Lecture 26, 4/26/22
Material corresponds to Ross §23.

Series of Functions

Definition Let $g_k : S \to \mathbb{R}$ be functions for all $k \in N$. The series $\sum g_k$ represents the sequence of functions (f_n), $f_n = \sum_{k=1}^{n} g_k$. If $f_n \to f$ uniformly, we say $\sum g_k$ converges to f uniformly.

Theorem If $g_k : S \to \mathbb{R}$ is continuous for all $k \in \mathbb{N}$ and $\sum g_k$ converges uniformly to f then f is continuous.

Theorem (Weierstrass M Test) Let $M_k \in \mathbb{R}$, $M_k \geq 0$ for all $k \in \mathbb{N}$ such that $\sum M_k$ converges. Let $g_k : S \to \mathbb{R}$ be a function such that $|g_k(x)| \leq M_k$ for all $x \in S$. Then $\sum g_k$ converges uniformly.

Power Series

Definition Let $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ (or $\mathbb{N} \cup \{0\}$.) Let $g_n(x) = a_n x^n$. The series of functions $\sum g_n = \sum a_n x^n$ is called a **power series**.

Definition (Radius of Convergence) Let $\sum a_n x^n$ be a power series.

- If $(|a_n|^{\frac{1}{n}})$ is not bounded, define $R = 0$.
- Otherwise, let $\beta = \limsup |a_n|^{\frac{1}{n}}$. If $\beta = 0$, define $R = \infty$.
- If $\beta > 0$, define $R = \frac{1}{\beta}$.

R is called the **radius of convergence** of $\sum a_n x^n$.

Recall that if $\left(\left|\frac{a_{n+1}}{a_n}\right|\right)$ converges, then

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim |a_n|^{\frac{1}{n}}.$$

Theorem Let $\sum a_n x^n$ be a power series with radius of convergence R.

a) If $|x| < R$, $\sum a_n x^n$ converges.

b) If $|x| > R$, $\sum a_n x^n$ does not converge.

Definition Let $\sum a_n x^n$ be a power series. The interval of convergence of $\sum a_n x^n$ is the set of $x \in \mathbb{R}$ such that $\sum a_n x^n$ converges.

Note: If $R = 0$ the interval of convergence is $\{0\}$. If $R = \infty$ iy is $(-\infty, \infty)$. Otherwise it can be $(-R, R)$, $[-R, R]$, $[-R, R)$ or $(-R, R]$.

Theorem Let $\sum a_n x^n$ be a power series with radius of convergence $R > 0$ (or $R = \infty$.) Let $0 < R_1 < R$ (or simply $0 < R_1$ if $R = \infty$). Then $\sum a_n x^n$ converges uniformly on $[-R_1, R_1]$.

Corollary $\sum a_n x^n$ converges to a continuous function on $(-R, R)$.