Lecture 20, 4/5/22
Material corresponds to Ross $\S 29, 30$.

Intermediate Value Theorem for Derivatives

Theorem Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a differentiable function. Then $f'(I)$ is an interval. That is, if $x_1, x_2 \in I$, $x_1 < x_2$, and y lies between $f'(x_1)$ and $f'(x_2)$ then there exists $x \in (x_1, x_2)$ such that $f'(x) = y$.

Corollary Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ a differentiable function. If $f'(x) \neq 0$ for all $x \in I$ then f is either strictly increasing or strictly decreasing.

L'Hospital's Theorem

Definition Let (s_n) be a sequence in $\mathbb{R}$.

1. (s_n) **diverges to infinity** ($s_n \to \infty$) if for all $M \in \mathbb{R}$ there exists $N \in \mathbb{R}$ such that if $n > N$ then $s_n > M$
2. (s_n) **diverges to negative infinity** ($s_n \to -\infty$) if for all $M \in \mathbb{R}$ there exists $N \in \mathbb{R}$ such that if $n > N$ then $s_n < M$.

Definition Let $E \subseteq \mathbb{R}$ and let $a \in \mathbb{R}$ such that there is a sequence in E converging to a. Let $f : E \to \mathbb{R}$ be a function.

1. $\lim_{x\to a} f(x) = \infty$ if for every sequence (s_n) in $E\backslash\{a\}$, $s_n \to a$ implies $f(s_n) \to \infty$.
2. $\lim_{x\to a} f(x) = -\infty$ if for every sequence (s_n) in $E\backslash\{a\}$, $s_n \to a$ implies $f(s_n) \to -\infty$.

Definition

1. Suppose $E \subseteq \mathbb{R}$ is not bounded above. Then $\lim_{x\to\infty} f(x) = L$ if for every sequence (s_n) in E, $s_n \to \infty$ implies $f(s_n) \to a$.
2. Suppose $E \subseteq \mathbb{R}$ is not bounded below. $\lim_{x\to-\infty} f(x) = L$ if for every sequence (s_n) in E, $s_n \to -\infty$ implies $f(s_n) \to L$.

Theorem Let $E \subseteq \mathbb{R}$ and let $a \in \mathbb{R}$ such that there is a sequence in E converging to a. Let $f : E \to \mathbb{R}$ be a function.

1. $\lim_{x\to a} f(x) = \infty$ if and only if for all $M \in \mathbb{R}$ there exists $\delta > 0$ such that $x \in E\backslash\{a\}$ and $|x - a| < \delta$ implies $f(x) > M$.
2. $\lim_{x\to a} f(x) = -\infty$ if and only if for all $M \in \mathbb{R}$ there exists $\delta > 0$ such that $x \in E\backslash\{a\}$ and $|x - a| < \delta$ implies $f(x) < M$.

Theorem

1. Let (s_n) be a sequence in $\mathbb{R}$ such that $s_n \neq 0$ for all $n \in \mathbb{N}$. Then $|s_n| \to \infty$ if and only if $\frac{1}{s_n} \to 0$

2. Let $E \subset \mathbb{R}$ and $a \in \mathbb{R}$ such that there is a sequence in E converging to a. Let $f : E \to \mathbb{R}$ be a function such that $f(x) \neq 0$ for all $x \in E$. Then $\lim_{x\to a} |f(x)| = \infty$ if and only if $\lim_{x\to a} \frac{1}{f(x)} = 0$.

Theorem (L'Hospital's Rule) Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions, and assume $g'(x) \neq 0$ for all $x \in (a, b)$. If

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \qquad \text{or} \qquad \lim_{x\to a} \frac{f'(x)}{g'(x)} = \pm\infty$$

and either

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \qquad \text{or} \qquad \lim_{x\to a} |g(x)| = \infty$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$