14)

a) Let $S_n = \frac{1}{n}$. Then $S_n \in \mathbb{R}$, $S_n \to 0$ but $f(S_n) = \frac{1/n}{1/n} = 1$ does not converge to $f(0) = 0$.

b) Choose $\varepsilon = 1/2$. Let $\delta > 0$. Choose $x = \frac{\delta}{2}$.

Then $|x - 0| = \frac{\delta}{2} < \delta$ but

$$|f(x) - f(0)| = \left| \frac{\delta/2}{\delta/2} - 0 \right| = 1 > \frac{1}{2} = \varepsilon.$$

17)

a) Let $x_0 \in \mathbb{R}$, $s_n \to x_0$.

Let $\varepsilon > 0$. There exists $N \in \mathbb{R}$ such that

$n > N$ implies $|s_n - x_0| < \varepsilon$.

Thus for $n > N$,

$$\big||s_n| - |x_0|\big| = \begin{cases} |s_n| - |x_0| \\ \text{or } |x_0| - |s_n| \end{cases} \leq |s_n - x_0| < \varepsilon.$$

($\triangle$ inequality; see also HW 1)

So $f(s_n) = |s_n| \to |x_0| = f(x_0)$.

b) Let $x_0 \in \mathbb{R}$, $\varepsilon > 0$. Choose $\delta = \varepsilon$.

If $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| = \big||x| - |x_0|\big| \leq |x - x_0| < \delta = \varepsilon.$$

18) Ross 17, 8

a) If $f(x) \geq g(x)$ then

$\frac{1}{2}(f(x)+g(x)) - \frac{1}{2}|f(x)-g(x)| = \frac{1}{2}(f(x)+g(x)-f(x)+g(x)) = g(x)$

$= \min\{f(x), g(x)\}.$

If $f(x) < g(x)$

$\frac{1}{2}(f(x)+g(x)) - \frac{1}{2}|f(x)-g(x)| = \frac{1}{2}(f(x)+g(x)+f(x)-g(x))$

$= f(x) = \min\{f(x), g(x)\}$

c) Since f, g are continuous, $f-g$, $\frac{1}{2}(f+g)$ are continuous. Since $|\cdot|$ is continuous (#17) $|f-g|$ is continuous (composition), so

$\min(f,g) = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$

is continuous.

19) Ross 17, 9

c) Let $\varepsilon > 0$. Let $\delta = \varepsilon$

If $x \in \mathbb{R}$, $x \neq 0$, $|x - 0| < \delta$, then

$$|f(x) - f(0)| = |x \sin(\tfrac{1}{x}) - 0| = |x|\,|\sin(\tfrac{1}{x})| \leq |x| < \delta = \varepsilon.$$

20)

Choose $\varepsilon = 1$. Let $\delta > 0$. Choose $x = \frac{2}{\delta}$, $y = x + \frac{\delta}{2}$.
Then $|x-y| < \delta$ but
$$|f(x) - f(y)| = |x^2 - y^2| = |x+y||x-y| = \left(\frac{2}{\delta} + \frac{\delta}{2}\right)\frac{\delta}{2}$$
$$\geq \left(\frac{2}{\delta}\right)\left(\frac{\delta}{2}\right) = 1.$$

(idea: If $x, y > 0$, $|x-y| = \frac{\delta}{2}$, then
$|x+y||x-y| = |x+y| \, \delta/2 \leq x\frac{\delta}{2}$; so need
$x\frac{\delta}{2} \geq 1$. Solve for x.)

21) $f: [0,2] \to \mathbb{R}$ is continuous.

Note $h: [0,1] \to \mathbb{R}$, $h(x) = x+1$ is continuous.

Since $h([0,1]) = [1,2] \subset [0,2]$,

$f \circ h : [0,1] \to \mathbb{R}$ is continuous; Also,

the restriction $f: [0,1] \to \mathbb{R}$ is continuous.

So $g: [0,1] \to \mathbb{R}$, $g(x) = f \circ h(x) - f(x)$
$= f(x+1) - f(x)$

is continuous.

$g(0) = f(1) - f(0)$ and $g(1) = f(2) - f(1)$
$= f(0) - f(1).$
$= -g(0)$

It follows that either

a) $g(0) > 0$, $g(1) < 0$

b) $g(1) < 0$, $g(0) > 0$

c) $g(1) = g(0) = 0$.

For cases a and b, 0 is between $g(0)$ and $g(1)$. Since $[0,1]$ is an interval, there exists $x_0 \in (0,1)$ such that $g(x) = 0$.

In case c) take $x_0 = 0$.

Thus $f(x_0+1) - f(x_0) = 0$

So $f(x_0+1) = f(x_0)$

and $|(x_0+1) - x_0| = 1$.

22)

a) $f(x) = 3x + 11$.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon/3$. Let $|x-y| < \delta$.

Then $|f(x) - f(y)| = |3x+11 - (3y+11)| = 3|x-y| < 3\delta = \varepsilon$.

b) $f: [0,3] \to \mathbb{R}$, $f(x) = x^2$

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{6}$.

Let $x, y \in [0,3]$, $|x-y| < \delta$. Then

$|x^2 - y^2| = |x+y||x-y| < 6\delta = \varepsilon$.

c) $f: [\frac{1}{2}, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$.

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{4}$. If $x, y \in [\frac{1}{2}, \infty)$,

$|x - y| < \delta$, then

$|\frac{1}{x} - \frac{1}{y}| = \frac{|x-y|}{xy} \leq 4|x-y| < 4\delta = \varepsilon$.

23) a) $f : [0,2] \to \mathbb{R}$, $f(x) = \frac{x}{x+1}$

Let $\varepsilon > 0$. Choose $\delta = \varepsilon$

If $x, y \in [0,2]$, $|x-y| < \delta$, then

$$\left| \frac{x}{x+1} - \frac{y}{y+1} \right| = \left| \frac{xy + x - xy - y}{(x+1)(y+1)} \right| = \frac{|x-y|}{(x+1)(y+1)} \leq |x-y|$$

$$< \delta = \varepsilon .$$

b) $f : [1, \infty) \to \mathbb{R}$, $f(x) = \frac{5x}{2x-1}$

Let $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{5}$

If $x, y \in [1, \infty)$ and $|x-y| < \delta$,

$$\left| \frac{5x}{2x-1} - \frac{5y}{2y-1} \right| = \left| \frac{5y - 5x}{(2x-1)(2y-1)} \right| \leq 5|x-y|$$

$$< 5\delta = \varepsilon .$$

24) $f:(0,1)\to\mathbb{R}$, $f(x)=x^2$

Choose $\varepsilon=1$, let $\delta>0$.

If $\delta<1$,

Choose $x=(\delta)^{1/3}$ $y=x-\frac{\delta}{2}$.

Then $0<\delta<x<1$ and

$0<\frac{\delta}{2}<y<x<1$, so $x,y\in(0,1)$

and $|x-y|=\frac{\delta}{2}<\delta$, but

$$\left|\frac{1}{x^2}-\frac{1}{y^2}\right|=\frac{|x+y||x-y|}{x^2y^2}\geq\frac{2y\left(\frac{\delta}{2}\right)}{x^2y^2}$$

$$=\frac{\delta}{x^2y}\geq\frac{\delta}{x^3}=1=\varepsilon.$$

If $\delta\geq 1$, choose $x=\frac{1}{2}$ and $y=\frac{1}{3}$

so $|x-y| = \frac{1}{6} < \delta$ and $\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = 5 > 1 = \varepsilon$.