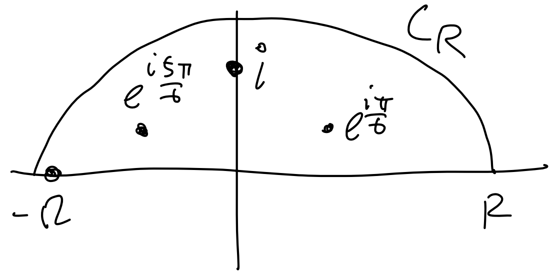
§86

4)

$$\int_0^\infty \frac{x^2}{x^6+1}\,dx$$

$$= \frac{1}{2} \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^6+1}\,dx \quad \text{since} \quad \frac{x^2}{x^6+1} \text{ is even}$$

$$f(z) = \frac{z^2}{z^6+1}$$



$$\int_{-R}^{R} \frac{x^2}{x^6+1}\,dx + \int_{C_R} f(z)\,dz = 2\pi i \left(\operatorname*{Res}_{z=e^{i\frac{\pi}{6}}} f(z) + \operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=e^{i\frac{5\pi}{6}}} f(z) \right)$$

$z^2 \neq 0$ at $z = e^{i\frac{\pi}{6}}, i,$ or $e^{i\frac{5\pi}{6}}$, and z^6+1 has a zero of order 1 at each, so $f(z)$ has a simple pole at each

$f'(z) = 6z^5 \neq 0$

and $$\operatorname*{Res}_{z=e^{i\frac{\pi}{6}}} f(z) = \frac{(e^{i\frac{\pi}{6}})^2}{6(e^{i\frac{\pi}{6}})^5} = \frac{1}{6e^{i\frac{\pi}{2}}} = -\frac{i}{6}$$

$$\operatorname*{Res}_{z=i} f(z) = \frac{(i)^2}{6(i)^5} = \frac{i}{6}$$

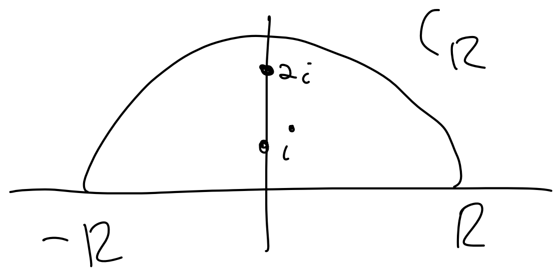
$$\operatorname*{Res}_{z=e^{i\frac{5\pi}{6}}} f(z) = \frac{(e^{i\frac{5\pi}{6}})^2}{6(e^{i\frac{5\pi}{6}})^5} = -\frac{i}{6}$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \pi R \,\frac{R^2}{R^6-1} = \frac{\pi \frac{1}{R^3}}{1-\frac{1}{R^3}} \to 0 \text{ as } R\to\infty$$

So $\displaystyle \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{x^6+1}\,dx = \frac{\pi}{3}$.

5) $$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{(x^2+1)(x^2+4)}\,dz$$

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$$



$$\int_{-R}^{R}\frac{x^2}{(x^2+1)(x^2+4)}\,dx + \int_{C_R} f(z)\,dz = 2\pi i\left(\operatorname{Res}_{z=2i} f(z) + \operatorname{Res}_{z=i} f(z)\right)$$

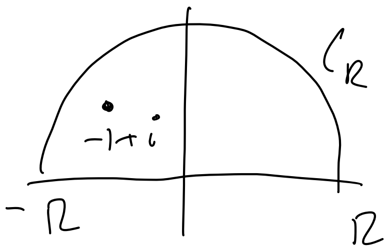
at $z=2i$, $\phi(z) = \dfrac{(z-2i)z^2}{(z^2+4)(z^2+1)} = \dfrac{z^2}{z+2i\,(z^2+1)}$, $\phi(2i) = \dfrac{-4}{4i(-3)} = -\dfrac{i}{3}$

at $z=i$ $\phi(z) = \dfrac{(z-i)z^2}{(z^2+1)(z^2+4)} = \dfrac{z^2}{(z+i)(z^2+4)}$, $\phi(i) = \dfrac{-1}{2i(3)} = \dfrac{i}{6}$

$$\left|\int_{C_R} f(z)\,dz\right| \le \pi R\cdot\frac{R^2}{(R^2-1)(R^2-4)} = \frac{\pi\frac{1}{R}}{(1-\frac{1}{R^2})(1-\frac{4}{R^2})} \to 0 \text{ as } R\to\infty$$

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{(x^2+1)(x^2+4)}\,dx = 2\pi i\left(-\frac{i}{3}+\frac{i}{6}\right)=\frac{\pi}{3}$$

7) $$\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^2+2x+2}$$



$$f(z)=\frac{1}{z^2+2z+2}=\frac{1}{(z+1-i)(z+1+i)}$$

$$\int_{-R}^{R}\frac{dx}{x^2+2x+2}+\int_{C_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1+i} f(z)$$

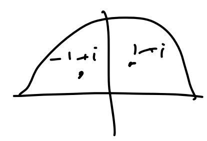
$$=2\pi i\,\frac{1}{(-1+i+1+i)}=\frac{2\pi i}{2i}=\pi$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{\pi R}{R^2-2R-2}\to 0 \text{ as } R\to\infty$$

§88

4) $$\int_{-\infty}^{\infty}\overset{\text{even}}{\frac{x\sin ax}{x^4+4}}\,dx = \operatorname{Im}\lim_{R\to\infty}\int_{-R}^{R}\frac{xe^{iax}}{x^4+4}\,dx$$

$$f(z)=\frac{ze^{iaz}}{z^4+4}$$



$$\int_{-R}^{R}\frac{xe^{iax}}{x^4+4}\,dx+\int_{C_R} f(z)\,dz = 2\pi i\left(\frac{(1+i)e^{ia(1+i)}}{(2+2i)(2)(2i)}+\frac{(-1+i)e^{ia(-1+i)}}{(-2+2i)(-2)(2i)}\right)$$

$$= 2\pi i \left(\frac{(1+i)\, e^{a(-1+i)}}{(2+2i)(2)(2i)} + \frac{(1-i)\, e^{a(-1-i)}}{(2-2i)(2)(-2i)} \right)$$

$$= 2\pi i \left(2\,\mathrm{Re}\left[\frac{(1+i)\, e^{a(-1+i)}}{8(-1+i)} \right] \right)$$

$$= \frac{\pi i}{2}\,\mathrm{Re}\left[\frac{e^{i\frac{\pi}{4}}\quad e^{a(-1+i)}}{e^{i\frac{3\pi}{4}}} \right]$$

$$= \frac{\pi i}{2}\,\mathrm{Re}\left[-i\, e^{a(-1+i)} \right]$$

$$= \frac{\pi i}{2}\, e^{-a} \sin(a)$$

$|e^{iaz}| = e^{-ay} \leq 1$ for $z \in C_R$ since $y \geq 0$.

$$\left| \int_{C_R} f(z)\,dz \right| \leq \pi R \cdot \frac{R}{R^4 - 1} \to 0 \quad \text{as} \quad R \to \infty.$$

b) $$\int_{-\infty}^{\infty} \frac{x \sin(x)}{(x^2+1)(x^2+4)}\,dx = \mathrm{Im} \lim_{R\to\infty} \int_{-R}^{R} \frac{x e^{ix}}{(x^2+1)(x^2+4)}\,dx \qquad f(z) = \frac{z e^{iz}}{(z^2+1)(z^2+4)}$$

$$\int_{-R}^{R} \frac{x e^{ix}}{(x^2+1)(x^2+4)}\,dx + \int_{C_R} f(z)\,dz = 2\pi i \left(\frac{i e^{-1}}{2i(3)} + \frac{2i e^{-2}}{(-3)(4i)} \right)$$

$$= 2\pi i \left(\frac{1}{6e} - \frac{1}{6e^2} \right)$$

$$= \pi i \left(\frac{e-1}{3e^2} \right)$$

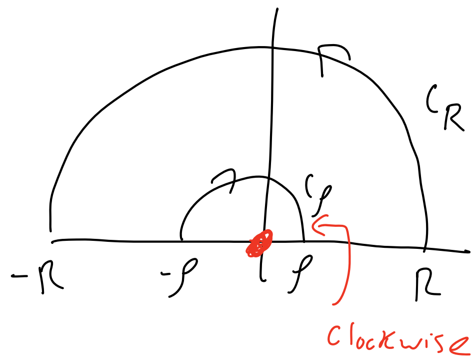
$$\left| \int_{C_R} f(z)dz \right| \leq \frac{\pi R \cdot R}{(R^2-1)(R^2-4)} \to 0 \quad \text{as } R\to\infty, \quad |e^{iz}| \leq 1$$

§ 9 |

1) $$\int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2}dx = \frac{1}{2}\left[\int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2}dx + \int_{-\infty}^0 \frac{\cos(ax)-\cos(bx)}{x^2}dx \right]$$

$$= \frac{1}{2} \lim_{R\to\infty} \lim_{\rho\to 0} \int_\rho^R + \int_{-R}^{\rho} \frac{\cos(ax)-\cos(bx)}{x^2}dx$$

$$f(z) = \frac{e^{iaz} - e^{ibz}}{z^2}$$



$$\int_{-R}^{\rho} f(z)dz - \int_{C_\rho} f(z)dz + \int_\rho^R f(z)dz + \int_{C_R} f(z)dz = 0$$

$$\left| \int_{C_R} f(z)dz \right| \leq \left(\frac{1+1}{R^2} \right) \pi R \to 0 \quad \text{as } R\to\infty \qquad (a,b \geq 0) \quad y \geq 0$$

$$\lim_{\rho\to 0} \int_{C_\rho} f(z)dz = \lim_{\rho\to 0} \int_{C_\rho} \left[\frac{a-b}{z} + g(z) \right] dz \qquad g(z) \text{ analytic and bounded}$$

$$= \lim_{\rho\to 0} (a-b) \int_{C_\rho} \frac{1}{z}dz = \pi i(a-b)$$

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2}\,dx = \frac{-1}{2} \operatorname{Re}\left\{ \pi i\,(a-b) \right\} = \frac{\pi(b-a)}{2}$$

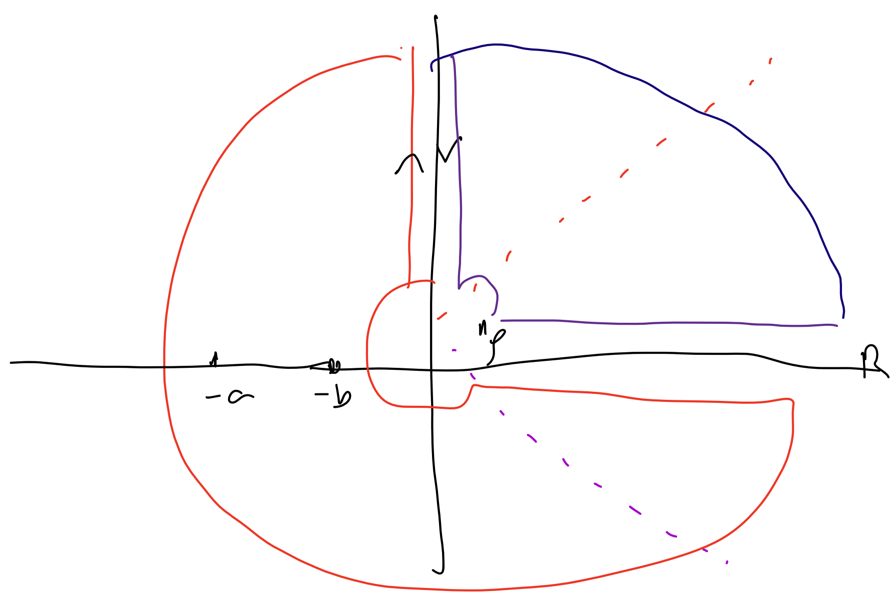
Use $a=0$, $b=2$, $1-\cos(2x) = 2\sin^2 x$ to show

$$\int_0^\infty \frac{2\sin^2 x}{x^2}\,dx = \pi .$$

4)

$$f_1(z) = \frac{e^{\frac{1}{3}\log(z)}}{(z+a)(z+b)} \qquad r>0,\ -\frac{\pi}{4} < \theta < \frac{7\pi}{4}$$

$$f_2(z) = \frac{e^{\frac{1}{3}\log(z)}}{(z+a)(z+b)} \qquad r>0,\ \frac{\pi}{4} < \theta \leq \frac{9\pi}{4}$$



for $i = 1$ or 2

$$\left| \int_{|z|=\rho} f_i(z)\,dz \right| \leq \frac{\rho^{\frac{1}{3}}}{(a-\rho)(b-\rho)} \to 0 \text{ as } \rho \to 0$$

$$\left| \int_{|z|=R} f_i(z)\,dz \right| \le \frac{R^{1/3}}{(R-a)(R-b)} = \frac{R^{-2/3}}{(1-\frac{a}{R})(1-\frac{b}{R})} \to 0 \text{ as } R \to \infty$$

So by Cauchy-Goursat

On the positive imaginary axis, $f_1 = f_2$, so adding $\int_{C_1} f_1(z)\,dz + \int_{C_2} f_2(z)\,dz$, since $f_2 = e^{\frac{2\pi i}{3}} f_1$ on the positive real axis, we have

$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx - e^{\frac{i2\pi}{3}} \int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx = 2\pi i \left(\frac{\sqrt[3]{a}\, e^{\frac{i\pi}{3}}}{(-a+b)} + \frac{\sqrt[3]{b}\, e^{\frac{i\pi}{3}}}{(-b+a)} \right)$$

$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx = \frac{2\pi(\sqrt[3]{a} - \sqrt[3]{b})}{a-b}\, i \left(\frac{-e^{\frac{i\pi}{3}}}{1-e^{\frac{i2\pi}{3}}} \right)$$

$$= \frac{2\pi(\sqrt[3]{a} - \sqrt[3]{b})}{a-b} \left(\frac{-i}{2i\sin(-\frac{\pi}{3})} \right)$$

$$= \frac{\pi(\sqrt[3]{a} - \sqrt[3]{b})}{a-b} \left(\cdot \frac{2}{\sqrt{3}} \right)$$