

The Integral

Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function.

Definitions

- Let $S \subseteq [a, b]$. Define $M(f, S) = \sup\{f(x) | x \in S\}$ and $m(f, S) = \inf\{f(x) | x \in S\}$.
- A **partition** of $[a, b]$ is a finite subset $P \subset [a, b]$ containing a and b , labeled

$$P = \{a = t_0 < t_1 < \dots < t_{n-1} < t_n = b\}.$$

- Given a partition P of $[a, b]$ the **upper and lower Darboux sums** of f are

$$U(f, p) = \sum_{k=1}^n M(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

$$L(f, p) = \sum_{k=1}^n m(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

- the **upper and lower Darboux integrals** of f are

$$U(f) = \inf\{U(f, P) | P \text{ is a partition of } [a, b]\}$$

$$L(f) = \sup\{L(f, P) | P \text{ is a partition of } [a, b]\}$$

- f is **integrable** if $U(f) = L(f)$. In that case the integral is defined as

$$\int_a^b f = U(f) = L(f).$$

Lemmas

- If P, Q are partitions of $[a, b]$ such that $P \subseteq Q$ then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

- If P, Q are partitions of $[a, b]$ then $L(f, P) \leq U(f, Q)$.
- $L(f) \leq U(f)$.

Theorem f is integrable if and only if for each $\epsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$