MATH 104

Integrals and derivatives of sequences of functions.

Q: if $f_n(x) \to f(x)$, is it true that

$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx \;?$$

A: not always!

<u>Thm</u> let $f_n, f : [a,b] \to \mathbb{R}$ be functions, $f_n \to f$ uniformly.

If $f_n(x)$ is integrable, then so is $f(x)$ and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Pf. Let $\varepsilon > 0$. Then there is $N \in \mathbb{R}$ such that

$$n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon/_{b-a} \quad \text{(uniform convergence)}$$

for $x \in [a,b]$

Let $P = \{a = t_0 < \cdots < t_k < \cdots < t_m = b\}$ be a partition of $[a,b]$.

Then for $n > N$,

$$U(f,P) = \sum_{k=1}^{m} M(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

$$\leq \sum_{k=1}^{m} \left(M(f_n, [t_{k-1}, t_k]) + \frac{\varepsilon}{b-a} \right)(t_k - t_{k-1}).$$

$$f(x) \leq f_n(x) + \frac{\varepsilon}{b-a}$$
$$\leq M(f_n, [t_{k-1}, t_k]) + \frac{\varepsilon}{b-a}$$

$$= \sum_{k=1}^{m} M(f_n, [t_{k-1}, t_k])(t_k - t_{k-1}) + \frac{\varepsilon}{b-a} \cdot (t_k - t_{k-1})$$

$$= U(f_n, P) + \frac{\varepsilon}{b-a} \underbrace{(t_m - t_0)}_{b-a} = U(f_n, P) + \varepsilon$$

Conclusion: $U(f, P) \leq U(f_n, P) + \varepsilon$

Similarly $L(f, P) \geq L(f_n, P) - \varepsilon$

Conclude (since P was arbitrary)

$$L(f_n) - \varepsilon \leq L(f) \leq U(f) \leq U(f_n) + \varepsilon$$

Since f_n is integrable $L(f_n) = U(f_n) = \int_a^b f_n(x)\,dx$.

let's subtract this value from each term in the inequality.

$$-\varepsilon \leq L(f) - \int_a^b f_n(x)\,dx \leq U(f) - \int_a^b f_n(x)\,dx \leq \varepsilon$$

so $$\left| U(f) - \int_a^b f_n(x)\,dx \right| \leq \varepsilon, \quad \left| L(f) - \int_a^b f_n(x)\,dx \right| \leq \varepsilon.$$

So as $n \to \infty$ (taking $\varepsilon \to 0$) we find

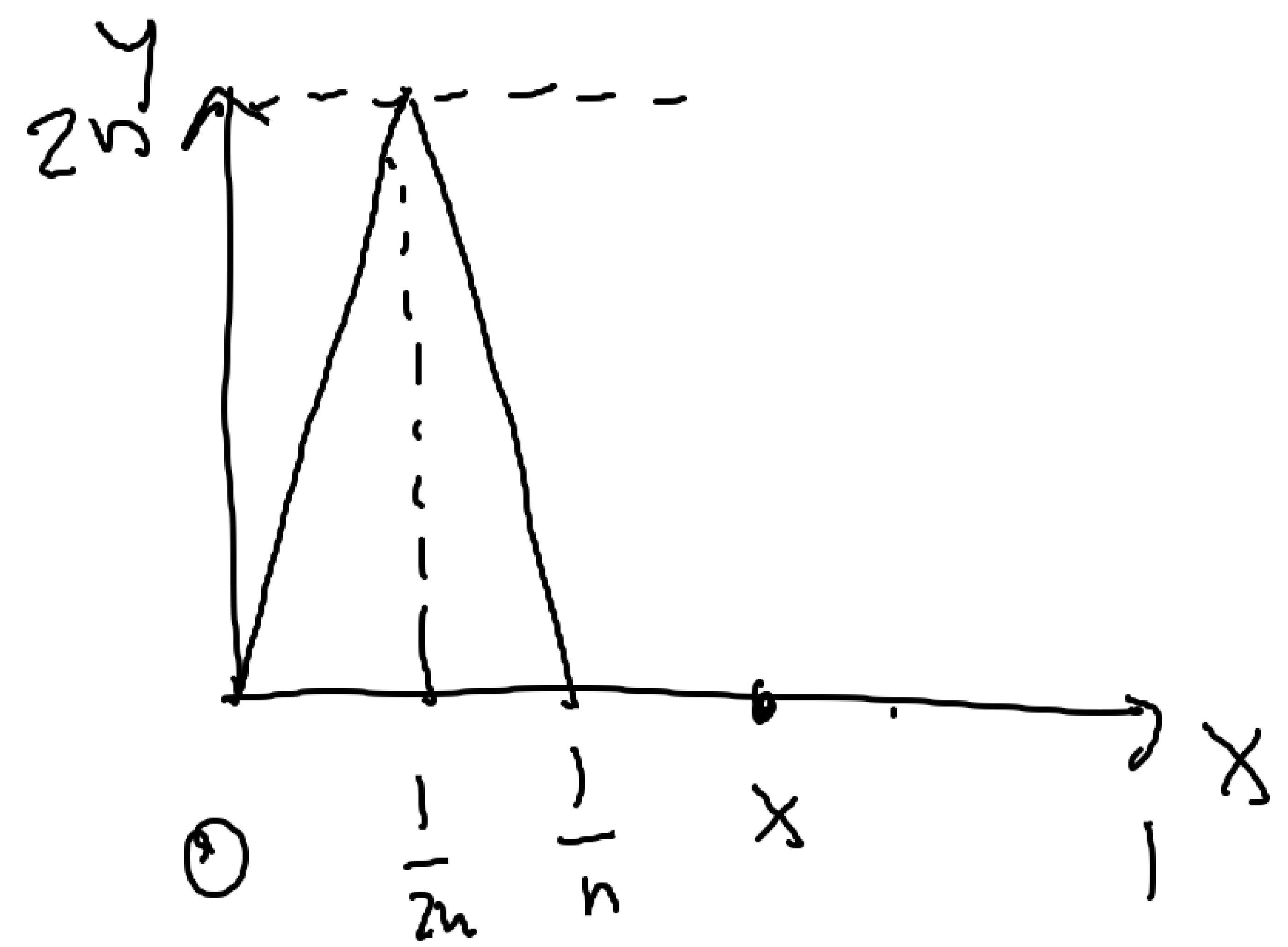
$$\int_a^b f_n(x)\,dx \to U(f) \quad \text{and also} \quad \int_a^b f_n(x)\,dx \to L(f)$$

Conclusion $U(f) = L(f)$ and so f is integrable.

$\uparrow$

This value is $\int_a^b f(x)\,dx$, and it the limit of $\int_a^b f_n(x)\,dx$ as $n \to \infty$.

Note that if $f_n \to f$ only pointwise, the theorem might not apply.



$$f_n(x) = \begin{cases} 4n^2 x & \text{if } 0 \le x \le \frac{1}{2n} \\ 4n^2\left(\frac{1}{n} - x\right) & \text{if } \frac{1}{2n} < x \le \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < x \le 1 \end{cases}$$

note that $f_n(x) \to 0$ is not uniform;

$\sup |f_n(x) - 0| = 2n$ which cannot be made less than ε.

$$\int_0^1 f_n(x) = \text{area of triangle} = \frac{1}{2} \text{ base} \cdot \text{height} = \frac{1}{2} \cdot \frac{1}{n} \cdot 2n = 1$$

$\lim_{n\to\infty} f_n(x) = 0$ for any x, so $f_n \to 0$ pointwise: $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1 \ne 0 = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$

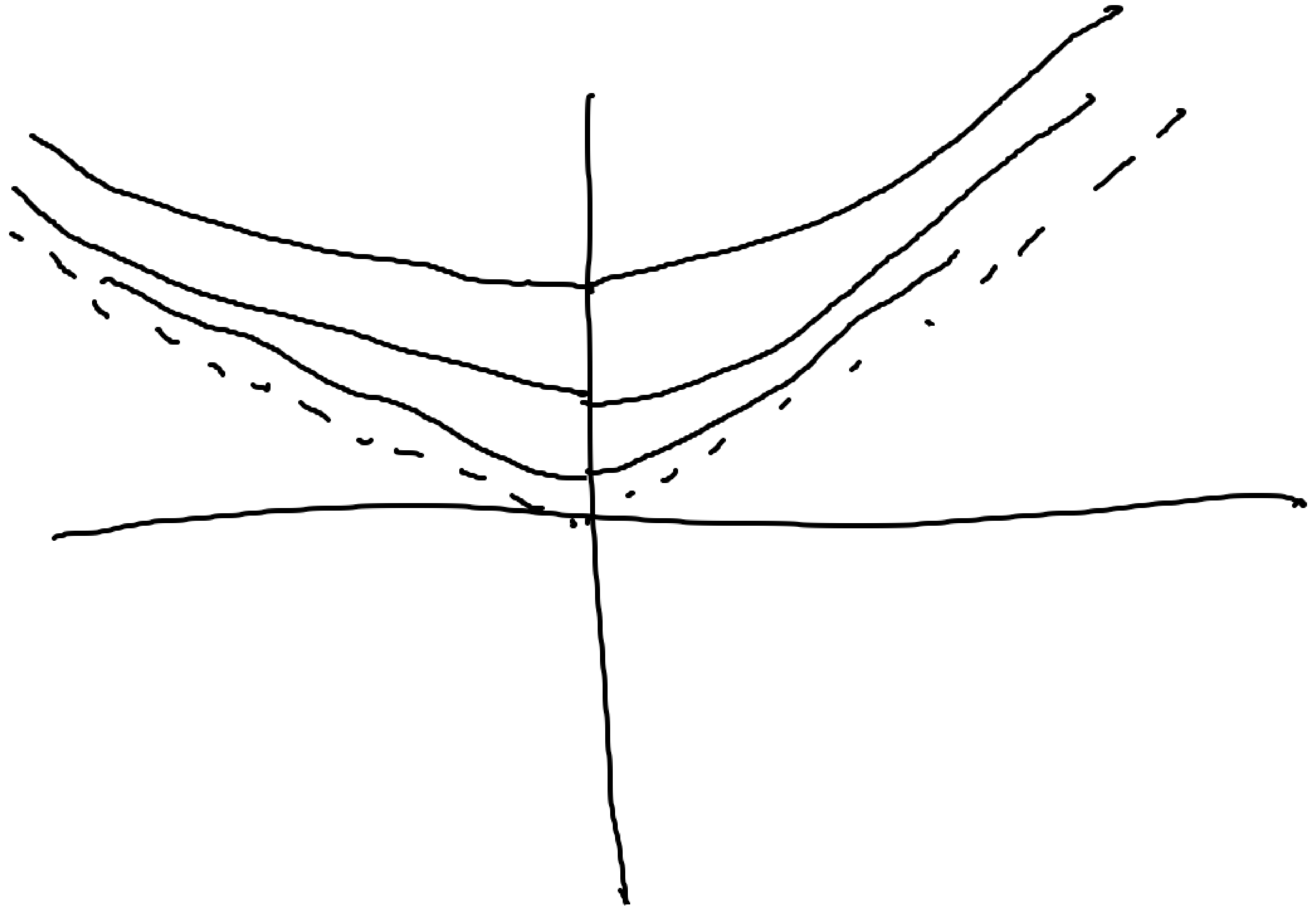
Thm let $f_n : [a,b] \to \mathbb{R}$ be continuous, and diff'ble on (a,b). If $f_n'(x) \to g(x)$ uniformly and $f_n(x_0)$ converges as $n \to \infty$ for some $x_0 \in [a,b]$ then there is $f : [a,b] \to \mathbb{R}$ s.t. $f_n \to f$ and $f'(x) = g(x)$ for all $x \in (a,b)$.

Note: need to assume $f_n(x_0)$ converges at some x_0

$f_n(x) = n$ $\quad f_n'(x) = 0$ $\quad$ so $f_n'(x)$ converges but $f_n(x)$ does not.

Note also: uniform convergence doesn't always preserve differentiability or derivatives

$\sqrt{x^2 + \frac{1}{n}} \longrightarrow |x|$ uniformly.



but $|x|$ is not diff'ble at zero even though $\sqrt{x^2+\frac{1}{n}}$ is.

$\frac{x^n}{n}$ on $[0,1]$

converges to 0 uniformly.

derivative of limit $f^{\underline{n}}$ is $\frac{d}{dx} 0 = 0$.

but $\lim\limits_{n\to\infty} \frac{d}{dx} \frac{x^n}{n} = \lim\limits_{n\to\infty} x^{n-1}$

$= \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$

$\underline{\text{Pf}}$ (special case $f_n'(x)$ integrable, $g(x)$ continuous, for general case see Rudin 7.17).

Define $f(x_0) = \lim\limits_{n\to\infty} f_n(x_0)$ (we were told an x_0 exists where this limit exists).

$$f(x) = \int_{x_0}^{x} g(y)\,dy + f(x_0)$$ (exists because $g(y)$ is continuous, hence integrable)

By fundamental thm of calculus $f'(x) = g(x)$

Let $\varepsilon > 0$, choose N s.t. $n > N$ implies

$$\left| f_n'(x) - g(x) \right| < \frac{\varepsilon}{2(b-a)} \quad \text{(uniform convergence of } f_n' \to g\text{).}$$

for $x \in [a,b]$,

and also $\left| f_n(x_0) - f(x) \right| < \frac{\varepsilon}{2}$

$$f_n(x) - f_n(x_0) = \int_{x_0}^{x} f_n'(x)\,dx \quad \text{(fund. thm. of calc.)}$$

first statement holds for $n > N_1$,
second holds for $n > N_2$
take $N = \max(N_1, N_2)$

So $$\left|f_n(x)-f(x)\right| = \Big| \underbrace{\int_{x_0}^{x} f_n'(y)\,dy + f_n(x_0)}_{f_n(x)} - \underbrace{\int_{x_0}^{x} g(y)\,dy - f(x_0)}_{f(x)} \Big|$$

$$\leq \Big| \int_{x_0}^{x} \underbrace{f_n'(y)-g(y)}_{|f_n'(y)-g(y)| \leq \frac{\varepsilon}{2(b-a)}}\,dy \Big| + \underbrace{\left| f_n(x_0) - f(x_0)\right|}_{\leq \frac{\varepsilon}{2}}$$

(since $|x-x_0| \leq b-a$)

$$\leq \frac{|x-x_0| \cdot \varepsilon}{2(b-a)} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So $f_n(x) \to f(x)$, in fact uniformly