Lecture 10, 2/17/22
Material corresponds loosely to Ross §13.

Compactness

Definition A set $E \subset \mathbb{R}$ is **sequentially compact** if every sequence (s_n) in E has a subsequence which converges to an element of E.

Theorem A subset of $\mathbb{R}$ is sequentially compact if and only if it is closed and bounded.

Metric Spaces

Definition A **metric space** is a set S together with a distance function d such that for all $a, b \in S$, $d(a, b) \in \mathbb{R}$ satisfies

1. $d(a, b) \geq 0$, and $d(a, b) = 0$ if and only if $a = b$.
2. $d(a, b) = d(b, a)$.
3. (Triangle Inequality) $d(a, b) \leq d(a, c) + d(c, b)$ for all $c \in S$.

Let S be a metric space with distance function d.

Definition

1. A **sequence** in S is a list of elements of $S : (s_n) = (s_1, s_2, s_3, ...)$ where $s_n \in S$ for all $n \in \mathbb{N}$.
2. (s_n) **converges** to $s \in S$ if for all $\epsilon \in \mathbb{R}$, $\epsilon > 0$ there exists $N \in \mathbb{R}$ such that $n > N$ implies $d(s_n, s) < \epsilon$.

Theorem If $s_n \to s$ and $s_n \to t$ then $s = t$.

Definition

1. For $s_0 \in S$ and $\epsilon \in \mathbb{R}$, $\epsilon > 0$,
$$B_\epsilon(s_0) = \{s \in S | d(s, s_0) < \epsilon\}.$$
2. A set $E \subseteq S$ is **bounded** if there exists $s \in S$, $M \in \mathbb{R}$, $M > 0$ such that $E \subseteq B_M(s)$.
3. A sequence (s_n) is **bounded** if the set $\{s_n | n \in \mathbb{N}\}$ is bounded.

Theorem Convergent sequences are bounded.

Theorem If $s_n \to s$ then any subsequence of (s_n) converges to s.

Definition A sequence (s_n) in S is **Cauchy** if for all $\epsilon \in \mathbb{R}$, $\epsilon > 0$ there exists $N \in \mathbb{R}$ such that $m, n > N$ implies $d(s_m, s_n) < \epsilon$.

Theorem If (s_n) converges, (s_n) is Cauchy.

Warning In some metric spaces, not every Cauchy sequence converges.

Definition A metric space S is called **complete** if every Cauchy sequence in S converges.