\pi}e^{i\pi} - 1}{1+i} = \frac{-e^{\pi} - 1}{1+i}$$

$$= \frac{(-e^{\pi}-1)(1-i)}{2}$$

$$= \frac{-e^{\pi}-1}{2} + i\,\frac{e^{\pi}+1}{2}$$

So

$$\int_0^{\pi} e^{t}\cos(t)\,dt = \frac{-e^{\pi}-1}{2} \qquad \int_0^{\pi} e^{t}\sin(t) = \frac{e^{\pi}+1}{2}$$

§43 2 — 4

1) a) $\int_{-b}^{-a} w(-t)\,dt = \int_{-b}^{-a} u(-t)\,dt + i\int_{-b}^{-a} v(-t)\,dt$

$= \int_{b}^{a} u(t)(-dt) + i\int_{b}^{a} v(t)(-dt)$ (sub $t \to -t$)

$= \int_{a}^{b} u(t)\,dt + i\int_{a}^{b} v(t)\,dt = \int_{a}^{b} w(t)\,dt$

$\phi: \mathbb{R} \to \mathbb{R}$

b) $\int_{a}^{b} w(t)\,dt$ $\quad t = \phi(\tau) \quad dt = \phi'(\tau)\,d\tau$

$\phi(\alpha) = a \quad \phi(\beta) = b$

$= \int_{a}^{b} u(t)\,dt + i\int_{a}^{b} v(t)\,dt$

$= \int_{\alpha}^{\beta} u(\phi(\tau))\,\phi'(\tau)\,d\tau + i\int_{\alpha}^{\beta} v(\phi(\tau))\,\phi'(\tau)\,d\tau$

(substitution; $\phi$ is real)

$= \int_{\alpha}^{\beta} \big(u(\phi(\tau)) + i\,v(\phi(\tau))\big)\,\phi'(\tau)\,d\tau$

$= \int_{\alpha}^{\beta} w(\phi(\tau))\,\phi'(\tau)\,d\tau$

4) $z(\tau) = x(\phi(\tau)) + i\, y(\phi(\tau))$

$$z(\tau)' = x'(\phi(\tau))\,\phi'(\tau) + i\, y'(\phi(\tau))\,\phi'(\tau)$$
$$= \left(x'(\phi(\tau)) + i y'(\phi(\tau))\right)\phi'(\tau)$$
$$= \left(z'(\phi(\tau))\right)\phi'(\tau)$$

5) $z(t) = x(t) + i y(t)$

$$f(z(t)) = u(x(t), y(t)) + i\, v(x(t), y(t))$$
$$(f \circ z)'(t) = (u_x x' + u_y y') + i(v_x x' + v_y y')$$
$$= (u_x x' - v_x y') + i(v_x x' + u_x y')$$
$$= (u_x + i v_x) x' + (u_x + i v_x) i y'$$
$$= (u_x + i v_x)(x' + i y') = f' \cdot z'$$

where $u_x, v_x, u_y, v_y$, $f'$ are evaluated at $z(t)$