Lecture 27, 4/28/22
Material corresponds to Ross §31.

Integrating and Differentiating Power Series

Lemma $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$ have the same radius of convergence.

Theorem Let $\sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. Then $\sum_{n=0}^{\infty} a_n x^n$ for $x \in (-R, R)$ converges pointwise to a differentiable function $f : (-R, R) \to \mathbb{R}$ such that

$$\int_0^{x_0} f = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x_0^{n+1} \text{ and } f'(x_0) = \sum_{n=1}^{\infty} n a_n x_0^{n-1}.$$

Taylor Series

Notation $f^{(n)}$ is the n^{th} derivative of f. If it exists, we say f has "derivatives to order n" or f is "n times differentiable". If $f^{(n)}$ exists for all $n \in \mathbb{N}$ we say f has "derivatives to all orders" or f is "infinitely differentiable." By convention, $f^{(0)} = f$.

Definition Let $I \subset \mathbb{R}$ be an open interval containing 0 and let $f : I \to \mathbb{R}$ be n-times differentiable .

- $\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$ is called the "Taylor Polynomial of order n" for f.
- $R_{n+1}(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$ is called the "remainder"
- If f has derivatives of all orders, $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ is called the "Taylor Series" for f.

Note $\lim_{n\to\infty} R_n(x) = 0$ if and only if $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \to f(x)$.

Theorem (Taylor's Theorem) Let $I \subset \mathbb{R}$ be an open interval containing 0 and let $f : I \to \mathbb{R}$ be n-times differentiable. Then for $x_0 \in I$, $x_0 \neq 0$, there exists y between 0 and x_0 such that

$$R_n(x_0) = \frac{f^{(n)}(y)}{n!} x_0^n.$$

Corrolary Let $I \subset \mathbb{R}$ be an open interval containing 0 and let $f : I \to \mathbb{R}$ be n-times differentiable for some $n \in \mathbb{N}$. If there exists $C \in \mathbb{R}$ such that $|f^{(n)}(x)| \leq C$ for all $x \in I$ then

$$\lim_{x\to 0} \frac{R_n(x)}{x^{n-1}} = 0.$$

Corollary Let $I \subset \mathbb{R}$ be an open interval containing 0 and let $f : I \to \mathbb{R}$ be differentiable to all orders. If there exists $C \in \mathbb{R}$ such that $|f^{(n)}(x)| \leq C$ for all $n \in \mathbb{N}$ and $x \in I$ then $\sum_{n=0}^{\infty} a_n x^x = f(x)$ for all $x \in I$.