Math 104 004 **Midterm** **October 26, 2021**

Name: ______________________________

SID: ______________________________

Instructions :

1. You have 80 minutes, 3:10pm-4:30pm.
2. No books, notes, or other outside materials are allowed.
3. There are 5 questions on the exam. Each question is worth 10 points.
4. You need to show all of your work and justify all statements. If you need more space, use the pages at the back of the exam or come get more paper at the front of the class. If you do so, please indicate which page your solution continues on.
5. Before you begin, take a quick look at all the questions on the exam, and start with the one you feel the most comfortable solving. It is more important to do the problems well that you know how to do, than it is to finish the whole exam.
6. While attempting any problem, do write something even if you are unable to solve it completely. You may get partial credit.

(Do not fill these in; they are for grading purposes only.)

1	
2	
3	
4	
5	
Total	

1. a) (5 pts) Prove that $[1, 2]$ is closed.

b) (5 pts) Let $r \in \mathbb{R}$. Use the denseness of $\mathbb{Q}$ to prove that there exists a sequence (s_n) such that $s_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$ and $s_n \to r$.

a) Let $x \in \mathbb{R} \setminus [1,2] = (-\infty, 1) \cup (2, \infty)$.

If $x < 1$, Since $|y-x| < 1-x \Rightarrow y < 1$,

$\{y \mid |y-x| < 1-x\} \subset (-\infty, 1)$.

If $x > 2$, $|y-x| < x-2 \Rightarrow y > 2$

So $\{y \mid |y-1| < x-2\} \subset (2, \infty)$.

Thus $\mathbb{R} \setminus [1,2]$ is open and $[1,2]$ is closed

b) See sol'ns to midterm 4.

$\frac{n^2}{2} - 3 > 0 \quad n^2 > 5$

2. (10 pts) For all $n \in \mathbb{N}$, define $s_n = \frac{1}{\sqrt{n}}$ if n is odd and $s_n = \frac{n+1}{n^2-3}$ if n is even.

   Find the limit of the sequence (s_n). Prove that (s_n) converges to your answer.

Let $\varepsilon > 0$. Choose $N = \max\left\{\frac{1}{\varepsilon^2}, \frac{4}{\varepsilon}, 2\right\}$.

Let $n > N$.

If n is odd,

$$|s_n - 0| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} \leq \varepsilon.$$

If n is even, since $n > 2$, $\frac{n^2}{2} > 3$, so

$$|s_n - 0| = \frac{n+1}{n^2-3} \leq \frac{2n}{n^2/2} = \frac{4}{n} < \frac{4}{N} \leq \varepsilon.$$

In either case, $|s_n - 0| < \varepsilon$, so $s_n \to 0$.

3. For all $n \in \mathbb{N}$, define $t_n = 3 + 1/n$ if n is odd and $t_n = (-1)^{n/2}$ if n is even.

   a) (8 pts) Find $\limsup t_n$ and $\liminf t_n$. Prove your answer.

   b) (2 pts) Does (t_n) converge? Prove your answer.

If N is even,

$$\{t_n \mid n > N\} = \left\{3 + \frac{1}{N+1}, (-1)^{\frac{N+2}{2}}, 3 + \frac{1}{N+3}, (-1)^{\frac{N+4}{2}}, \dots\right\}$$

$$= \{-1, 1\} \cup \left\{3 + \frac{1}{N+1}, 3 + \frac{1}{N+3}, \dots\right\}$$

$\inf\{t_n \mid n > N\} = -1 = \min\{t_n \mid n > N\}$.

Since $3 + \frac{1}{N+1} > 1, -1$, and $3 + \frac{1}{n}$ for $n > N+1$,

$3 + \frac{1}{N+1} = \sup\{t_n \mid n > N\} = \max\{t_n \mid n > N\}$.

Similarly, if N is odd, $-1 = \inf\{t_n \mid n > N\}$ and $3 + \frac{1}{N+3} = \sup\{t_n \mid n > N\}$.

$\left(\dots, 3 + \frac{1}{N+3}, 3 + \frac{1}{N+2}, 3 + \frac{1}{N+5}, 3 + \frac{1}{N+4}, \dots\right)$ converges to 3

$(-1, -1, \dots, -1, \dots)$ converges to -1

So $\limsup t_n = 3 \neq \liminf t_n = -1$, and (t_n) does not converge.

4. (10 pts) Define $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \frac{1}{1+x}$ if $x < -1$ and $g(x) = -x^2$ if $x \geq -1$. Prove that g is not continuous.

Let $x_n = -1 - \frac{1}{n} \to -1$

$g(x_n) = -n$ does not converge

(not bounded)

So g is not continuous at -1.

5. a) (4 pts) Prove that $f : [3, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x-2}$ is uniformly continuous.

b) (4 pts) Find a sequence (x_n) in $(2, \infty)$ such that (x_n) is Cauchy but $\left(\frac{1}{x_n - 2}\right)$ is not Cauchy.

c) (2 pts) Prove that $f : (2, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x-2}$ is not uniformly continuous.

a) Let $\varepsilon > 0$. Choose $\delta = \varepsilon$

Let $|x-y| < \delta$.

$$\left| \frac{1}{x-2} - \frac{1}{y-2} \right| = \left| \frac{y-x}{(x-2)(y-2)} \right| < \frac{\delta}{(3-2)(3-2)} = \delta$$

b) $X_n = 2 + \frac{1}{n} \to 2$, so it is Cauchy

$\frac{1}{x_n - 2} = n$ is not bounded, so not Cauchy

c) If f were uniformly continu, since (x_n) is Cachy, $f(x_n) = \frac{1}{x_n - 2}$ would be Cauchy.

So f is not uniformly continuous.