Lecture 15, 3/8/22
Material corresponds to Ross $\S 19, 21$.

Uniform Continuity

Theorem If $f : S \to \mathbb{R}$ is uniformly continuous and (x_n) is a Cauchy sequence in S then $(f(x_n))$ is Cauchy.

Theorem A function $f : (a, b) \to \mathbb{R}$ is uniformly continuous if and only if it can be extended to a continuous function $f : [a, b] \to \mathbb{R}$.

Continuity in Metric Spaces

Let S be a metric space with distance function d.
Let S^* be a metric space with distance function d^*.

Definition Let $E \subseteq S$. A function $f : S \to S^*$ **is continuous at** $x_0 \in E$ if

- for all sequences (x_n) in E, $x_n \to x_0$ implies $f(x_n) \to f(x_0)$.
- OR EQUIVALENTLY, If for all $\epsilon >$ there exists $\delta > 0$ such that $x \in E$, $d(x, x_0) < \delta$ implies $d^*(f(x), f(x_0)) < \epsilon$.

Definition Let $E \subseteq S$. A function $f : E \to S^*$ is **uniformly continuous** if for all $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in E$, $d(x, y) < \delta$ implies $d^*(f(x), f(y)) < \epsilon$.

Theorem Let $E \subseteq S$ be sequentially compact and let $f : E \to S^*$ be continuous. Then:

1. $f(E)$ is sequentially compact.
2. f is uniformly continuous.

Theorem Let $E \subseteq S$ be sequentially compact and let $f : E \to \mathbb{R}$ be continuous. Then there exists $x_0, x_1 \in E$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in E$.