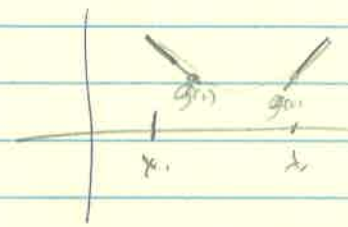
Intermediate Value Thm for derivatives

Let I be an open interval, $f: I \to \mathbb{R}$ diff.
Then $f'(I)$ is an interval.

Proof

Let $x_1, x_2 \in I$, $c \in \mathbb{R}$ such that $f'(x_1) < c < f'(x_2)$
Define $g: I \to \mathbb{R}$, $g(x) = f(x) - cx$.
Then $g' = f' - c$; we show that g achieves its min
at $x_0 \in (x_1, x_2)$, so $0 = g'(x_0) = f'(x_0) - c$.

Note $g'(x_1) < 0 < g'(x_2)$



There exists $\delta_1 > 0$ such that for $x \in (x_1, x_1 + \delta)$,
$\frac{g(x) - g(x_1)}{x - x_1} < 0$, so $g(x) < g(x_1)$.
Thus $g(x_1) \neq \min g(I)$.
There exists $\delta_2 > 0$ such that for $x \in (x_2 - \delta_2, x_2)$
$\frac{g(x) - g(x_2)}{x - x_2} > 0$ so $g(x) < g(x_2)$, so $g(x_2) \neq \min g(I)$.
$[x_1, x_2]$ cpct,
But g is continuous on, so there exists $x_0 \in (x_1, x_2)$ such that
$g(x_0) = \min g(I)$; then $g'(x_0) = 0$.

Cor Let $I \subseteq \mathbb{R}$ be an open interval, $f: I \to \mathbb{R}$ diff.
If $f'(x) \neq 0$ for all $x \in I$, f is either strictly increasing or strictly decreasing.

Proof Suppose there exists $x_1, x_2 \in I$ such that $f'(x_1) < 0$ and $f'(x_2) > 0$.
Then there exists $x \in I$ such that $f'(x) = 0$, a contradiction.
So either $f'(x) < 0$ for all x, or $f'(x) > 0$ for all x.

L'Hospital's Rule

Let $f, g : (a,b) \to \mathbb{R}$ be diff, $g'(x) \neq 0$ for all $x \in (a,b)$.
Assume $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$.

If either

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} |g(x)| = \infty$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$

Lemma (Cauchy MVT) Let $f, g : [a,b] \to \mathbb{R}$ be cont, and diff on (a,b).
Then there is an $x \in (a,b)$ such that $f'(x)(g(b)-g(a)) = g'(x)(f(b)-f(a))$

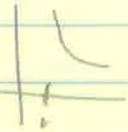
Proof: Apply MVT to $h(x) = f(x)(g(b)-g(a)) - g(x)(f(b)-f(a))$

(if $g'(x)$, $g(b)-g(a) \neq 0$, $\frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)}$)

Rudin 5.13 Idea of Proof of L'Hospital

$f, g : (a,b) \to \mathbb{R}$ diff, $g'(x) \neq 0$ for all $x \in (a,b)$, $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$

$\lim_{x\to a} |g(x)| = \infty$ Then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$



Let $\varepsilon > 0$. There exists $\delta > 0$ such that for $x \in (a, a+\delta)$
$g(x) > 0$ and $\left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon$, so $\frac{f'(x)}{g'(x)} < L + \varepsilon$.

Let $y \in (a, a+\delta)$ then $x \in (a, y)$. Then exists $x_0 \in (x, y)$
such that $f'(x_0)(g(y) - g(x)) = g'(x_0)(f(y) - f(x))$.

So $g' \neq 0$, $g \to \infty$, g is decr > $g(x)-g(y) > 1$ → Thus $f(x) = f(y) + \frac{f'(x_0)}{g'(x_0)}(g(x) - g(y)) < f(y) + (L+\varepsilon)(g(x)-g(y))$

$$\frac{f(x)}{g(x)} < \frac{f(y)}{g(x)} + (L+\varepsilon)\left(1 - \frac{g(y)}{g(x)}\right)$$

$\lim_{x\to a} \frac{f(x)}{g(x)} \leq L + \varepsilon$. So this holds for $\varepsilon > 0$, $\lim_{x\to a} \frac{f(x)}{g(x)} \leq L$

Def (negative infinity) ($S_n \to -\infty$)

1) A sequence (S_n) "diverges to infinity" ($S_n \to \infty$) if for all $M \in \mathbb{R}$ there exists $N \in \mathbb{R}$ such that $n > N$ implies $S_n > M$ ($S_n < M$)

2) $f: E \to \mathbb{R}$, a the limit of a sequence in E, then $\lim_{x \to a} f(x) = \infty$ ($-\infty$) if, for every sequence (S_n) in E $S_n \to a$ implies $f(S_n) \to \infty$ ($-\infty$)

3) $f: E \to \mathbb{R}$, E not bounded above (below)
$\lim_{x \to \infty} f(x) = L$ ($-\infty$) if for all sequences (S_n) in E, $S_n \to \infty$ ($-\infty$) implies $f(S_n) \to L$.

Fact $S_n \neq 0$,

1) $|S_n| \to \infty$ if and only if $\frac{1}{S_n} \to 0$

2) $\lim_{x \to a} |f(x)| = \infty$ if and only if $\lim_{x \to a} \frac{1}{f(x)} = 0$ ($f(x) \neq 0$)

$f: (a, \infty)$ 3) $\lim_{x \to \infty} f(x) = L$ if and only if $\lim_{x \to 0} f(\frac{1}{x}) = L$.
$(0, \frac{1}{a})$

4) $\lim_{x \to a} f(x) = \infty$ if and only if for all $M \in \mathbb{R}$ there exists $\delta > 0$ such that for $x \in E$, $|x - a| < \delta$ implies $f(x) > M$.