Lecture 19, 3/31/22
Material corresponds to Ross §29.

Mean Value Theorem

Theorem Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be differentiable at $x_0 \in I$. If f assumes its maximum at x_0 ($f(x_0) = \max f(I)$, that is, $f(x_0) \geq f(x)$ for all $x \in I$) or f assumes its minimum at x_0 ($f(x_0) = \min f(I)$, that is, $f(x_0) \leq f(x)$ for all $x \in I$) then $f'(x_0) = 0$.

Lemma (Rolle's Theorem) Let $f : [a, b]$ be a continuous function which is differentiable on (a, b). If $f(a) = f(b)$ then there exists $x \in (a, b)$ such that $f'(x) = 0$.

Meant Value Theorem (MVT) Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Then there exists $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Corollary Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a differentiable function. If $f' : I \to \mathbb{R}$ is bounded then f is uniformly continuous.

Corollary Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a differentiable function. Let $a, b \in I, a < b$.

1. If $f'(x) > 0$ for all $x \in I$ then $f(a) < f(b)$ (f is strictly increasing).
2. If $f'(x) \geq 0$ for all $x \in I$ then $f(a) \leq f(b)$ (f is increasing).
3. If $f'(x) < 0$ for all $x \in I$ then $f(a) > f(b)$ (f is strictly decreasing).
4. If $f'(x) \leq 0$ for all $x \in I$ then $f(a) \geq f(b)$ (f is decreasing).
5. If $f'(x) = 0$ for all $x \in I$ then $f(a) = f(b)$ (f is constant).

Theorem (Intermediate Value Theorem for Derivatives) Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a differentiable function. Then $f'(I)$ is an interval. That is, if $x_1, x_2 \in I$, $x_1 < x_2$, and y lies between $f'(x_1)$ and $f'(x_2)$ then there exists $x \in (x_1, x_2)$ such that $f'(x) = y$.