Practice Problems

These problems mostly cover the second half of the semester. They do not cover every topic, and are meant as extra practice, not a comprehensive review.

1. Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions such that $\int_a^b f = \int_a^b g$. Prove that $f(x) = g(x)$ for some $x \in [a, b]$.

2. Find the Taylor series for $f : (-\infty, 1) \to \mathbb{R}$, $f(x) = \sqrt{1-x}$. Use Taylor's theorem to prove that the Taylor series converges to f for $x \in (-1, 0)$.

3. Define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \frac{x}{1+nx^2}$. Find $f : \mathbb{R} \to \mathbb{R}$ such that $f_n \to f$ uniformly. Prove that $\lim_{n\to\infty} f_n'(x) = f'(x)$ for all $x \in \mathbb{R}$ except $x = 0$.

4. For each $k \in \mathbb{N}$ define

$$g_k : [-\pi, \pi] \to \mathbb{R} \text{ by } g_k(x) = (\sin(x))^2(\cos(x))^{2k}.$$

a) Find $f : [-\pi, \pi] \to \mathbb{R}$ such that $\sum g_k \to f$ pointwise.

b) Does $\sum g_k \to f$ uniformly?

c) If we change the domain to $[\pi/4, 3\pi/4]$, does $\sum g_k \to f$ uniformly?

5. Define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{\sin(x)}$ and $g(x) = \int_0^{x^2} f$. Find $g'(x)$.

6. Let $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ such that $\sum a_n 2^n$ converges. Prove that the sequence of functions $\sum a_n x^n$ converges uniformly on $[-1, 1]$.

7. Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be differentiable such that that $f'(x) \neq 0$ for all $x \in I$.

   a) Prove that there exists an "inverse function", that is, a function $g : f(I) \to \mathbb{R}$ such that $g \circ f(x) = x$ for all $x \in I$ and $f \circ g(y) = y$ for all $y \in f(I)$.

   b) Prove that g is continuous.

8. Let S be a metric space with distance function d. Fix a point $x_0 \in S$. Define $f : S \to \mathbb{R}$ by $f(x) = d(x, x_0)$. Prove that f is uniformly continuous.

9. Let E be a nonempty connected subset of $\mathbb{R}$ such that $E \subset \mathbb{Q}$. Prove that E has exactly one element.