§ 14

2) a)

$(x+iy)^3 + x+iy + 1$

$= x^3 + 3ix^2y - 3xy^2 - iy^3 + x + iy + 1$

$= \left(x^3 - 3xy^2 + x + 1 \right) + i \left(3x^2y - y^3 + y \right)$

b) $\dfrac{(x-iy)^2}{x+iy} = \dfrac{(x^2 - y^2 - i2xy)(x-iy)}{x^2+y^2}$

$= \dfrac{x^3 - xy^2 - 2xy^2}{x^2+y^2} + i \left(\dfrac{-x^2y + y^3 - 2x^2y}{x^2+y^2} \right)$

$= \dfrac{x^3 - 3xy^2}{x^2+y^2} + i \left(\dfrac{y^3 - 3x^2y}{x^2+y^2} \right)$

3) $$\frac{z^2+2z\bar{z}+\bar{z}^2}{4} + \frac{z^2-2z\bar{z}+\bar{z}^2}{4} - \frac{z-\bar{z}}{i}$$

$$+i\left(z+\bar{z} - \frac{z^2-\bar{z}^2}{2i} \right)$$

$$= \frac{z^2}{2} + \frac{\bar{z}^2}{2} + iz - i\bar{z} + iz + i\bar{z} - \frac{z^2}{2} + \frac{\bar{z}^2}{2}$$

$$= \bar{z}^2 + 2iz$$

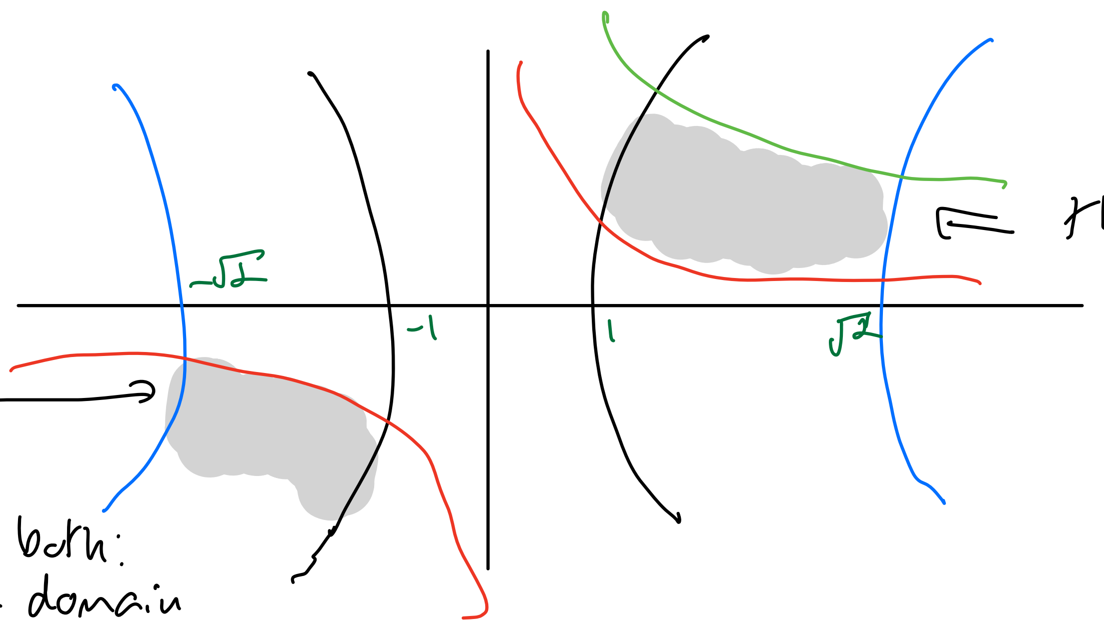
4) $$z + \frac{1}{z} = re^{i\theta} + \frac{1}{re^{i\theta}} = re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$= \left(r+\frac{1}{r}\right)\cos\theta + i\left(r-\frac{1}{r}\right)\sin\theta$$

$x^2-y^2=1$ $\quad x^2-y^2=2$

$2xy=1$ $\quad 2xy=2$

5) $u = x^2-y^2 \qquad v = 2xy$

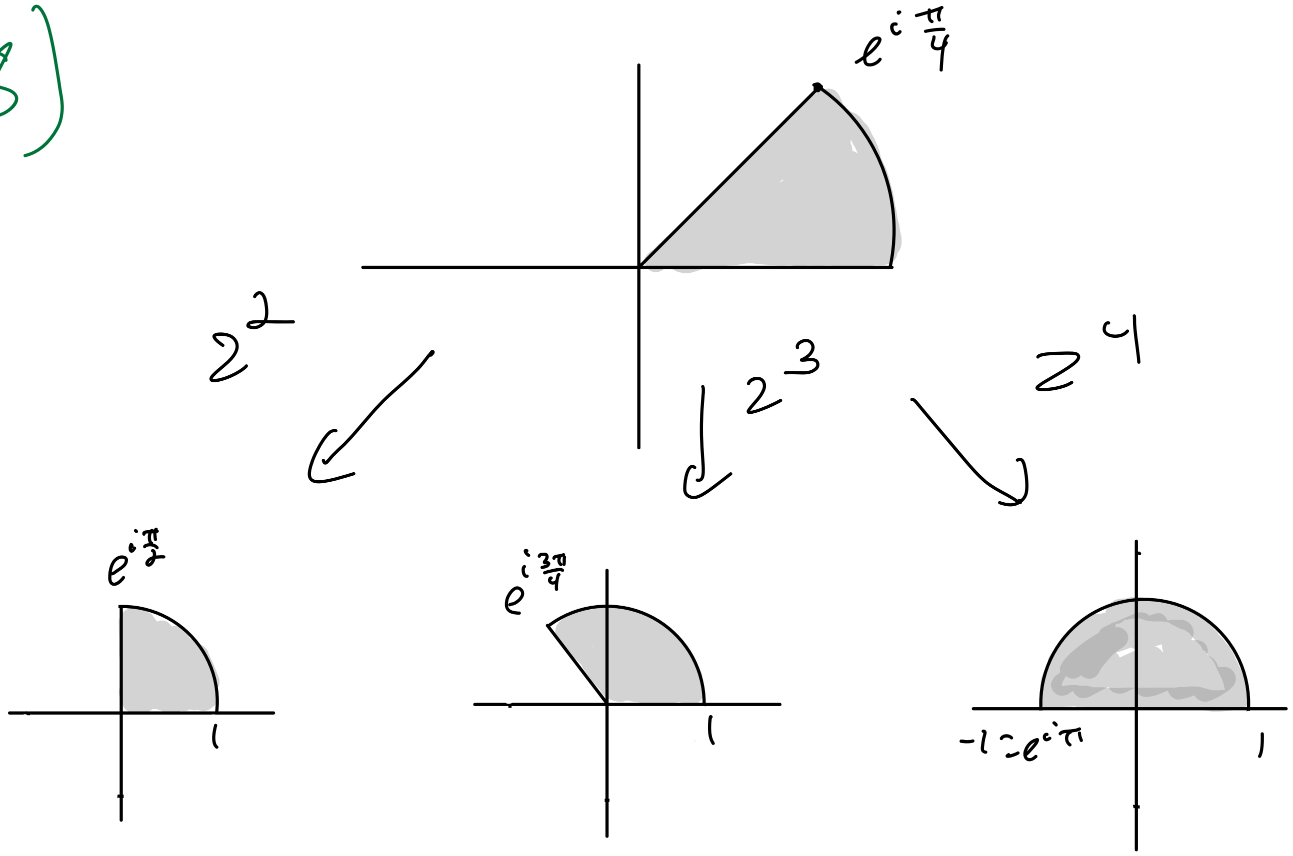


$\Leftarrow$ this

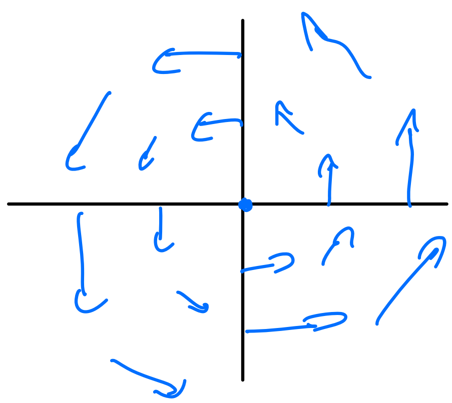
or this $\rightarrow$

(but not both: not a domain

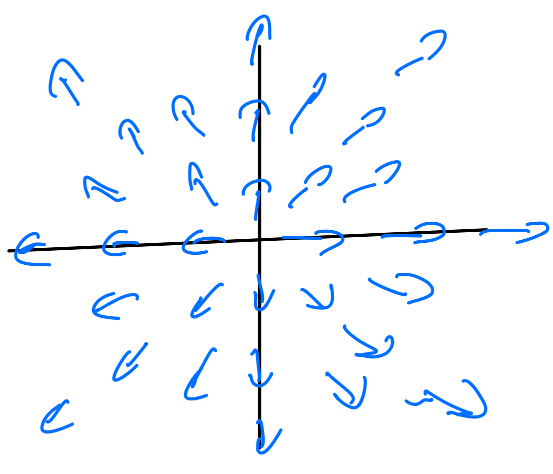
8)



9) $iz = \begin{pmatrix} -y \\ x \end{pmatrix}$



$$\frac{z}{|z|} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \end{pmatrix}$$



§18

1) a) If $|z-z_0|<\varepsilon$,

then $|Re(z)-Re(z_0)| = |Re(z-z_0)| \le |z-z_0| < \varepsilon.$

b) If $|z-z_0|<\varepsilon$ then

$|\bar{z}-\bar{z}_0| = |\overline{z-z_0}| = |z-z_0| < \varepsilon$

c) If $|z-0|<\varepsilon$ then

$\left|\frac{\bar{z}^2}{z}-0\right| = \frac{|z|^2}{|z|} = |z| < \varepsilon.$

5) $\left(\frac{x}{\bar{x}}\right)^2 = (1)^2 = 1 \quad \left(\frac{iy}{\overline{iy}}\right)^2 = (-1)^2 = 1$

$\left(\frac{x+ix}{x-ix}\right)^2 = \left(\frac{(1+i)(1+i)}{(1-i)(1+i)}\right)^2 = \left(\frac{2i}{2}\right)^2 = -1$

If the limit existed, it would equal $\lim_{x\to 0}\left(\frac{x}{\bar{x}}\right)^2 = 1$. But for $\varepsilon=1$ and all $\delta>0$

$\left|\frac{\delta}{2}+i\frac{\delta}{2}\right| = \frac{\delta}{\sqrt{2}} < \delta$ while $\left|\left(\frac{\delta+i\delta}{\delta-i\delta}\right)^2-1\right| = |-1-1| = 2 > \varepsilon.$

8) $0<|z-z_0|<\delta \iff 0<|(z-z_0)-0|<\delta$

$|f(z)-w_0|<\varepsilon \iff |f((z-z_0)+z_0)-w_0|<\varepsilon$

9) Let $\varepsilon>0$
Choose $\delta>0$ such that

$0<|z-z_0|<\delta \Rightarrow |f(z)|<\frac{\varepsilon}{M}$ and $|g(z)|<M$.

Thus $0<|z-z_0|<\delta \Rightarrow |f(z)g(z)|<\varepsilon$.

10)

a) $\lim_{z\to 0} \frac{4\left(\frac{1}{z}\right)^2}{\left(\frac{1}{z}-1\right)^2} = \lim_{z\to 0} \frac{4}{(1-z)^2} = 4$

b) $\lim_{z\to 1} (z-1)^3 = 0$

c) $\lim_{z\to 0} \frac{\frac{1}{z}-1}{\left(\frac{1}{z}\right)^2+1} = \lim_{z\to 0} \frac{z-z^2}{1+z^2} = 0$.

11) a) if $c=0$

$$\lim_{z\to 0} \frac{d}{a(\frac{1}{z})+b} = \lim_{z\to 0} \frac{dz}{a+bz} = 0$$

b) if $c \neq 0$

$$\lim_{z\to 0} \frac{\frac{a}{z}+b}{\frac{c}{z}+d} = \lim_{z\to 0} \frac{a+bz}{c+dz} = \frac{a}{c}$$

$$\lim_{z\to -\frac{d}{c}} \frac{cz+d}{az+b} = \frac{0}{-\frac{ad}{c}+b} = 0$$ because

$-\frac{ad}{c}+b \neq 0 \Rightarrow ad-bc \neq 0.$

13) Let $\varepsilon > 0$. Since S is unbounded,

$S \not\subset |z| \leq \frac{1}{\varepsilon}$, so ~~there~~ there is $z \in S$ with

$|z| > \frac{1}{\varepsilon}$; thus z is in that neighborhood

of ∞.

§ 20

8) a) $$\lim_{\Delta z \to 0} \frac{Re(z+\Delta z) - Re(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{Re(\Delta z)}{\Delta z}$$

$\Delta z = x$ $\qquad$ $\Delta z = iy$

$$\lim_{x \to 0} \frac{x}{x} = 1 \neq \lim_{y \to 0} \frac{0}{iy} = 0.$$

b) $$\lim_{\Delta z \to 0} \frac{Im(\Delta z)}{\Delta z}$$

$$\lim_{x \to 0} \frac{Im(x)}{x} = 0 \neq \lim_{\cancel{x}\, y \to 0} \frac{Im(iy)}{iy} = \frac{1}{i}$$

9) $\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{\bar{z}^2}{z^2}$

by the same argument as §1.8 #5 that limit does not exist

$$\frac{\bar{z}^2}{z} = \frac{(x^2 - y^2 - 2xyi)(x - iy)}{x^2 + y^2}$$

$$= \frac{x^3 - xy^2 - 2xy^2}{x^2 + y^2} + i\left(\frac{-x^2y + y^3 - 2x^2y}{x^2 + y^2}\right)$$

$$= \frac{x^3 - 3xy^2}{x^2 + y^2} + i\left(\frac{y^3 - 3x^2y}{x^2 + y^2}\right)$$

$$u_y = \frac{-6x^3y - 6xy^3 - (x^3 - 3xy^2)(2y)}{(x^2 + y^2)^2}$$

$$= \frac{-8x^3y}{(x^2 + y^2)^2} \quad x = y \quad \frac{-8x^4}{4x^4} = -2$$

§24

1) a) $f(z) = x - iy$

$u_x = 1 \neq v_y = -1$

b) $f(z) = 2yi$

$u_x = 0 \neq v_y = 2$

c) $f(z) = 2x + ixy^2$

$u_x = 2 = u_y = 2xy \Rightarrow xy = 1$

$u_y = 0 = -v_x = y^2 \Rightarrow y = 0$

no point satisfies both equations

d) $f(z) = e^x \cos(y) - e^x \sin(y)\, i$

$u_x = e^x \cos(y) = v_y = -e^x \cos(y)$

$\Rightarrow \cos(y) = 0$

$u_y = -e^x \sin(y) = v_x = e^x \sin(y)$

$\Rightarrow \sin(y) = 0$

no y satisfies both

2) a) $f(z) = 2 - y + ix$

u, v have continuous partials everywhere (polynomials in x and y)

$u_x = 0 = v_y \qquad u_y = -1 = -v_x$

$f' = i$

3) a) $f(z) = \dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}$ if $z \neq 0$

$u_x = \dfrac{y^2 - x^2}{(x^2+y^2)^2} = v_y \qquad u_y = \dfrac{-2xy}{(x^2+y^2)^2} = -v_x$

$$f' = \frac{(y^2 - x^2) + i\,2xy}{(x^2+y^2)^2} = \frac{-\bar{z}^2}{|z|^4} = \frac{-\bar{z}^2}{z^2\bar{z}^2}$$

$$= -\frac{1}{z^2}$$

partials are continuous if $(x^2+y^2) \neq 0$; rational functions

b) $u_x = 2x \quad u_y = 0 \quad v_x = 0 \quad v_y = 2y$
Continuous everywhere; CR eqs hold if $x=y$
if so $\quad f'(z) = 2x$

c) $f(z) = xy + iy^2$
$u_x = y \quad u_y = x \quad v_x = 0 \quad v_y = 2y$
Continuous everywhere, CR hold if $x=y=0$
if so $\quad f' = y = 0.$

Additional 1

Assume $x_n \to x$, $y_n \to y$.
Let $\varepsilon > 0$; choose N such that
$n > N$ implies $|x_n - x| < \frac{\varepsilon}{2}$, $|y_n - y| < \frac{\varepsilon}{2}$.
Then $|z_n - z| = |(x_n - x) + i(y_n - y)|$

$$\le |x_n - x| + |y_n - y||i|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \cdot 1 = \varepsilon.$$

So $z_n \to z$.

Now assume $z_n \to z$. Let $\varepsilon > 0$

choose N such that

$n > N$ implies $|z_n - z| < \varepsilon$.

Then $|x_n - x| = |\mathrm{Re}(z_n - z)| \leq |z_n - z| < \varepsilon$

So $x_n \to x$ and

$|y_n - y| = |\mathrm{Im}(z_n - z)| \leq |z_n - z| < \varepsilon$

so $y_n \to y$.

Additional 2

(should have included $f(0) = 0$)

a) $\displaystyle \lim_{\Delta z \to 0} \frac{\frac{\overline{\Delta z}^2}{\Delta z} - 0}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}^2}{\Delta z^2}$

This is (almost) the same function as §18 #5, and so limit does not exist (same argument)

b) $\dfrac{\bar{z}^2}{z} = \dfrac{x^2 - y^2 - 2xyi}{x + iy} = \dfrac{(x^2-y^2-2xyi)(x-iy)}{x^2+y^2}$

$$= \frac{x^3 - xy^2 - 2xy^2}{x^2+y^2} + i\,\frac{-yx^2 + y^3 - 2x^2y}{x^2+y^2}$$

$$= \frac{x^3 - 3xy^2}{x^2+y^2} + i\,\frac{y^3 - 3x^2y}{x^2+y^2} \qquad = u + iv$$

$(x,y) \neq (0,0)$

$u(0,0) = v(0,0) = 0$

$(x,y) \neq (0,0)$

$$u_y = \frac{-6xy\,(x^2+y^2) - (x^3-3xy^2)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-8x^3y}{(x^2+y^2)^2}$$

$(x,y) = (0,0)$

$u(0,y) = 0$, so $\displaystyle\lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = 0.$

$$\lim_{x\to 0} u_y(x,0) = 0$$

$\times$

$$\lim_{x\to 0} u_y(x,x) = \lim_{x\to 0} \frac{-8x^4}{(2x^2)^2} = -2$$

(i.e. same argument as §18 #5)

So, $\lim_{(x,y)\to(0,0)} u_y(x,y)$ does not exist.