Def 1) A sequence of functions (f_n) is an list of functions which with the same domain: $f_n: S \to \mathbb{R}$ for each $n \in \mathbb{N}$.

2) A sequence of functions (f_n), $f_n: S \to \mathbb{R}$ converges pointwise to a function $f: S \to \mathbb{R}$ if for all $x \in S$, $(f_n(x))$ converges to $f(x)$
("$f_n \to f$ pointwise") seq. in $\mathbb{R}$ $\mathbb{R}$

3) A sequence of functions (f_n), $f_n: S \to \mathbb{R}$ converges uniformly to a function $f: S \to \mathbb{R}$ if
for all $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that $n > N$, $x \in S$ implies $|f_n(x) - f(x)| < \varepsilon$.
("$f_n \to f$ uniformly")

Fact: If $f_n \to f$ uniformly then $f_n \to f$ pointwise
Proof: Let $x \in S$, $\varepsilon > 0$. Then exists N s.t. $n > N$ implies $|f_n(x) - f(x)| < \varepsilon$, so $f_n(x) \to f(x)$

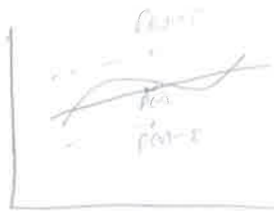
The difference: pointwise converges: For all $x \in S$, $\varepsilon > 0$, there exists N_x s.t. $n > N$ then $|f_n(x) - f(x)| < \varepsilon$
might depend on x

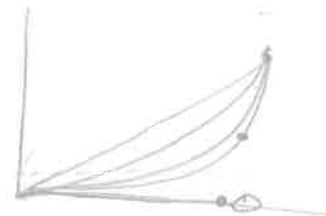
Example 1) $f_n: [0,1] \to \mathbb{R}$, $f_n(x) = x^n$ $f: [0,1]$, $f_n(x) = 0$ if $x < 1$
$f_n(1) = 1$

$f_n \to f$ pointwise: If $x < 1$, $x^n \to 0$ if $x = 1$, $x^n = 1 \to 1$.

2) $g_n: [0,1] \to \mathbb{R}$, $g_n(x) = \frac{x}{n}$ $f: [0,1] \to \mathbb{R}$, $f(x) = 0$
Let $\varepsilon > 0$. Choose $N = \frac{1}{\varepsilon}$. If $n > N$, $x \in [0,1]$, $\left|\frac{x}{n} - 0\right| = \frac{x}{n} \le \frac{1}{n} < \frac{1}{N} = \varepsilon$.

ε def: for $\varepsilon > 0$ we can N s.t. $n \geq N$ ... f_n ... ε ... f





Eg 6.1 $\quad f_n \not\to f$ uniformly.

Choose $\varepsilon = 1/2$. Let $N \in \mathbb{N}$. So f_{N+1} ($= x^{N+1}$) is continuous, there exists $\delta > 0$ such that if $|x-1| < \delta$, $|x^n - 1| < \frac{1}{2}$, so $x^n > 1 - \frac{1}{2} = 1/2$.

Let $x = 1 - \delta/2$; then $|f_{N+1}(x) - f(x)| = x^n > \frac{1}{2} = \varepsilon$.

Theorem Let $f_n, f : S \to \mathbb{R}$ be functions such that $f_n \to f$ uniformly. If f_n is continuous at $x_0 \in S$ for all $n \in \mathbb{N}$, then f is continuous at x_0.

Proof Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon/3$ for all $x \in S$. So f_{N+1} is continuous at x_0, choose $\delta > 0$ such that $x \in S$, $|x - x_0| < \delta$ implies $|f_{N+1}(x) - f_{N+1}(x_0)| < \varepsilon/3$.

If $x \in S$, $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| \leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(x_0)| + |f_{N+1}(x_0) - f(x_0)|$$

$$< \underset{\text{(uniform)}}{\tfrac{\varepsilon}{3}} + \underset{\text{(cont)}}{\tfrac{\varepsilon}{3}} + \underset{\text{(cont unif)}}{\tfrac{\varepsilon}{3}} = \varepsilon$$

So f is continuous.

Lemma Let $f_n, f : S \to \mathbb{R}$ be functions. Then $f_n \to f$ uniformly if and only if $\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| \mid x \in S\} = 0$.

Def A sequence of functions (f_n), $f_n : S \to \mathbb{R}$ is uniformly Cauchy if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n, m \geq N$, $x \in S$ implies $|f_n(x) - f_m(x)| < \varepsilon$.

Pf: If $f_n \to f$ unif, (f_n) is unif Cauchy

Thm If a sequence of functions (f_n), $f_n: S \to \mathbb{R}$ are unif Cauchy, then $\exists$ $f: S \to \mathbb{R}$ such that $f_n \to f$ unif.

Proof

For each $x \in S$, $(f_n(x))$ is Cauchy, so it conv; define $f(x) = \lim_{n\to\infty} f_n(x)$ (thus $f_n \to f$ pointwise)

Let $\varepsilon > 0$. Then $\exists$ N s.t. for $n, m > N$, $x \in S$, $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$.

Fix $n > N$. For all $m > N$, $x \in S$ $f_n(x) - \frac{\varepsilon}{2} \le f_m(x) \le f_n(x) + \varepsilon/2$

Then $f_n(x) - \varepsilon/2 \le f(x) = \lim f_m(x) \le f_n(x) + \varepsilon/2$, or

$|f(x) - f_n(x)| \le \varepsilon/2 < \varepsilon$ for all $x \in S$. This proves $f_n \to f$ unif.

Def Given fns $g_k: S \to \mathbb{R}$, for all $k \in \mathbb{N}$, the series of functions $\sum g_k$ represents the sequence of fns (f_n), $f_n = \sum_{k=1}^{n} g_k$.

If $f_n \to f$ unif (pointwise), we say $\sum g_k$ conv to f unif (pointw).

Thm If $g_k: S \to \mathbb{R}$ is continuous at $x_0 \in S$ for all $k \in \mathbb{N}$ & $\sum g_k$ conv unif to f then f is cont. at x_0.

Proof $f_n = \sum_{k=1}^{n} g_k$ is cont. at x_0.

Thm (Weierstrass M test) Let $g_k: S \to \mathbb{R}$ be functs, $M_k \in \mathbb{R}$ s.t. that $|g_k(x)| \le M_k$ for all $x \in S$. If $\sum M_k$ conv

then $\sum g_k$ converges uniformly