Integration

Let $f : [a,b] \to \mathbb{R}$ be bounded.

Notation $S \subseteq [a,b]$

$M(f,S) = \sup f(S) = \sup\{f(x) \mid x \in S\}$

$m(f,S) = \inf f(S)$

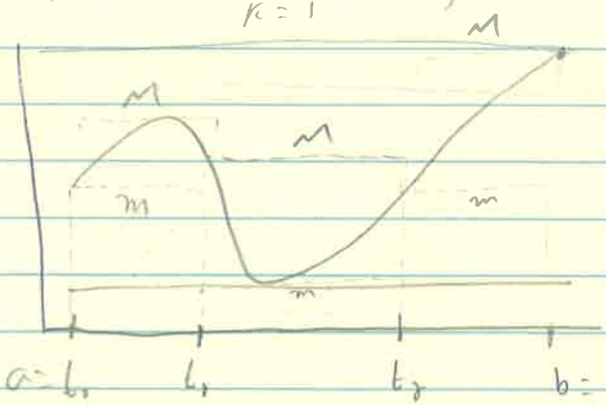
A partition of $[a,b]$ is a finite subset $P \subset [a,b]$ such that $a, b \in P$.

Notation: $P = \{a = t_0 < t_1 < \dots < t_{n-1} < t_n = b\}$.

Def "Darboux Sums" Given a partition P of $[a,b]$

$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1})$ ≈ overestimate

$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1})$ ≈ underestimate



Recall If $A \subseteq B$, $\inf B \le \inf A \le \sup A \le \sup B$

$m(f,[a,b]) \le m(f,[t_{k-1},t_k]) \le M(f,[t_{k-1},t_k]) \le M(f,[a,b])$

so $m(f,[a,b])(b-a) \le L(f,P) \le U(f,P) \le M(f,[a,b])(b-a)$

upper, lower bounds independent of P

Def $U(f) = \inf\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$

$L(f) = \sup\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$

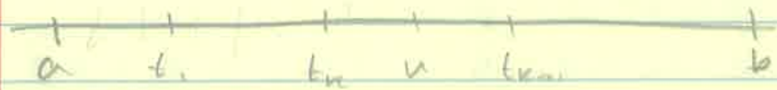
Def f is integrable on $[a,b]$ if $U(f) = L(f)$. Then $\int_a^b f = U(f) = L(f)$.

Lemma If $P \subseteq Q$ then
$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$$

Proof

Let $P = \{a = t_0 < t_1 < \dots < t_{k-1} < t_k < \dots < t_n = b\}$

$Q = \{a = t_0 < t_1 < \dots < t_{k-1} < u < t_k < \dots < t_n = b\}$



$$L(f,Q) - L(f,P) = m(f,[t_k,u])(u - t_{k-1}) + m(f,[u,t_{k+1}])(t_{k+1} - u)$$
$m(f,[t_k,u]) \ge m(f,[t_k,t_{k-1}])$
$$- \; m(f,[t_{k-1},t_k])(t_k - t_{k-1})$$
$$\ge m(f,[t_{k-1},t_k])(u - t_{k-1} + t_k - u - t_k + t_{k-1}) = 0$$
Similar for U. Repeat to add more points.

Lemma If P, Q are partitions of $[a,b]$ then
$$L(f,P) \le U(f,Q)$$

Proof $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$ so
$$L(f,P) \le L(f,P \cup Q) \le U(f,P \cup Q) \le U(f,Q).$$

Lemma $L(f) \le U(f)$

Let P be a partition of $[a,b]$. For all partitions Q of $[a,b]$
$L(f,P) \le U(f,Q)$. So $L(f,P)$ is a lower bound for $\{U(f,Q) \mid Q$ part of $[a,b]\}$
Thus $L(f,P) \le U(f)$.
This is true for all partitions P, so $U(f)$ is an upper bound for $\{L(f,P) \mid P$ part of $[a,b]\}$
Thus $L(f) \le U(f)$.

Thm f is integrable if and only if
For all $\varepsilon > 0$ there exists a partition P_ε of $[a,b]$ such that
$$U(f,P) - L(f,P) < \varepsilon.$$

Proof

Step 1 Assume f is integrable. Let $\varepsilon > 0$.
Since $L(f) - \frac{\varepsilon}{2} < L(f)$, there exists a partition P_1 of $[a,b]$ such that $L(f) - \frac{\varepsilon}{2} < L(f,P_1) \le L(f)$

Similarly, there exists a partition P_2 such that $U(f,P_2) < U(f) + \frac{\varepsilon}{2}$
Let $P = P_1 \cup P_2$. Then

$$L(f) - \frac{\varepsilon}{2} < L(f,P_1) \le L(f,P) \le U(f,P) \le U(f) + \frac{\varepsilon}{2}$$

So $U(f,P) - L(f,P) < U(f) + \frac{\varepsilon}{2} - L(f) + \frac{\varepsilon}{2} = \varepsilon$
$\therefore$ Since $U(f) = L(f)$.

Step 2 Assume for all $\varepsilon > 0$, there exists P_ε such that $U(f,P) - L(f,P) < \varepsilon$
Then $U(f) \le U(f,P) < L(f,P) + \varepsilon \le L(f) + \varepsilon$.
Since this is true for all $\varepsilon > 0$, $U(f) \le L(f)$, so $U(f) = L(f)$.

Define $f : [0,2] \to \mathbb{R}$, $f(x) = x$.
$P_n = \{t_0 = 0 < t_1 = \frac{1}{n} < \dots < t_k = \frac{k}{n} < \dots t_{2n} = 2\}$
Since f is increasing $M(f,[t_{k-1},t_k]) = t_k$ $\quad m(f,[t_{k-1},t_k]) = t_{k-1}$
so

$$U(f,P) = \sum_{k=1}^{2n} t_k(t_k - t_{k-1}) = \sum_{k=1}^{2n} \frac{k}{n}\left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^2}\sum_{k=1}^{2n} k$$

$$= \frac{2n(2n+1)}{2n^2} = 2 + \frac{1}{n}$$

$$L(f,P) = \sum_{k=1}^{2n} t_{k-1}(t_k - t_{k-1}) = \sum_{k=1}^{2n} \frac{k-1}{n^2} = 2 + \frac{1}{n} - \frac{2n}{n^2} = 2 - \frac{1}{n}$$

So finally $2 - \frac{1}{n} = L(f,P_n) \le L(f) \le U(f) \le U(f,P_n) = 2 + 1/n$.
So $U(f) = L(f) = 2$.