Lecture 22, 4/12/22
Material corresponds to Ross §33.

Properties of Integrals

Let $f : [a,b] \to \mathbb{R}$ be a function.

Theorem

a) If f is monotone, f is integrable.

b) If f is continuous, f is integrable.

Let $f, g : [a,b] \to \mathbb{R}$ be integrable.

Theorem (linearity)

a) Let $c \in \mathbb{R}$. Then cf is integrable and $\in_a^b cf = f \int_a^b f$.

b) $f + g$ is integrable and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Theorem (order)

a) If $f(x) \le g(x)$ for all $x \in [a,b]$ then $\int_a^b f \le \int_a^b g$.

b) $|f|$ is integrable and $\left| \int_a^b f \right| \le \int_a^b |f|$.

c) If f is continuous, $f(x) \ge 0$ for all $x \in [a,b]$, and $\int_a^b f = 0$ then $f(x) = 0$ for all $x \in [a,b]$.

Theorem Let $f : [a,b] \to \mathbb{R}$ be a function and let $c \in (a,b)$. If $f : [a,c] \to \mathbb{R}$ and $f : [c,b] \to \mathbb{R}$ are integrable then $f : [a,b] \to \mathbb{R}$ is integrable and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Theorem Let $f, g : [a,b] \to \mathbb{R}$ be functions. If f is integrable and $f(x) = g(x)$ for all $x \in [a,b] \setminus S$, where S is finite, then g is integrable and $\int_a^b f = \int_a^b g$.

Definition

1. $f : [a,b] \to \mathbb{R}$ is **piecewise monotone** if it is bounded and there exists a partition

$$P = \{a = t_0 < t_1 < ... < t_n = b\}$$

of $[a,b]$ such that $f : (t_{k-1}, t_k) \to \mathbb{R}$ is monotone for all $k = 1, ..., n$.

2. $f : [a, b] \to \mathbb{R}$ is **piecewise continuous** if there exists a partition

$$P = \{a = t_0 < t_1 < ... < t_n = b\}$$

of $[a, b]$ such that $f : (t_{k-1}, t_k) \to \mathbb{R}$ is uniformly continuous for all $k = 1, ..., n$.

Theorem A piecewise monotone or piecewise continuous function is integrable.