Lecture 24, 4/19/22
Material corresponds to Ross §24.

Sequences of Functions

Definition A sequence (f_n) of functions $f_n : S \to \mathbb{R}$ **converges pointwise** to a function $f : S \to \mathbb{R}$ if for all $x \in S$ $(f_n(x))$ converges to $f(x)$. We write "$f_n \to f$ pointwise."

Definition A sequence (f_n) of functions $f_n : S \to \mathbb{R}$ **converges uniformly** to a function $f : S \to \mathbb{R}$ if for all $\epsilon > 0$ there exists $N \in \mathbb{R}$ such that $n > N$ and $x \in S$ implies $|f_x(x) - f(x)| < \epsilon$. We write "$f_n \to f$ uniformly."

Fact If $f_n \to f$ uniformly then $f_n \to f$ pointwise.

Theorem Let $f_n, f : S \to \mathbb{R}$ be functions for all $n \in \mathbb{N}$ such that $f_n \to f$ uniformly. If f_n is continuous at $x_0 \in S$ for each $n \in \mathbb{N}$ then f is continuous at x_0.

Lemma Let $f, f_n : S \to \mathbb{R}$ be functions for all $n \in \mathbb{N}$. Then $f_n \to f$ uniformly if and only if

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| \mid x \in S\} = 0.$$

Definition A sequence of functions (f_n), $f_n : S \to \mathbb{R}$, is **uniformly Cauchy** if for all $\epsilon > 0$ there exists $N \in \mathbb{R}$ such that $n, m > N$ implies $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in S$.

Theorem If (f_n), $f_n : S \to \mathbb{R}$ is a uniformly Cauchy sequence of functions, then there exists $f : S \to \mathbb{R}$ such that $f_n \to f$ uniformly.

Definition Let $g_k : S \to \mathbb{R}$ be functions for all $k \in N$. The series $\sum g_k$ represents the sequence of functions (f_n), $f_n = \sum_{k=1}^{n} g_k$. If $f_n \to f$ uniformly (or pointwise), we say $\sum g_k$ converges to f uniformly (or pointwise).