Lecture 5, 2/1/22
Material corresponds to Ross §7 − 8.

Sequences

Theorem (Triangle inequality) If $a, b, c \in \mathbb{R}$ then

$$|a-b| \leq |a-c| + |c-d|.$$

Theorem (Limits are Unique) If $s_n \to s$ and $s_n \to t$ then $s = t$.

Theorem Let $S \subset \mathbb{R}$. If $r = \sup S$ or $r = \inf S$ then there is a sequence (s_n) such that $s_n \in S$ for all $n \in \mathbb{N}$ and $s_n \to r$.

Example

$$\frac{2n^2 - 7n}{n^2 - n + 8} \to 2$$

Let $\epsilon > 0$. Choose $N = \max\left\{2, \frac{42}{\epsilon}\right\}$. Let $n > N$. Then

$$\left|\frac{2n^2 - 7n}{n^2 - n + 8} - 2\right| = \left|\frac{5n + 16}{n^2 - n + 8}\right| \leq \frac{5n + 16}{n^2 - n} \leq \frac{21n}{n^2 - n}$$

because $n > 1$, so $|n^2 - n + 8| = n^2 - n + 8 \geq n^2 - n$ and $5n + 16 \leq 5n + 16n$. Furthermore,

$$\frac{21n}{n^2 - n} < \frac{21n}{n^2/2} = \frac{42}{n}$$

since $n > N \geq 2$ implies $n^2 - n > n^2/2$. Finally,

$$\frac{42}{n} < \frac{42}{N} \leq \epsilon$$

since $n > N$ and $N \geq 42/\epsilon$. Stringing the inequalities together we have

$$\left|\frac{2n^2 - 7n}{n^2 - n + 8} - 2\right| < \epsilon.$$