Lecture 13, 3/1/22
Material corresponds to Ross §17, 20.

Properties of Continuous Functions

Theorem ($\epsilon - \delta$ property)

1. Given $f : S \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that there is a sequence in S converging to x_0,

$$\lim_{x \to x_0} f(x) = L$$

   if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that $x \in S \backslash \{x_0\}$ and $|x - x_0| < \delta$ imply $|f(x) - L| < \epsilon$.

2. $f : S \to \mathbb{R}$ is **continuous at** $x_0 \in S$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that $x \in S$, $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$.

Theorem If $f : S \to \mathbb{R}$ is continuous at $x_0 \in S$ and $E \subseteq S$ is a subset such that $x_0 \in E$, then the restriction $f : E \to \mathbb{R}$ is continuous at x_0.

Definition

1. $f : S \to \mathbb{R}$, $E \subset S$, then the **image** of E is $f(E) = \{f(x) \in \mathbb{R} |\ x \in E\}$.

2. $f : S \to \mathbb{R}$, $f(S) \subseteq T$, $g : T \to \mathbb{R}$, the **composition** $g \circ f : S \to \mathbb{R}$ is defined by $g \circ f(x) = g(f(x))$.

Theorem

1. Given $f : S \to \mathbb{R}$, $f(S) \subseteq T$, and $g : T \to \mathbb{R}$, if $\lim_{x \to x_0} f(x) = L$, $L \in T$, and g is continuous at L, then $\lim_{x \to x_0} g \circ f(x) = g(L)$.

2. Let $f : S \to \mathbb{R}$ be continuous at $x_0 \in S$, $f(S) \subseteq T$, and let $g : T \to \mathbb{R}$ be continuous at $f(x_0) \in T$. Then $g \circ f : S \to \mathbb{R}$ is continuous at $x_0 \in S$.

Theorem If S is sequentially compact and $f : S \to \mathbb{R}$ is continuous, then $f(S)$ is sequentially compact.