$g_k : S \to \mathbb{R}$, $k \in \mathbb{N}$, $\sum g_k$ reps the seque (f_n), $f_n = \sum_{k=1}^{n} g_k$

Thm If g_k is cont for all k, and $\sum g_k$ converges unif. to f, then f is continuous.

Proof: $f_n = \sum_{k=1}^{n} g_k$ is continuous.

Thm (Weierstrass M test) Let $g_k : S \to \mathbb{R}$ be a fn, $M_k \in \mathbb{R}$, for all $k \in \mathbb{N}$. If $|g_k(x)| \le M_k$ for all $x \in S$ and $\sum M_k$ converges, then $\sum g_k$ converges uniformly.

Proof
Since $\sum M_k$ converges, it is Cauchy. Let $\varepsilon > 0$, then exists $N \in \mathbb{R}$ such that $n > m > N$ implies $\left|\sum_{k=m+1}^{n} M_k\right| < \varepsilon$.
For all $x \in S$,
$|f_n(x) - f_m(x)| = \left|\sum_{k=m+1}^{n} g_k(x)\right| \le \sum_{k=m+1}^{n} |g_k(x)| \le \sum_{k=m+1}^{n} M_k < \varepsilon$.

So (f_n) is uniformly Cauchy, and conv uniformly.

Example $g_k : [0, 3] \to \mathbb{R}$, $g_k = \left(\frac{\sin(x)}{2}\right)^k$ $\quad |g_k(x)| \le \frac{1}{2^k}$, $\sum \frac{1}{2^k}$ converges, so $\sum g_k$ conv unif.

Indeed, for fixed x, $g_k(x)$ is a geometric series $\sum_{k=0}^{\infty} g_k(x) = \frac{1}{1 - \frac{\sin(x)}{2}}$

Common type of series of fns is a Power Series

Def $a_k \in \mathbb{R}$ for all $k \in \mathbb{N} \cup \{0\}$ let $g_k(x) = a_k x^k$. $\sum_{k=0} g_k = \sum a_k x^k$ is called a power series

when does $\sum a_k x^k$ conv pointw? unif?

Def (Radius of conv R) Let $\sum a_n x^n$ be a power series.

a) If $|a_n|^{1/n}$ is unbounded, define $R = 0$. "R = radius of conv of pow ser $\sum a_n x^n$"

Otherwise let $\beta = \limsup |a_n|^{1/n}$

b) If $\beta = 0$, define $R = \infty$

c) If $\beta > 0$, define $R = 1/\beta$

Thm Let $\sum_{k=0}^{\infty} a_k x^k$ be a power series w/ radius of conv R.
a) If $|x| < R$, $\sum_{k=0}^{\infty} a_k x^k$ converges
b) If $|x| > R$, $\sum_{k=0}^{\infty} a_k x^k$ does not converge

Proof Case 1 $R \in (0, \infty)$. By root test,
$\limsup |a_k x^k|^{1/k} = \limsup (|a_n|^{1/n} |x|) = |x| B$.
So $\sum a_k x^k$ conv if $|x| B = \frac{|x|}{R} < 1$, div if $\frac{|x|}{R} > 1$.

Case 2 $R = \infty$, $B = 0$, so $|x| B = 0 < 1$ for all x.

Case 3 $R = 0$. $|a_n|^{1/n} |x|$ is unbounded for $x \neq 0$, so $\sum a_n x^n$ does not converge for $x \neq 0$.

So $\sum$ conv pointwise on $(-R, R)$
Note: anything can happen at $x = \pm R$

Example
$\sum_{k=0}^{\infty} x^k$; $a_k = 1$ $|a_k|^{1/k} = 1 \to 1$, $R = 1$; at $\sum (\pm 1)^k$ does not conv
Conv pointwise on $(-1, 1)$ (to $\frac{1}{1-x}$)

$\sum_{k=1}^{\infty} \frac{x^k}{k}$ $a_k = \frac{1}{k}$ $|a_k|^{1/k} = \frac{1}{k^{1/k}} \to 1$, $R = 1$ $\sum \frac{1}{k}$ does not conv
but $\sum \frac{(-1)^n}{k}$ does; conv pointwise on $[-1, 1)$ (to $-\ln(1-x)$)

Def Interval of conv I of $\sum a_n x^k$ is set of $x \in \mathbb{R}$ s.t. $\sum a_n x^n$ conv.
Note If radius of conv is R, $I = (-R, R)$ or $[-R, R)$ or $(-R, R]$ or $[-R, R]$.

Thm Let $\sum a_k x^k$ have radius of conv $R > 0$ (could be ∞)
If $0 < R_1 < R$ then $\sum a_n x^n$ conv unif on $[-R_1, R_1]$

Proof
For all $x \in [-R_1, R_1]$, $|a_n x^n| \le a_n R_1^n$.
By the root test, $\limsup |a_n R_1^n|^{1/n} = R_1 \limsup |a_n|^{1/n} = R_1 B = \begin{cases} 0 \\ \frac{R_1}{R} \end{cases} < 1$

Cor Let $\sum a_k x^k$ have rad of conv $R>0$. Then $\sum a_k x^k$ converges to a continuous func on $(-R, R)$

Proof Let $|x_0| < R$; chose R_1 s.t. $|x_0| < R_1 < R$. Then $\sum a_k x^k$ conv unifly on $[-R_1, R_1]$, so the limit is cont at $x_0 \in [-R_1, R_1]$.

Lemma $\sum_{0}^{\infty} a_k x^k$ and $\sum_{1} k a_k x^{k-1}$ have the same radius of convergence

Proof $\sum k a_k x^{k-1}$ conv if and only if $\sum k a_k x^k$ conv

$\limsup |k a_k|^{1/k} = \limsup |k|^{1/k} |a_k|^{1/k} = \limsup |a_k|^{1/k}$

because $|k|^{1/k} \to 1$

Thm Let $\sum a_n x^n$ have radius of conv $R > 0$.
Let $f(x) = \sum a_n x^n$ for $x \in (-R, R)$.
Then for all $x_0 \in (-R, R)$

$$\int_0^{x_0} f = \sum_{k=0}^{\infty} \frac{a_n}{n+1} x_0^{n+1} \qquad f'(x_0) = \sum_{n=1}^{\infty} n a_n x_0^{n-1}$$

Proof

Ex. $x_0 < R$, $\sum a_n x^n \to f$ unif. on $[0, x]$. Thus

$$\int_0^{x_0} f = \lim_{n \to \infty} \int_0^{x_0} \sum_{k=0}^{n} a_k x^k = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{a_k x^{k+1}}{k+1} = \sum \frac{a_k x^{k+1}}{k+1}$$

Let $|x_0| < R_1 < R$.

Ex $\left(\sum_{k=0}^{n} a_k x^k\right)' = \sum_{k=0}^{n} k a_k x^{k-1}$ conv unif on $[-R_1, R_1]$.

and $\sum_{k=0}^{n} a_k x^k \to f$ pointwise, f is diff at x_0 and
$f'(x_0) = \sum k a_k x_0^{k-1}$.