Lecture 11, 2/22/22
Material on series corresponds to Ross §14.

Open and Closed sets in a general Metric Space

Let S be a metric space with distance function d.
Definition 1

- A set $E \subset S$ is closed if for every sequence (s_n) in E, $s_n \to s$ implies $s \in E$.
- A set E is open if its complement $S \backslash E = \{s \in S | s \notin E\}$ is closed.

Definition 2

- A set $E \subset S$ is open if for all $s \in E$ there exists $\epsilon \in \mathbb{R}$, $\epsilon > 0$ such that $B_\epsilon(s) \subset E$.
- A set E is closed if its complement $S \backslash E = \{s \in S | s \notin E\}$ is open.

Compactness

Definition A set $E \subset S$ is **sequentially compact** if every sequence (s_n) in E has a subsequence which converges to an element of E.

Theorem A sequentially compact set is closed and bounded.

Warning There are closed and bounded sets (even in complete metric spaces) which are not sequentially compact.

Series

Given an infinite list of numbers $a_k \in \mathbb{R}$ for all $k \in \mathbb{N}$ the symbol $\sum a_k$ (or $\sum_{k=m}^{\infty} a_k$), called a **series**, represents the sequence (s_n) where

$$s_n = \sum_{k=1}^{n} a_l \quad \left(\text{or } \sum_{k=m}^{n} a_k \right).$$

If $s_n \to s$ we say "$\sum a_k$ converges to s" and write $\sum a_k = s$.

Fact $\sum a_k$ is Cauchy if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{R}$ such that if $n \geq m > N$ then $\left|\sum_{k=m+1}^{n} a_k\right| < \epsilon$.

Theorem A series converges if and only if it is Cauchy.

Corollary If $\sum a_k$ converges then $a_k \to 0$.

Limit Comparison Test If $|b_k| \leq a_k$ for all $k \in \mathbb{N}$ and $\sum a_k$ converges then $\sum b_k$ converges.

Definition A series $\sum a_k$ **converges absolutely** if $\sum |a_k|$ converges.

Theorem If $\sum a_k$ converges absolutely then $\sum a_k$ converges.

Ratio Test Let $a_k \neq 0$ for all $k \in \mathbb{N}$.

- If $\limsup \left|\frac{a_{k+1}}{a_k}\right| < 1$ then $\sum a_k$ converges absolutely.
- If $\liminf \left|\frac{a_{k+1}}{a_k}\right| > 1$ or does not exist then $\sum a_k$ does not converge.

Root Test

- If $\limsup |a_k|^{\frac{1}{k}} < 1$ then $\sum a_k$ converges absolutely.
- If $\limsup |a_k|^{\frac{1}{k}} > 1$ or does not exist then $\sum a_k$ does not converge.